A VAN KAMPEN THEOREM FOR WEAK JOINS

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1. *Introduction*

The infinite Hawaiian earring $\mathscr{X}$ is the subspace of the plane consisting of the infinite family of circles X_i of radius $1/i$ and centre $(1/i, 0)$, mutually tangent to the y-axis at the origin. The question which motivated this paper is: what is the fundamental group of $\mathscr{X}$?

At first glance, $\mathscr{X}$ might be confused with the join $\mathbf{X}_\infty = \bigvee_{i=1}^{\infty} \mathbf{X}_i$ of the X_i at the origin y, that is, $\mathbf{X}_\infty$ is the quotient of the disjoint union of the X_i by the relation which identifies all the copies y_i of y. Indeed, the obvious bijection $\mathbf{X}_\infty \to \mathscr{X}$ is continuous and a local homeomorphism except at y. In both spaces, the neighbourhoods of this point are of the form $\bigcup_{i=1}^{\infty} U_i$ where U_i is a neighbourhood of y_i in X_i. But neighbourhoods in $\mathscr{X}$ must also satisfy $U_i = X_i$ for almost all i.

This difference in the topology at the distinguished point has a striking effect on the fundamental group. According to Van Kampen's theorem, the fundamental group of a join of spaces $(\mathbf{X}, \mathbf{y}) = \bigvee_{i \in I} (X_i, y_i)$ is naturally isomorphic to the free product $*_{i \in I}\, \pi_1(X_i, y_i)$ provided that each X_i is first countable and locally simply connected. (Although usually applied when I is finite, when the hypothesis of first countability may be omitted, the theorem is valid for any countable index set I). Hence $\pi_1(\mathbf{X}_\infty)$ is the free group generated by the elements represented by the circles X_i. In particular, any based loop in $\mathbf{X}_\infty$ is homotopic to a *finite* concatenation of loops each lying in a single X_i. This fails spectacularly in $\mathscr{X}$. As a first example involving infinite concatenation, there is a loop which for each $i \geqslant 1$ circles X_i on the interval $[1/2^i, 1/2^{i-1}]$. In fact, these intervals may be replaced by any sequence of intervals with disjoint interiors. That even such behaviour is relatively simple is shown by the example of the loop which circles X_1 on $[\frac{1}{3}, \frac{2}{3}]$, X_2 on $[\frac{1}{9}, \frac{2}{9}]$ and $[\frac{7}{9}, \frac{8}{9}]$, etc. None of these loops is homotopic to a finite concatenation of loops each lying in a single X_i, nor to a rectifiable loop. Indeed, the last example cannot be rectified at any point of the residual Cantor set on which it sits at y. Moreover, even though the X_i and $\mathbf{X}_\infty$ are locally simply connected at their distinguished points, $\mathscr{X}$ is not.

The infinite Van Kampen theorem of the title identifies the fundamental group of $\mathscr{X}$. In fact, without added difficulty, one can allow X_i to be an arbitrary space which is first countable and locally simply connected at y_i. Briefly (see § 2 for more details), the groups $\mathbf{G}_i = *_{j \leqslant i}\, \pi_1(X_j)$ form in a natural way an inverse system of groups whose limit we denote by $\mathscr{G}$. Elements of $\pi_1(X_j)$ are called letters of type j. Elements of $\mathbf{G}_i$ are then reduced words in the letters of type $j \leqslant i$ and

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elements of $\mathcal{G}$ are sequences $\mathfrak{s} = (\mathbf{s}_i)$ of these satisfying suitable compatibility conditions. We call $\mathfrak{s}$ *locally eventually constant* if, for each j, the number of letters of type j in the word $\mathbf{s}_i$ is bounded independently of i, and let $\mathscr{C}$ denote the subgroup these form. Using the Van Kampen isomorphisms between $\pi_1(\mathbf{X}_i)$ and $\mathbf{G}_i$, we construct a map β: $\pi_1(\mathscr{X}) \to \mathscr{C}$ and show in (4.1) that

INFINITE VAN KAMPEN THEOREM. *The map β is injective and its image is $\mathscr{C}$, that is, $\pi_1(\mathscr{X}) \cong \mathscr{C}$.*

In fact, we show, with respect to the natural topologies on $\pi_1(\mathscr{X})$ and $\mathscr{C}$, that β is a homeomorphism onto $\mathscr{C}$.

