Collective Effects

in

Equilibrium and Nonequilibrium Physics

Website: http://cncs.bnu.edu.cn/mccross/Course/

Caltech Mirror: http://haides.caltech.edu/BNU/

Today's Lecture: Symmetry Aspects of Patterns

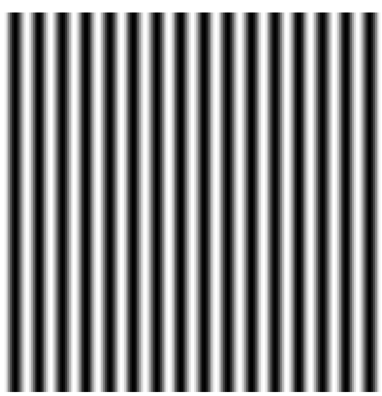
Outline

- Broken symmetry, phase variable and Goldstone modes
- Phase equation near stationary instabilities
- Phase equations far from onset
- Topological defects
- Amplitude and phase equations for oscillatory instabilities

Phase Dynamics

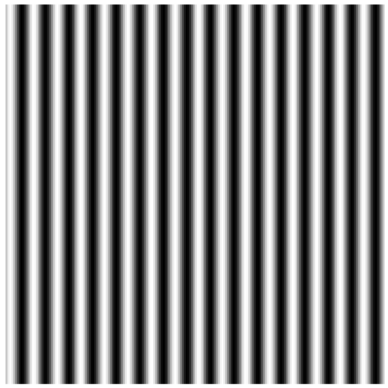
- The local structure of a stripe pattern is $\propto \cos(\mathbf{q}\cdot\mathbf{x}+\theta)+$ harmonics.
- A constant phase change is just a spatial shift of the pattern.
- A phase change that varies slowly in space (over a length η^{-1}, say, with η small) will evolve slowly in time.
- For small enough η the phase variation is slow compared with the relaxation of other degrees of freedom such as the magnitude or the internal structure, and a particularly simple description is obtained.
- The phase variable describes the symmetry properties of the system: the connection between symmetry and slow dynamics is known as Goldstone's theorem.
- Near onset θ is simply the phase of the complex amplitude, and an equation for the phase dynamics can be derived from the amplitude equation for $\eta \ll \varepsilon$ (Pomeau and Manneville, 1979)

Effect of a Phase Change



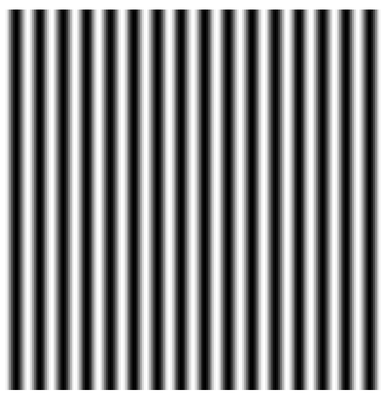
$\cos(\mathbf{q}\cdot\mathbf{x}+\theta);\qquad \mathbf{q}=(1,0),\ \theta=0$

Effect of a Phase Change



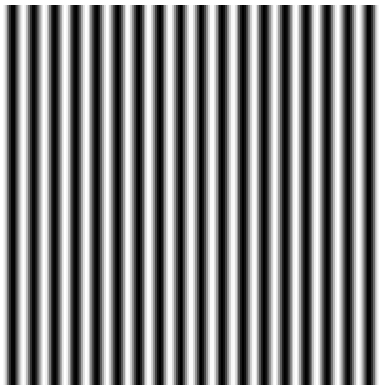
$\cos(\mathbf{q}\cdot\mathbf{x}+\theta);\qquad \mathbf{q}=(1,0),\ \theta=1$

Effect of a Phase Change



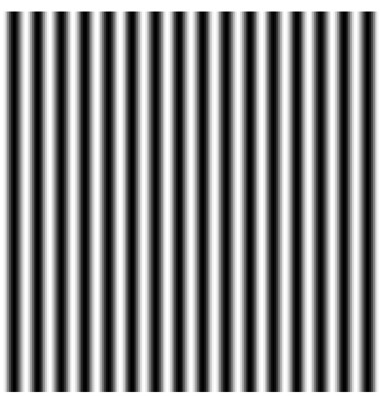
$\cos(\mathbf{q}\cdot\mathbf{x}+\theta);\qquad \mathbf{q}=(1,0),\ \theta=0$

Effect of a Phase Change



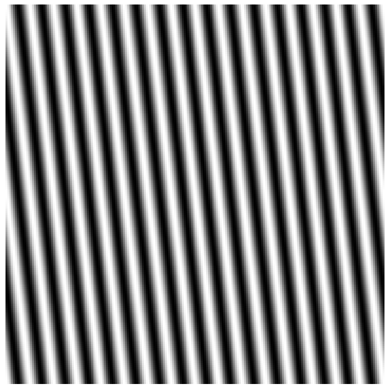
$\cos(\mathbf{q}\cdot\mathbf{x}+\theta);\qquad \mathbf{q}=(1,0),\ \theta=\mathbf{Q}\cdot\mathbf{x},\ \mathbf{Q}=(1/8,0)$

Effect of a Phase Change



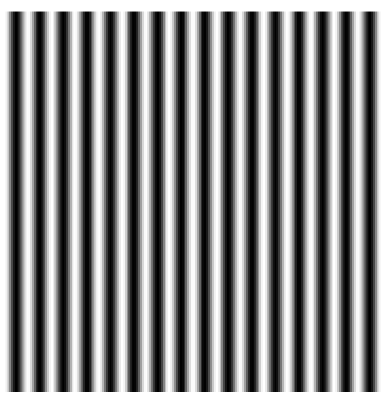
$\cos(\mathbf{q}\cdot\mathbf{x}+\theta);\qquad \mathbf{q}=(1,0),\ \theta=0$

Effect of a Phase Change



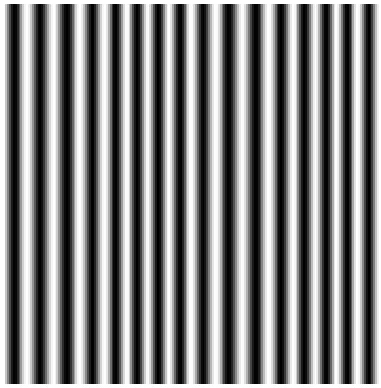
$\cos(\mathbf{q}\cdot\mathbf{x}+\theta);\qquad \mathbf{q}=(1,0),\ \theta=\mathbf{Q}\cdot\mathbf{x},\ \mathbf{Q}=(0,1/8)$

Effect of a Phase Change



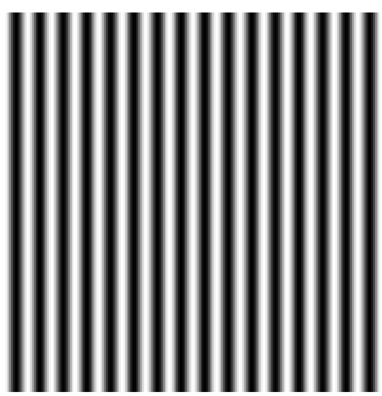
$\cos(\mathbf{q}\cdot\mathbf{x}+\theta);\qquad \mathbf{q}=(1,0),\ \Phi=0$

Effect of a Phase Change



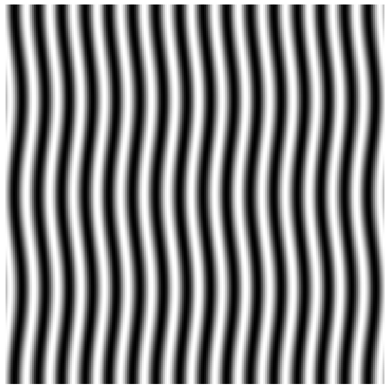
$\cos(\mathbf{q}\cdot\mathbf{x}+\theta);\qquad \mathbf{q}=(1,0),\ \theta=\cos(\mathbf{Q}\cdot\mathbf{x}),\ \mathbf{Q}=(1/8,0)$

Effect of a Phase Change



$\cos(\mathbf{q}\cdot\mathbf{x}+\theta);\qquad \mathbf{q}=(1,0),\ \theta=0$

Effect of a Phase Change



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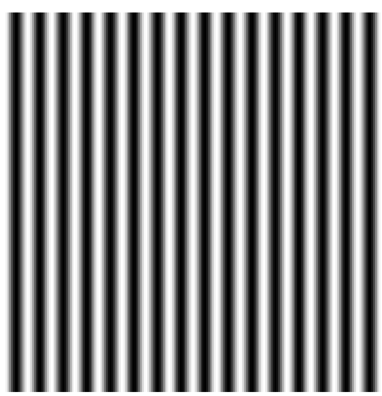
Effect of a Phase Change



$$\cos(\mathbf{q}\cdot\mathbf{x}+\theta); \qquad \mathbf{q}=(1,0),\ \theta=0$$

Phase Dynamics Near Onset

- Near threshold the phase reduces to the phase of the complex amplitude, and the phase equation can be derived by *adiabatically eliminating* the relatively fast dynamics of the magnitude.
- Basic assumption: we are looking at the dynamics driven by gradual spatial variations of the phase, i.e. derivatives of θ are small.
- For simplicity also assume that we are looking at small deviations from a straight stripe pattern, so that the phase perturbations may also be considered small.
- This leads to the *linear phase diffusion equation* first derived by Pomeau and Manneville (1979). We will consider the full nonlinear phase equation in the more general context away from threshold.