This theorem was first stated by H. B. Griffiths in **[1]**. Unfortunately his proof of the most delicate assertion—the injectivity of β—contains an essential error. (The problem occurs in the application of Lemma 6.6 to conclude on p. 475 that, in Griffiths's notation: $\mathbf{ml}(\xi) = \xi$. The hypothesis of Lemma 6.6, that 'each point in $X^{(m)}$ remains in $X^{(m)}$ during the homotopy' cannot be realized. Taking the $X^{(n)}$ to be circles, a simple counter-example is given by the loop l which circles $X^{(2)}$ clockwise on $[0, \frac{1}{4}]$ and counterclockwise on $[\frac{3}{4}, 1]$, and which goes halfway around $X^{(1)}$ on $[\frac{1}{4}, \frac{1}{2}]$, then returns to the base point in the opposite direction on $[\frac{1}{2}, \frac{3}{4}]$. This loop is nullhomotopic but if H: $[0, 1] \times [0, 1] \to X$ is a homotopy with $H(t, 0) = l(t)$ and $H(t, 1) = y$ then for every $t \in (0, 1)$ there is a u such that $H(t, u)$ lies in $X^{(2)}$ but not at y. Moreover, without the quoted hypothesis the map ψ defined in Lemma 6.6 need not be continuous.) In finding our own proof, we were led rather far from Griffiths's point of view and it seemed to us worthwhile to recast the entire proof. We have also tried, by a more systematic use of characteristic universal properties to both simplify the argument, and separate its algebraic and topological components. The only genuinely new results are the purely algebraic material of § 2 from Corollary 2.7 on and the constructions in § 3.

One point deserves special mention. The proof of the injectivity of β has an unavoidably ineffective character stemming from the existence of homotopy classes not supported by any finite subcollection of the X_i. Here this character is isolated in Proposition 2.7 whose proof requires a set-theoretic version of the Bolzano–Weierstrass theorem: an inverse system of non-empty finite sets has a non-empty inverse limit. The geometric consequence is that given a nullhomotopic loop $\not{p}$ in $\mathscr{X}$, there is no effective procedure to construct a homotopy from $\not{p}$ to the constant loop. Again, this contrasts sharply with the usual Van Kampen theorem which is completely effective.

The paper is organized as follows: § 2 contains the algebraic preliminaries and § 3 the topological ones including a review of the usual Van Kampen theorem for joins; after a brief discussion of the topology of $\mathscr{X}$, § 4 proceeds to the construction of β and the proof of the main theorem.

The problem of identifying $\pi_1(\mathscr{X})$ was originally posed to us as a challenge by Sammy Eilenberg during a Columbia colloquium dinner. His intuition told him that Griffiths's proof must be incomplete—without pinpointing the lacuna—because it had no ineffective element. On the occasion of his seventieth birthday and his retirement from Columbia, we take pleasure in dedicating to him this answer to his question.

Notational conventions. For the reader's convenience, we use three fonts: italic at the level of individual spaces (X_i, y_i) and related objects or individual groups G_i and their elements; boldface, at the level of joins, $\mathbf{X}_i = \bigvee_{j \le i} X_j$, and free products $\mathbf{G}_i = *_{j \le i} G_j$; and script at the level of inverse limits of spaces and groups. The letter P is reserved for a space of suitably tame loops in the typographically analogous X, and Q for a semi-group giving rise to G. As for maps, we have, with some exceptions, used a Latin letter for a map of points, loops, or semi-groups and the corresponding Greek letter for an induced group homomorphism. We use '$\hat{\ }$' to denote taking homotopy classes of loops in topological contexts and reduction of words in algebraic ones. Doubly indexed families of maps between simply indexed families of objects obey the convention that f_j^i takes A_i into A_j.

2. *Products of groups*

Here we will discuss various product constructions on groups which will arise in our study of fundamental groups. We begin with a brief review of the free product to fix our notation for §§ 3–4. For more details, we suggest [**3**]. If G_i is a collection of groups indexed by I, the free product of the G_i, denoted $\mathbf{G} = *_{i \in I} G_i$ is constructed as follows. A word is defined to be a finite, possibly empty, ordered set $\mathbf{q} = ((q_1, i_1), \ldots, q_l, i_l))$ of letters (q_k, i_k) where $i_k \in I$ and $q_k \in G_{i_k}$. The ordered set $(i_1, \ldots, i_l)$ is called the type of $\mathbf{q}$ and the integer l, the length of $\mathbf{q}$. We denote by $\mathbf{Q}$ the set of all words and by $\mathbf{e}$ the empty word. If $\mathbf{r} = ((r_1, j_1), \ldots, (r_m, j_m))$ is another word, the *juxtaposition* $\mathbf{q} \circ \mathbf{r}$ of $\mathbf{q}$ and $\mathbf{r}$ is the word $((q_1, i_1), \ldots, (q_l, i_l), (r_1, j_1), \ldots, (r_m, j_m))$ and the *reversal* $\bar{\mathbf{q}}$ is the word $((q_l^{-1}, i_l), (q_{l-1}^{-1}, i_{l-1}), \ldots, (q_1^{-1}, i_1))$. The pair $(\mathbf{Q}, \circ)$ is a semi-group with identity $\mathbf{e}$ and $\bar{\ }$ is an anti-involution of $\mathbf{Q}$. We say $\mathbf{r}$ is an *elementary simplification* of $\mathbf{q}$ of type i if $\mathbf{r}$ is obtained from $\mathbf{q}$ by replacing two consecutive letters (q_k, i) (q_{k+1}, i) by the single letter $(q_k q_{k+1}, i)$ or by deleting a letter (e_i, i). We denote by $\Rrightarrow$ the transitive relation and by $\equiv$ the equivalence relation generated by elementary simplifications and call the $\equiv$-equivalence classes *simplification classes.*

Straightforward computations show that both $\circ$ and $\bar{\ }$ are well defined on the quotient $\mathbf{G} = \mathbf{Q}/\equiv$ and that $\mathbf{q} \circ \bar{\mathbf{q}} \equiv \bar{\mathbf{q}} \circ \mathbf{q} \equiv \mathbf{e}$. Therefore, $\circ$ defines a group operation on $\mathbf{G}$ with inverse given by $\bar{\ }$. We denote by $\hat{\ }$: $\mathbf{Q} \to \mathbf{G}$ the quotient map. A word in $\mathbf{Q}$ is called *reduced* if it has no elementary simplifications, and $\mathbf{S}$ will denote the set of all reduced words. Clearly, if $\mathbf{q}$ is any word in $\mathbf{Q}$, then a finite sequence of elementary simplifications replaces $\mathbf{q}$ by a reduced word $\mathbf{s} = R(\mathbf{q})$. This word is the unique shortest word in the simplification class of $\mathbf{q}$ and hence is independent of the chosen sequence of simplifications. Therefore $\mathbf{S}$ forms a set of representatives for the simplification classes and the natural bijection $\mathbf{G} \to \mathbf{S}$ fits into the diagram of surjective homomorphisms of semi-groups with anti-involution

$$\begin{array}{ccc} & \mathbf{Q} & \\ {}^{R}\swarrow & & \searrow^{\hat{\ }} \\ \mathbf{S} & \longleftarrow & \mathbf{G} \end{array}$$

By transport of structure, $\mathbf{S}$ becomes a group. Its operation is loosely described as 'juxtaposition followed by simplifications'.

We remark that nowhere have we assumed that I is finite. The use of finite words means that all constructions above involve only finitely many of the G_i and hence can be checked as if I were finite. This finiteness can be expressed more formally, by saying that $\mathbf{G}$ is the direct limit of the free products indexed by the finite subsets of I with respect to the natural system of inclusions.

We now wish to modify this construction so as to allow certain types of 'infinite words'. The basic idea is to replace the direct limit above with an inverse limit. To keep the notation simpler, we henceforth fix the index set I to be the natural numbers $\mathbb{N}$, but by standard arguments all the results which follow have analogues for I any directed index set of countable cofinality.

For each $i \in \mathbb{N} \cup \{\infty\}$, we denote by $\mathbf{Q}_i$ the semigroup of finite words in the letters of type at most i, and by $\mathbf{S}_i$ and $\mathbf{G}_i$, the corresponding groups of reduced words and simplification classes. We will suppress the subscript i on the maps $\hat{}: \mathbf{Q}_i \to \mathbf{G}_i$ and R: $\mathbf{Q}_i \to \mathbf{S}_i$ since all these are obtained by restricting the corresponding maps from $\mathbf{Q}_\infty$. For $j \leqslant i$, and for $\mathbf{q}_i \in \mathbf{Q}_i$, we define $T^i_j(\mathbf{q}_i) \in \mathbf{Q}_j$ to be the word obtained from $\mathbf{q}_i$ by deleting all letters of type greater than j. The T^i_j are morphisms of semi-groups preserving the anti-involutions and satisfy $T^j_k \circ T^i_j = T^i_k$ if $i \geqslant j \geqslant k$. Thus $\{\mathbf{Q}_i, T^i_j\}$ becomes an inverse system of semi-groups with anti-involution. Note that again the maps T^i_j: $\mathbf{Q}_i \to \mathbf{Q}_j$ are all restrictions of the analogous map $T_j = T^\infty_j$: $\mathbf{Q}_\infty \to \mathbf{Q}_j$.