Derivation of Phase Equation from Amplitude Equation I

Consider the (scaled) amplitude equation

$$\partial_T \bar{A} = \bar{A} + \left(\partial_X - \frac{i}{2}\partial_Y^2\right)^2 \bar{A} - \left|\bar{A}\right|^2 \bar{A}$$

Look at small perturbations about the state $\bar{A} = a_K e^{iKX}$ with $a_K^2 = 1 - K^2$,

$$\bar{A} = a e^{iKX} e^{i\theta}, \qquad a = a_K + \delta a$$

Expand in

- small phase perturbations θ and amplitude perturbations δa
- low order derivatives of θ (up to second order)

Then using

$$e^{-iKX} e^{-i\theta} \partial_T A = \partial_T a + i a \partial_T \theta,$$

the real part of the equation gives the dynamical equation for a, and the imaginary part of the equation gives the dynamical equation for θ.

Derivation of Phase Equation from Amplitude Equation II

Real part

$$\partial_T \delta a = -2a_K^2 \delta a - 2K a_K \partial_X \theta + \partial_X^2 \delta a$$

For time variations on a T-scale much longer than unity, the term on the left hand side is negligible, and δa is said to adiabatically follow the phase perturbations. The term in $\partial_X^2 \delta a$ will lead to phase derivatives that are higher than second order, and so can be ignored. Hence

$$a_K \delta a \simeq -K \partial_X \theta.$$

Imaginary part

$$a_K \partial_T \theta \simeq 2K \partial_X \delta a + a_K \partial_X^2 \theta + a_K K \partial_Y^2 \theta.$$

Eliminating δa and using $a_K^2 = 1 - K^2$ gives

$$\partial_T \theta = \left[\frac{1 - 3K^2}{1 - K^2}\right] \partial_X^2 \theta + K \partial_Y^2 \theta.$$

the phase *diffusion equation* in scaled units.

Derivation of Phase Equation from Amplitude Equation III

Returning to the unscaled units we get the phase diffusion equation for a phase variation $\theta = kx + \delta\theta$

$$\partial_t \theta = D_\parallel \partial_x^2 \theta + D_\perp \partial_y^2 \theta$$

with diffusion constants for the state with wave number $q = q_c + k$ (with k related to K by $k = \xi_0^{-1} \varepsilon^{1/2} K$)

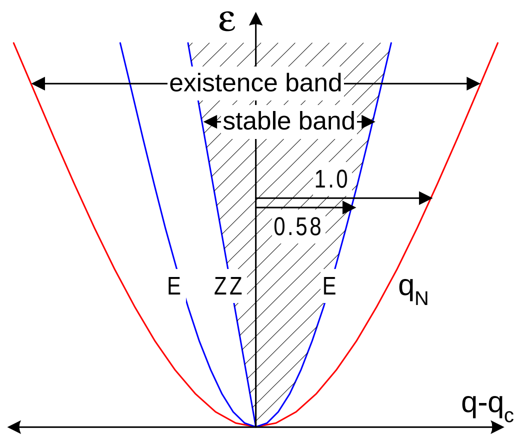
$$D_\parallel = (\xi_0^2 \tau_0^{-1}) \frac{\varepsilon - 3\xi_0^2 k^2}{\varepsilon - \xi_0^2 k^2}$$

$$D_\perp = (\xi_0^2 \tau_0^{-1}) \frac{k}{q_c}.$$

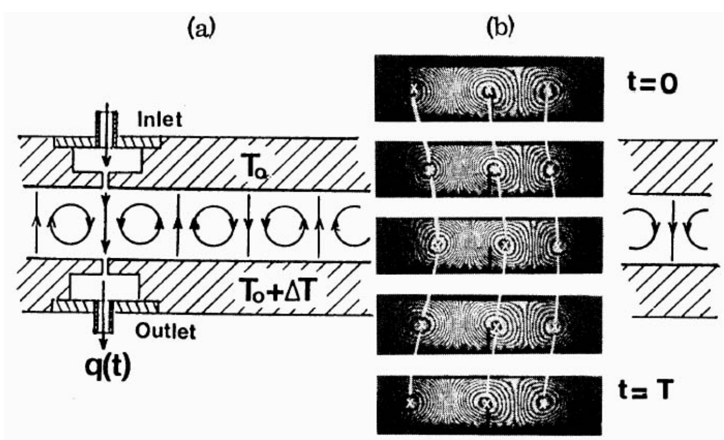
A negative diffusion constant leads to exponentially growing solutions, i.e. the state with wave number $q_c + k$ is *unstable* to long wavelength phase perturbations for

$\lvert\xi_0 k\rvert > \varepsilon^{1/2}/\sqrt{3}$	$D_\parallel < 0$: longitudinal (Eckhaus)
$k < 0$	$D_\perp < 0$: transverse (ZigZag)