We can perform an analogous construction for reduced words: τ^i_j: $\mathbf{S}_i \to \mathbf{S}_j$ is defined by omitting letters of type greater than j, and then simplifying to a reduced word. This determines an inverse system of groups $\{\mathbf{S}_i, \tau^i_j\}$ and R induces a map of inverse systems. Again one checks that the τ^i_j are well defined on simplification classes, and hence that they induce maps π^i_j: $\mathbf{G}_i \to \mathbf{G}_j$ for $i \geqslant j$. The data $\{\mathbf{G}_i, \pi^i_j\}$ is then also an inverse system of groups and fits into the following diagram of inverse systems of semi-groups with anti-involution:

(2.1)
$$\begin{array}{ccc} & \{\mathbf{Q}_i, T^i_j\} & \\ {}^{R}\swarrow & & \searrow{}^{\wedge} \\ \{\mathbf{S}_i, \tau^i_j\} & \longleftarrow & \{\mathbf{G}_i, \pi^i_j\}. \end{array}$$

We let $\mathcal{Q}$, $\mathcal{S}$, and $\mathcal{G}$ be the inverse limits of these inverse systems and let

(2.2)
$$\begin{array}{ccc} & \mathcal{Q} & \\ {}^{R}\swarrow & & \searrow{}^{\wedge} \\ \mathcal{S} & \longleftarrow & \mathcal{G} \end{array}$$

be the triangle induced by (2.1). Elements of G, for example, are compatible sequences $\mathscr{g} = (\mathbf{g}_i)$, that is, sequences such that $\mathbf{g}_i \in \mathbf{G}_i$ and $\pi^i_j(\mathbf{g}_i) = \mathbf{g}_j$. With the obvious set of projections π_i: $\mathcal{G} \to \mathbf{G}_i$, $\mathcal{G}$ solves the universal problem

(2.3) Given a group $\mathcal{F}$ and homomorphisms α_i: $\mathcal{F} \to \mathbf{G}_i$ compatible with the π^i_j, there is a unique homomorphism β: $\mathcal{F} \to \mathcal{G}$ such that $\pi_i \circ \beta = \alpha_i$.

Two warnings are in order here. First, while a sequence $\mathscr{s} = (\mathbf{s}_i)$ in $\mathcal{S}$ may be viewed as a sequence of elements $\mathbf{s}_i \in \mathbf{Q}_i$, it will not usually yield an element of $\mathbf{Q}$. Indeed, the $\mathbf{s}_i$ are only weakly compatible in the sense that $T^i_j(\mathbf{s}_i) \Rrightarrow \mathbf{s}_j$. The second warning is that $\hat{}: \mathcal{Q} \to \mathcal{G}$ or equivalently $R: \mathcal{Q} \to \mathcal{S}$ is not surjective (unless only finitely many G_i are non-trivial). An example of a sequence in $\mathcal{S}$ which is not even represented by a sequence in $\mathcal{Q}$ is given in (2.5).

We remark that the natural inclusions σ_i^j: $\mathbf{Q}_j \to \mathbf{Q}_i$ split the projections π_j^i: $\mathbf{Q}_i \to \mathbf{Q}_j$. We will also use σ_i^j to denote the induced sections which split the maps τ_j^i and π_j^i.

We say that an element $\mathcal{g} = (\mathbf{q}_i) \in \mathcal{Q}$ is *eventually constant* if there is a $j \in \mathbb{N}$ such that for all $k \geq j$ we have $\mathbf{q}_k = \sigma_k(\mathbf{q}_j)$. The sub-semigroup of $\mathcal{Q}$ consisting of eventually constant sequences is naturally isomorphic to $\mathbf{Q}_\infty$ and its image under the map $\hat{}: \mathcal{Q} \to \mathcal{G}$ is naturally isomorphic to $\mathbf{G}_\infty$.

We next present some constructions involving sequences in $\mathcal{Q}$ which will justify our claim that these are 'infinite words'. Fix $\mathbf{q} = (\mathbf{q}_i) \in \mathcal{Q}$. Define $\mathbf{q}_\infty$ to be the union over i of the letters of type i in $\mathbf{q}_i$. Thus $\mathbf{q}_\infty$ is of at most countable cardinality and it contains only finitely many letters of each type. The set $\mathbf{q}_\infty$ has a natural order: if r is a letter of type i in $\mathbf{q}_\infty$ and r' a letter of type i', then $r < r'$ if, whenever $k \geq \max(i, i')$, then r lies to the left of r' in $\mathbf{q}_k$. This order is also the only one on $\mathbf{q}_\infty$ with respect to which the natural inclusions of $\mathbf{q}_i$ into $\mathbf{q}_\infty$ are all order preserving. For later use, we remark that

LEMMA 2.4. *There is a set W_∞ of disjoint open intervals of* (0, 1) *which has the same order type at* $\mathbf{q}_\infty$. (*The order on W_∞ is that induced by position in* (0, 1).)

Proof. The proof which goes back to Sierpinski [**4**, Theorem 2, p. 21] depends only on the countability of $\mathbf{q}_\infty$. Let $r_1, r_2, \ldots, r_n, \ldots$ be a denumeration of $\mathbf{q}_\infty$. Given $I_1, \ldots, I_{n-1}$ with $I_k = (a_k, b_k)$ and $b_k < a_l$ if $r_k < r_l$, pick $a_n < b_n$ so that $a_n > b_k$ if $r_n \geq r_k$ and $b_n < a_k$ if $r_n < r_k$. Then let $W_\infty = \{I_1, I_2, \ldots\}$: sending r_k to I_k yields the required order isomorphism. We will denote by W_i the image in W_∞ of the letters in $\mathbf{q}_i$.

We now turn to the problem of identifying the image $\hat{\mathcal{Q}}$ of $\mathcal{Q}$ in $\mathcal{G}$, elements of which we will also loosely think of as 'infinite words'. For any $\mathbf{q} \in \mathbf{Q}_\infty$ and $j \in \mathbb{N}$, we define $\#(\mathbf{q}, j)$ to be the number of letters of type j in $\mathbf{q}$. Clearly $\#(\mathbf{q}, j) = \#(T_i(\mathbf{q}), j)$ for any $i \geq j$ and if $\mathbf{q} \Rrightarrow \mathbf{q}'$ then $\#(\mathbf{q}, j) \geq \#(\mathbf{q}', j)$ with equality if and only if $\mathbf{q}'$ is obtained from $\mathbf{q}$ by reductions of type different from j. If $\mathcal{s} = (\mathbf{s}_i) \in \mathcal{S}$ and $j \in \mathbb{N}$, we define $\#(\mathcal{s}, j) = \sup_{i \geq j} \#(\mathbf{s}_i, j)$. We say that $\mathcal{s}$ is *locally eventually constant* if $\#(\mathcal{s}, j) < \infty$ for every $j \in \mathbb{N}$. Clearly $\#(\mathcal{s} \circ \mathcal{s}', j) \leq \#(\mathcal{s}, j) + \#(\mathcal{s}', j)$ and $\#(\bar{\mathcal{s}}, j) = \#(\mathcal{s}, j)$. Thus the locally eventually constant sequences form a subgroup $\mathcal{L}$ of $\mathcal{S}$. We denote by $\mathcal{E}$ the corresponding subgroup of $\mathcal{G}$.

Let us first observe that $R(\mathcal{Q}) \subset \mathcal{L}$ and that $\mathcal{L} \subset \mathcal{S}$, with $\mathcal{L} \neq \mathcal{S}$. Indeed suppose $\mathcal{G} = (\mathbf{q}_i) \in \mathcal{Q}$ and $R(\mathbf{q}_i) = \mathbf{s}_i$. Then $\#(\mathbf{q}_i, j) \geq \#(\mathbf{s}_i, j)$ so that if $\mathcal{s} = R(\mathcal{g})$, then

$$\begin{aligned}\#(\mathcal{s}, j) &= \sup_{i \geq j} \#(\mathbf{s}_i, j)\\ &\leq \sup_{i \geq j} \#(\mathbf{q}_i, j)\\ &= \sup_{i \geq j} \#(T_j^i(\mathbf{q}_i), j)\\ &= \#(\mathbf{q}_j, j)\\ &< \infty.\end{aligned}$$

This shows that $R(\mathscr{Q}) \subset \mathscr{L}$. On the other hand, if g_i is a non-trivial element of G_i and

$$\mathrm{s}_i = \prod_{j<k<i} [g_j, g_k] \tag{2.5}$$

then the sequence $\mathscr{s} = (\mathrm{s}_i) \in \mathscr{S}$. Since $\#(\mathscr{s}, j) = \infty$ for all j, this shows that $\mathscr{L} \subset \mathscr{S}$ when all the G_i are non-trivial. A similar argument shows that $\mathscr{L} \subset \mathscr{S}$ (and $\mathscr{L} \neq \mathscr{S}$) when infinitely many G_i are non-trivial.