Stability Balloon Near Threshold

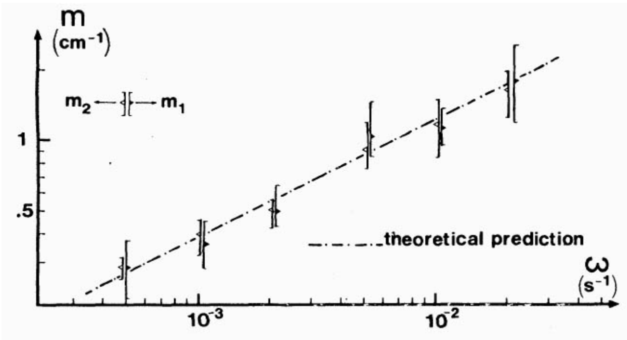


Experimental Test of Phase Diffusion



Wesfried and Croquette (1980)

Experimental Test of Phase Diffusion

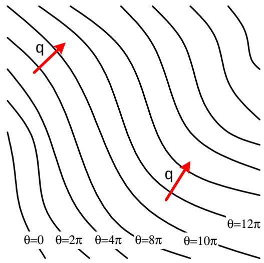


Solution to phase equation with periodic driving at frequency ω

$$\theta(x,t) = \theta_0 e^{-m_1|x|}\cos(m_2|x| - \omega t) \text{ with } m_1 = m_2 = \sqrt{\omega/2D_\parallel}$$

Phase Dynamics Away From Threshold (MCC and Newell, 1984)

Away from threshold the other degrees of freedom relax even more quickly, and so idea of a slow phase equation remains.



- pattern is given by the lines of constant phase θ of a local stripe solution;
- wave vector $\mathbf{q}$ is the gradient of this phase $\mathbf{q} = \nabla\theta$.

A nonlinear saturated straight-stripe solution with wave vector $\mathbf{q} = q\hat{\mathbf{x}}$ is

$$\mathbf{u} = \mathbf{u}_q(\theta, z, t) \qquad \theta = qx$$

For slow spatial variations of the wave vector over a length scale η^{-1} this leads to the ansatz for a pattern of slowly varying stripes

$$\mathbf{u} \approx \mathbf{u}_q(\theta, z, t) + O(\eta), \qquad \mathbf{q} = \nabla\theta(\mathbf{x})$$

where $\mathbf{q} = \mathbf{q}(\eta\mathbf{x})$ so that $\nabla\mathbf{q} = O(\eta)$.

We can develop an equation for the phase variation by expanding in η

$$\tau(q)\partial_t\theta = -\nabla \cdot [\mathbf{q}B(q)]$$

The form of the equation derives from symmetry and smoothness arguments, and expanding up to second order derivatives of the phase.

The parameters $\tau(q)$, $B(q)$ are system dependent functions depending on the equations of motion, $\mathbf{u}_q$, etc.

Small Deviations from Stripes

$$\tau(q)\partial_t\theta = -\nabla \cdot [\mathbf{q}B(q)]$$

For $\theta = qx + \delta\theta$ this reduces to

$$\partial_t\delta\theta = D_\parallel(q)\partial_x^2\delta\theta + D_\perp(q)\partial_y^2\delta\theta$$

with

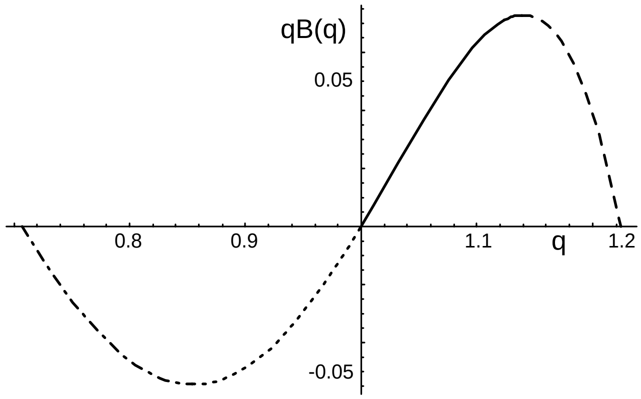
$$D_\perp(q) = -\frac{B(q)}{\tau(q)}$$

$$D_\parallel(q) = -\frac{1}{\tau(q)}\frac{d(qB(q))}{dq}$$

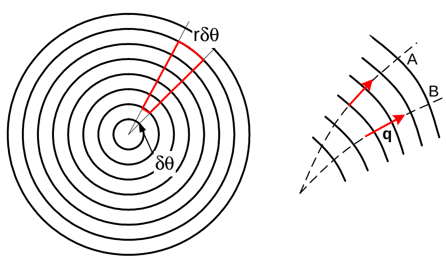
A negative diffusion constant signals instability:

- $[qB(q)]' < 0$: Eckhaus instability
- $B(q) < 0$: zigzag instability

Phase Parameters for the Swift-Hohenberg Equation



Application: Wave Number Selection by a Focus

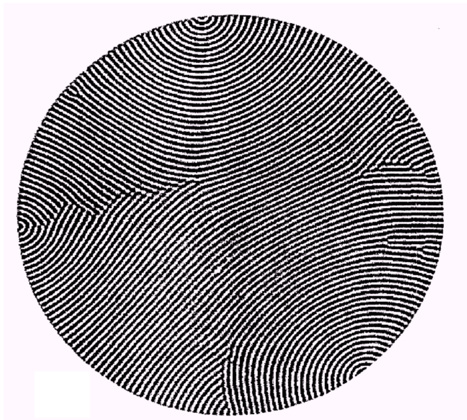


$$\nabla\cdot(\mathbf{q}B(q))=0 \qquad \Rightarrow \qquad \oint B(q)\mathbf{q}\cdot\hat{\mathbf{n}}\,dl=\mathbf{0}$$

$$qB(q)=\frac{C}{r}\underset{r\to\infty}{\rightarrow}0$$

i.e. $q\to q_f$ with $B(q_f)=0$, the wave number of the zigzag instability!