The next lemma explains (cf. (iii)) the terminology 'locally eventually constant'.

LEMMA 2.6. *Fix $\mathscr{s} = (\mathrm{s}_i) \in \mathscr{S}$. The following are equivalent:*

(i) $\mathscr{s} \in \mathscr{L}$;

(ii) *for all $j \in \mathbb{N}$, there is an $i = i(j)$ such that if $h \geq i$, then s_i is obtained from $T_i^h(\mathrm{s}_h)$ by simplifications of type greater than j;*

(iii) *for all $j \in \mathbb{N}$, there is an $i = i(j)$ such that if $h \geq i$, then $T_j^h(\mathrm{s}_h) = T_j^i(\mathrm{s}_i)$.*

Proof. First we observe that if $h \geq i \geq j$, then

(1) $\#(\mathrm{s}_h, j) = \#(T_i^h(\mathrm{s}_h), j) \geq \#(\mathrm{s}_i, j)$,

(2) equality holds in (1) if and only if s_i is obtained from $T_i^h(\mathrm{s}_h)$ by simplification of type other than j.

(i)$\Rightarrow$(ii). Given $\mathscr{s} \in \mathscr{L}$ and $j \in \mathbb{N}$ choose $i \geq j$ such that $\#(\mathrm{s}_i, k) = \#(\mathscr{s}, k)$ for all $k \leq j$. Then if $h \geq i$, we have, using (1), that

$$\#(\mathscr{s}, k) \geq \#(\mathrm{s}_h, k) \geq \#(\mathrm{s}_i, k) = \#(\mathscr{s}, k)$$

for any $k \leq j$. Applying (2), we see that s_i is obtained from $T_i^h(\mathrm{s}_h)$ by simplifications of type greater than j.

(ii)$\Rightarrow$(iii). Since s_i is obtained from $T_i^h(\mathrm{s}_h)$ by simplifications of type greater than j, we have

$$T_j^i(\mathrm{s}_i) = T_j^i(T_i^h(\mathrm{s}_h)) = T_j^h(\mathrm{s}_h).$$

(iii)$\Rightarrow$(i). For each $j \in \mathbb{N}$ choose $i = i(j)$ as in (iii). Then

$$\begin{aligned}\#(\mathscr{s}, j) &= \sup_{h \geq i} \#(\mathrm{s}_h, j)\\ &= \sup_{h \geq i} \#(T_j^h(\mathrm{s}_h), j)\\ &= \sup_{h \geq i} \#(T_j^i(\mathrm{s}_i), j)\\ &= \#(T_j^i(\mathrm{s}_i), j) < \infty.\end{aligned}$$

COROLLARY 2.7. (i) $R(\mathscr{Q}) = \mathscr{L}$.
(ii) $\hat{\mathscr{Q}} = \mathscr{E}$.

Thus the elements of $\mathscr{G}$ which are represented by compatible sequences in $\mathscr{Q}$ are exactly those which are locally eventually constant.

Proof. By definition, (i) and (ii) are equivalent. Since $R(\mathcal{Q}) \subset \mathcal{L}$ it remains to show that $\mathcal{L} \subset R(\mathcal{Q})$. Given $\mathfrak{s} = (\mathbf{s}_i) \in \mathcal{L}$ choose an increasing sequence $i(1) \leqslant i(2) \leqslant \ldots$ so that if $h \geqslant i(j)$, then

$$T_j^h(\mathbf{s}_h) = T_j^{i(j)}(\mathbf{s}_{i(j)})$$

and let $\mathbf{q}_j = T_j^{i(j)}(\mathbf{s}_{i(j)})$. Then

$$T_k^j(\mathbf{q}_j) = T_k^j(T_j^{i(j)}(\mathbf{s}_{i(j)})) = T_k^{i(k)}(\mathbf{s}_{i(k)}) = \mathbf{q}_k$$

and

$$R(\mathbf{q}_j) = R(T_j^{i(j)}(\mathbf{s}_{i(j)}) = R(\mathbf{s}_j) = \mathbf{s}_j.$$

Therefore, $\mathfrak{q} = (\mathbf{q}_j) \in \mathcal{Q}$ and $R(\mathcal{Q}) = \mathcal{L}$.