Defects



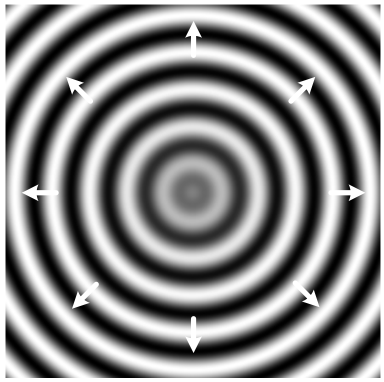
Defects and Broken Symmetries

There are *two* broken symmetries: rotational and translational.
Correspondingly there are two types of topological defects:

- Rotational (associated with direction of wavevector **q**)
  - ◇ Focus, disclination (point defect)
  - ◇ Grain boundary (line defect)
- Translational (associated with phase θ)
  - ◇ Dislocation (point defect)

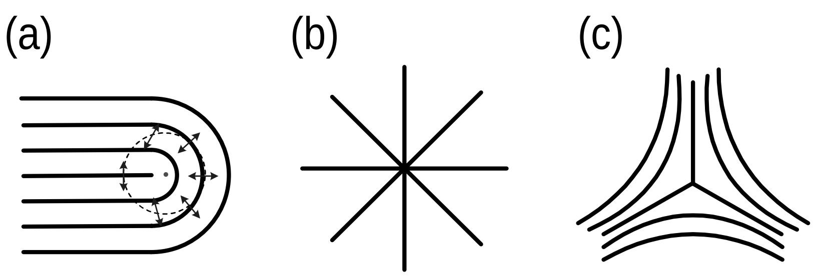
Unfortunately the rotational defects cannot be considered independently of the broken translational symmetry, and this complicates the discussion.

Focus/target defect



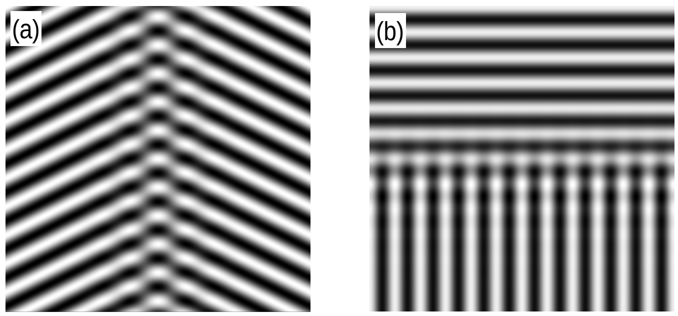
Wavevector winding number = 1

Disclinations

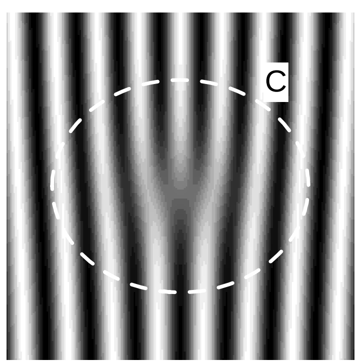


Winding numbers: (a) $\frac{1}{2}$; (b) 1; (c) -1

Grain Boundaries

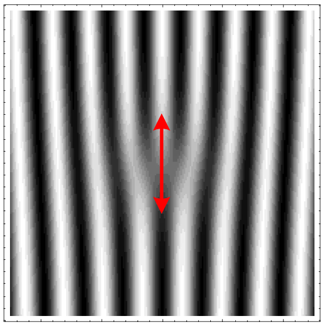


Dislocation



$$\text{Phase winding number } = \frac{1}{2\pi}\oint \nabla\theta \cdot \mathbf{dl} = 1$$

Dislocation Climb



Climb motion is through symmetry related states and is smooth

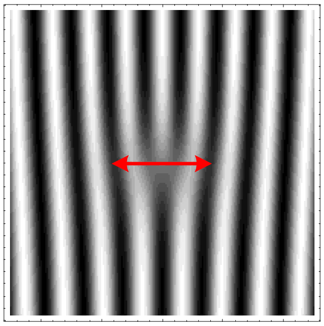
Climb velocity

$$v_d \approx \beta(q - q_d)$$

What is q_d? Easy in equilibrium systems, not in ones far from equilibrium.

Convection experiments (from website of Eberhard Bodenschatz)

Dislocation Glide



Glide motion involves stripe pinch off, and is pinned to the periodic structure

Dislocation motion is important in the relaxation of patterns.

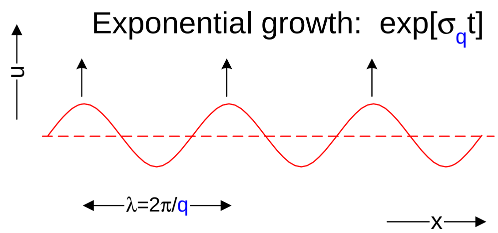
Oscillatory Instabilities

The same set of ideas can be applied to oscillatory instabilities $\mathrm{Im}\,\sigma(\mathbf{q} = \mathbf{q}_c) \neq 0$

- Now the amplitude equation is the Complex Ginzburg-Landau (CGL) equation (complex coefficients)
- Phase equation can be used to understand shocks

I will briefly discuss the case of an instability to spatially uniform oscillations ($q_c = 0$)

Linear Instability



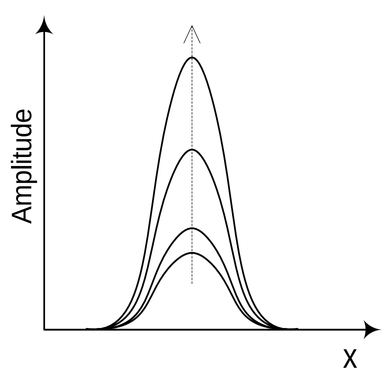
$$\delta \mathbf{u_q}(\mathbf{x}_\perp, z, t) = \mathbf{u_q}(z)\, e^{i\mathbf{q}\cdot\mathbf{x}_\perp}\, e^{\sigma_\mathbf{q} t}$$

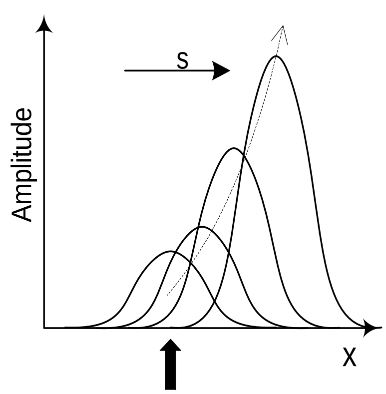
If $\omega_c = -\operatorname{Im}\sigma_{q_c} \neq 0$ we have an instability to

- for $q_c = 0$: a nonlinear oscillator which also supports travelling waves
- for $q_c \neq 0$: a wave pattern (standing or travelling)

Important new concept: absolute v. convective instability

Absolute and Convective Instability





Conditions for Convective and Absolute Instability

- **Convective instability:** same as condition for instability to Fourier mode

$$\mathrm{Max}_{\mathbf{q}}\ \mathrm{Re}\,\sigma(\mathbf{q}) = 0$$

- **Absolute instability:** for a growth rate spectrum σ_q, the system is absolutely unstable if

$$\mathrm{Re}\,\sigma(\mathbf{q}_s) = 0$$

where $\mathbf{q}_s$ is a *complex* wave vector given by the solution of the stationary phase condition

$$\frac{d\sigma_{\mathbf{q}}}{d\mathbf{q}} = 0$$

Derivation of Condition for Absolute Instability

In the linear regime the disturbance growing from any given initial condition $u_p(\mathbf{x}, t = 0)$ can be expressed as

$$u_p(x,t) = \int_{-\infty}^{\infty} dq\, e^{iqx+\sigma_q t} \int_{-\infty}^{\infty} dx'\, u_p(x',0) e^{-iqx'}$$

Rewrite the integral as

$$u_p(x,t) = \int_{-\infty}^{\infty} dx' u_p(x',0) \int_{-\infty}^{\infty} dq\, e^{iq(x-x')+\sigma_q t}$$

For large time and at fixed distance the integral can be estimated using the stationary phase method: the integral is dominated by the region around the complex wave number $q = q_s$ given by the solution of

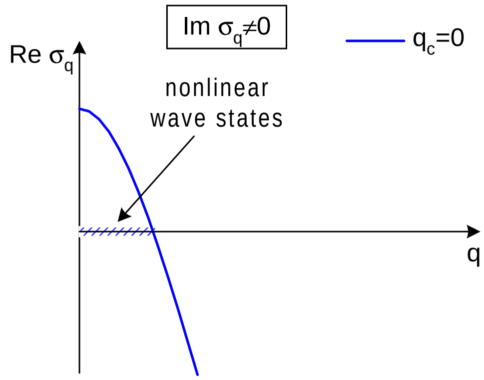
$$\frac{d\sigma_q}{dq} = 0$$

Estimating the integral from the value of the integrand at the stationary phase point gives

$$u_p(x = 0, t) \sim e^{\sigma_{q_s} t}$$

Thus the system will be absolutely unstable for $\mathrm{Re}\,\sigma_{q_s} > 0$.