The importance of this result will be to link the *algebraic* condition that $\mathfrak{s}$ is locally eventually constant with a condition—membership in $\mathcal{Q}$—which will have *geometric* significance in the applications to fundamental groups.

We close this section by developing some results about simplifications of words with trivial reduction. Fix such a word $\mathbf{q} \in \mathbf{Q}_\infty$. A *cancellation* C of $\mathbf{q}$ is an equivalence relation on the letters of $\mathbf{q}$ satisfying the following conditions.

(2.8) (i) The letters of each equivalence class c of C are of a single type $i = i(c)$ called the *type* of c.

(ii) The product of the letters in c (taken in $G_{i(c)}$ in the order induced by the order on $\mathbf{q}$) is trivial.

(iii) Order the classes of C by $c < c'$ if the right-most letter of c appears to the left of the right-most letter of c' in $\mathbf{q}$. If $c < c'$, then in $\mathbf{q}$ *either*

(a) every letter in c appears to the left of every letter in c', *or*

(b) every letter in c appears between two consecutive letters in c'.

A cancellation C exists for any $\mathbf{q}$ with $R(\mathbf{q}) = \mathbf{e}$. First suppose that $\mathbf{q}'$ is an elementary simplification of $\mathbf{q}$. If two consecutive letters in $\mathbf{q}$ are multiplied, there is a natural order-preserving surjection from the letters of $\mathbf{q}$ to those of $\mathbf{q}'$. If a letter (e_i, i) is deleted, then there is an order-preserving bijection between the remaining letters of $\mathbf{q}$ and those of $\mathbf{q}'$. In either case if a letter in $\mathbf{q}$ has an image in $\mathbf{q}'$, we call this image its *offspring*. We say that a letter r' in $\mathbf{q}'$ is a *descendent* of the letter r in $\mathbf{q}$ if there is a sequence of elementary simplifications $\mathbf{q} = \mathbf{q}_0, \mathbf{q}_1, \ldots, \mathbf{q}_n = \mathbf{q}'$ and letters $r = r_0, r_1, \ldots, r_n = r'$ with r_i in $\mathbf{q}_i$ and r_{i+1} the offspring of r_i for each i. Given a sequence of elementary simplications from $\mathbf{q}$ to e, we define an equivalence relation C on the letters of $\mathbf{q}$ by $r_1 \sim r_2$ if r_1 and r_2 have a common descendent. Under each elementary simplification, these equivalence classes also descend and the product of the elements in a given class is the same as the product of the elements in its offspring class. Since every class eventually descends to the empty class, each of these products is trivial. Thus C satisfies (ii). If (iii) failed for C, there would be classes c and c' and consecutive letters r_j, r_{j+1} in c and r'_k, r'_{k+1} in c' appearing in $\mathbf{q}$ in the order r_j, r'_k, r_{j+1}, r'_{k+1}. This would contradict the construction: multiplication of descendents of r_j and r_{j+1} must follow deletion of a descendent of r'_k, which must follow multiplication of descendents of r'_k and r'_{k+1}, which must follow deletion of a descendent of r_{j+1}, which must follow the desired multiplication of r_j and r_{j+1}. Conversely, every cancellation C arises (non-canonically) via this construction; for example, by successively multiplying the elements in each class c and deleting the last descendent, taking the classes in the order specified by (2.8)(iii).

If $T_j(\mathbf{q}) = \mathbf{q}'$ and C is a cancellation of $\mathbf{q}$ then the classes in C of type at most j form a cancellation of q' which we denote $T_j(C)$, or $T_j^i(C)$ if we wish to emphasize that $\mathbf{q} \in \mathbf{Q}_i$ for some i. The last result we need is that compatible sequences of words have compatible sequences of cancellations. Precisely,

PROPOSITION 2.9. *If $\mathcal{q} = (\mathbf{q}_i) \in \mathcal{Q}$ and if $\hat{\mathcal{q}} = e \in \mathcal{G}$, then there are cancellations C_i of $\mathbf{q}_i$ such that whenever $i > j$, we have $T_j^i(C_i) = C_j$.*

Proof. Let Γ_i be the set of all cancellations of $\mathbf{q}_i$, which by hypothesis is a non-empty finite set. Deletion defines maps T_j^i: $\Gamma_i \to \Gamma_j$, and the collection $\{\Gamma_i, T_j^i\}$ forms an inverse system of non-empty finite sets. The proposition is therefore immediate from the following well-known lemma.