Oscillatory Intability: Complex Ginzburg-Landau



1d: $$\partial_T \bar{A} = (1+ic_0)\bar{A} + (1+ic_1)\partial_X^2 \bar{A} - (1-ic_3)\left|\bar{A}\right|^2 \bar{A}$$

2d: $$\partial_T \bar{A} = (1+ic_0)\bar{A} + (1+ic_1)\nabla_\perp^2 \bar{A} - (1-ic_3)\left|\bar{A}\right|^2 \bar{A}$$

Simulations of the CGL Equation

General equation (2d)

$$\partial_T \bar{A} = (1+ic_0)\bar{A} + (1+ic_1)\nabla_\perp^2 \bar{A} - (1-ic_3)\left|\bar{A}\right|^2 \bar{A}$$

Case simulated:

- $c_0 = -c_3$ (no loss of generality) for simplicity of plots
- $c_1 = 0$ (choice of parameters)

$$\partial_T \bar{A} = (1-ic_3)\bar{A} + \nabla_\perp^2 \bar{A} - (1-ic_3)\left|\bar{A}\right|^2 \bar{A}$$

Simulations…

Nonlinear Wave Patterns

As well as uniform oscillations, CGL equation supports travelling wave solutions, but with properties that are strange to those of us brought up on linear waves:

- Waves annihilate at shocks rather than superimpose
- Waves disappear at boundaries rather than reflect (not shown)
- Defects: importance as persistent sources
- Spiral defects play a conspicuous role, because they are topologically defined persistent sources
- Shocks between spiral defects lead to exponential decay of interaction (not $1/r$ as in real amplitude equation)
- Instabilities can lead to spatiotemporal chaos

Wave Solutions

$$\partial_T \bar{A} = (1 + ic_0)\bar{A} + (1 + ic_1)\nabla_\perp^2 \bar{A} - (1 - ic_3)\left|\bar{A}\right|^2 \bar{A}$$

Travelling wave solutions

$$\bar{A}_K(\mathbf{X}, T) = a_K e^{i(\mathbf{K}\cdot\mathbf{X} - \Omega_K T)}$$

$$a_K^2 = 1 - K^2 \qquad \Omega_K = -(c_0 + c_3) + (c_1 + c_3)K^2$$

Group speed

$$S = d\Omega_K/dK = 2(c_1 + c_3)K$$

Standing waves, based on the addition of waves at $\mathbf{K}$ and $-\mathbf{K}$ can be constructed, but they are unstable towards travelling waves

Stability Analysis

$$\bar{A}_K(\mathbf{X}, T) = (a_K + \delta a)e^{i(\mathbf{K}\cdot\mathbf{X} - \Omega_K T + \delta\theta)}$$

For *small*, *slowly varying* phase perturbations

$$\partial_T \delta\theta + S\partial_X \delta\theta = D_\parallel(K)\partial_X^2 \delta\theta + D_\perp(K)\partial_Y^2 \delta\theta$$

with longitudinal and transverse diffusion with constants

$$D_\parallel(K) = (1 - c_1 c_3)\frac{1 - \nu K^2}{1 - K^2} \qquad D_\perp(K) = (1 - c_1 c_3)$$

with

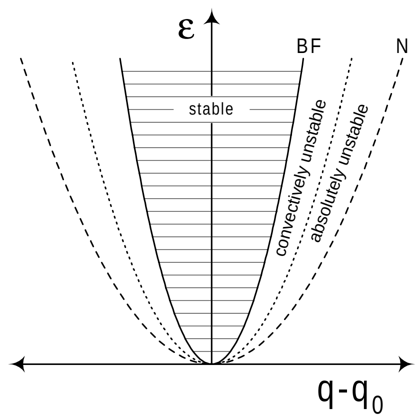
$$\nu = \frac{3 - c_1 c_3 + 2c_3^2}{1 - c_1 c_3}$$

- $D_\parallel = 0 \Rightarrow$ Benjamin-Feir instability (longitudinal sideband instability analogous to Eckhaus) for

  $$|K| \geq \Lambda_B = \nu^{-1}$$

  leaving a stable band of wave numbers with width a fraction ν^{-1} of the existence band.
- For $1 - c_1 c_3 < 0$ *all* wave states are unstable (Newell)

Stability Balloon



Nonlinear Phase Equation

For slow phase variations about spatially uniform oscillations (now keeping all terms up to second order in derivatives)

$$\partial_T \theta = \Omega + \alpha \nabla_\perp^2 \theta - \beta (\vec{\nabla}_\perp \theta)^2$$

with

$$\alpha = 1 - c_1 c_3$$
$$\beta = c_1 + c_3$$
$$\Omega = c_0 + c_3$$

Can be used to understand shocks

Cole-Hopf Transformation

The Cole-Hopf transformation

$$\chi(X, Y, T) = \exp[-\beta\theta(X, Y, T)/\alpha]$$

transforms the nonlinear phase equation into the *linear* equation for χ

$$\partial_T \chi = \alpha \nabla_X^2 \chi$$

Plane wave solutions

$$\chi = \exp\left[(\mp\beta K X + \beta^2 K^2 T)/\alpha\right]$$

correspond to the phase variations

$$\theta = \pm K X - \beta K^2 T$$

Cole-Hopf Transformation (cont)

Since the χ equation is *linear*, we can superimpose a pair of these solutions

$$\chi = \exp\left[(-\beta K X + \beta^2 K^2 T)/\alpha\right] + \exp\left[(+\beta K X + \beta^2 K^2 T)/\alpha\right]$$

The phase is

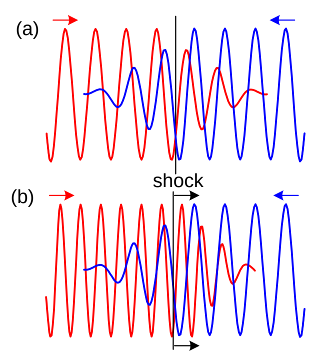
$$\theta = -\beta K^2 T - \frac{\alpha}{\beta} \ln[2\cosh(\beta K X/\alpha)].$$

For large $|X|$ the phase is given by (assuming βK positive)

$$\theta \to -K X - \beta K^2 T - \tfrac{\alpha}{\beta}\exp(-2\beta K X/\alpha) \quad \text{for} \quad X \to +\infty$$

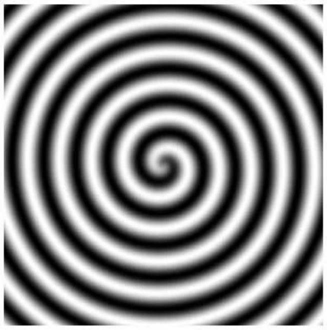
i.e. left moving waves plus exponentially decaying right moving waves with the decay length $\alpha/2\beta K$. Similarly for $X \to -\infty$ get left moving waves with exponentially small right moving waves.

Shocks



- Shocks are sinks, not sources
- For positive group speed shocks between waves of different frequency move so that the higher frequency region expands

Spiral Defects



m-armed spiral: $$\oint \nabla\theta \cdot \mathbf{dl} = m \times 2\pi$$

$$\bar{A} = a(R)e^{i(K(R)R+m\theta-\Omega_s T)}$$

with for $R \to \infty$

$$a(R) \to a_K \qquad K(R) \to K_s \qquad \text{with} \qquad \Omega_K(K_s) = \Omega_s$$

Uniqueness

A key question is whether there is a family of spirals giving a continuous range of possible frequencies Ω_s, or there is a unique spiral structure with a prescribed frequency that selects a particular wave number (or possibly a discrete set of possible spirals).

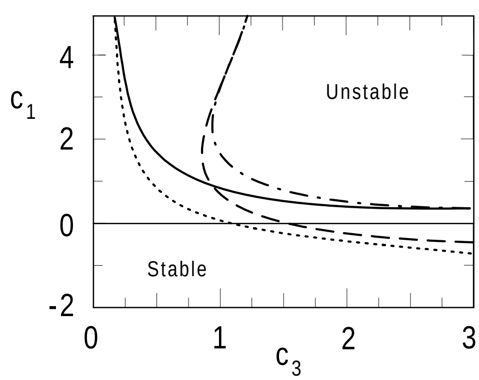
A perturbative treatments of the CGLE for small $c_1 + c_3$ about the real amplitude equation predicts a unique stable spiral structure, with a wave number K_s that varies as (Hagan, 1982)

$$K_s \to \frac{1.018}{|c_1 + c_3|} \exp[-\frac{\pi}{2\,|c_1 + c_3|}].$$

Stability Revisited

- Wave number of nonlinear waves determined by spirals
- Only BF stability of waves at K_s relevant to stability of periodic state
- Convective instability may not lead to breakdown
- Core instabilities may intervene

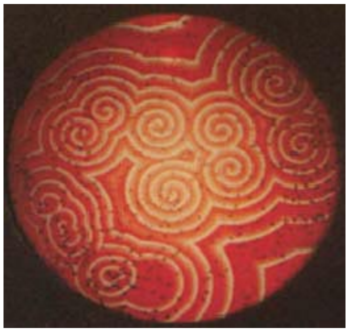
Stability lines of the CGLE (unstable states are towards larger positive c_1c_3)



solid line: Newell criterion $c_1c_3 = 1$
dotted line: convective Benjamin-Feir instability of spiral-selected wavenumber
dashed line: absolute instability of spiral selected wavenumber
dashed-dotted line: absolute instability of whole wavenumber band

Waves in Excitable Media

Waves in reaction-diffusion systems such as chemicals or heart tissue show similar properties to waves in the CGL



[From Winfree and Strogatz (1983) and the website of G. Bub, McGill]