LEMMA 2.10. *If Γ is the inverse limit of an inverse system $\{\Gamma_i, T_j^i\}$ of non-empty finite sets then $\Gamma \neq \varnothing$.*

Proof. Let $\Gamma_i^h = T_i^h(\Gamma_h)$. The sets Γ_i^h for $h \geqslant i$ form a decreasing sequence of non-empty subsets of Γ_i. Hence $\Gamma_i^\infty = \bigcap_{h \geqslant i} \Gamma_i^h$ is non-empty. Since $T_i^h(\Gamma_h^\infty) = \Gamma_i^\infty$, we can inductively choose elements $C_i \in \Gamma_i^\infty$ with $T_i^{i+1}(C_{i+1}) = C_i$. The sequence (C_i) is then compatible, and hence yields a point in the inverse limit.

It will be convenient to amalgamate the elements C_i of a compatible sequence of cancellations of a sequence $\mathcal{q} \in \mathcal{Q}$. Namely, the C_i determine an equivalence relation C_∞ on the letters in $\mathbf{q}_\infty$. Each C_∞ equivalence class is finite and properties described in (2.8) hold for these classes. We call an equivalence relation with these properties a *cancellation* of $\mathbf{q}_\infty$.

COROLLARY 2.11. *If $\mathcal{q} \in \mathcal{Q}$ and $\hat{\mathcal{q}} = e \in \mathcal{G}$, then there is a cancellation C_∞ for $\mathbf{q}_\infty$.*

3. *Geometrically finite loops and Van Kampen's theorem*

Let (X,y) be a pointed topological space. If U is a neighbourhood of y in X, we let $U' = U \setminus \{y\}$. If p: $[0, 1] \to X$ is a loop in X based at y, then $p^{-1}(y)$ is a closed subset of $[0, 1]$. Its complement, $p^{-1}(X')$ is therefore the union of a collection of disjoint open intervals. We denote this collection of intervals by W_p. The set W_p has a natural order induced by the ordering of $(0, 1)$. We remark that quite often, for example, if X is a circle, any collection of disjoint open intervals in $(0, 1)$ arises as W_p for a suitable p. Therefore, using the argument of Lemma 2.4, we see that W_p may have any countable order type. If the topology at y of X is nice then any loop p is homotopic to a loop p' with $W_{p'} = \{(0, 1)\}$. For example, it suffices that X be locally simply connected at y, and that given any neighbourhood U of y in X, there is a neighbourhood V of y in X with $V \subset U$ and V' path connected in U'. For the figure eight, or indeed any non-trivial join, this fails: given $n > 0$, there are homotopy classes in the figure eight with the property that if p is any representing loop, $\#W_p \geqslant n$. The infinite Hawaiian ring provides an example of a space in which loops with W_p of arbitrary order type are required to represent all homotopy classes.

We will call a loop p for which W_p is finite a *geometrically finite loop*. The next lemma gives a criterion that all homotopy classes in $\pi_1(X, y)$ have geometrically finite representatives.

LEMMA 3.1. *Suppose that X is locally simply connected and first countable at y and that p is a loop in X based at y. Then there is a based homotopy H: $[0, 1] \times [0, 1] \to X$ from p to a geometrically finite loop p'. Moreover, H can be chosen so that if $p(s) = y$ then $H(s, t) = y$ for all $t \in [0, 1]$.*

Proof. We may as well assume that W_p is infinite and then let $I_j = (a_j, b_j)$ for $j \in \mathbb{N}$ be a denumeration of the intervals in W_p. We first observe that if U is a neighbourhood of y in X, then p takes I_j into U for all but finitely many j. Indeed, otherwise there would be sequences α_j and t_j satisfying $\alpha_j \leq t_j \leq \alpha_{j+1}$, $p(\alpha_j) = y$, and $p(t_j) \notin U$. By compactness of the interval, these sequences would have a common accumulation point at which p could not be continuous.

Now using the hypotheses on X and y choose decreasing sequences $U_1 \supset U_2 \supset \dots$ and $V_1 \supset V_2 \supset \dots$ of open neighbourhoods of y, each forming a basis for the topology of X at y and with the property that a loop in V_i can be contracted inside U_i. For convenience, we set $U_0 = V_0 = X$. Now if $I \in W_p$, let $i(I)$ be the greatest index i for which $p(I) \subset V_i$. This defines a function i: $W_p \to \mathbb{N}$ which by the observation above is finite-to-one.

For each $I \in W_p$, with $i(I) \geq 1$, we fix a homotopy h_I: $\bar{I} \times [2^{-i(I)}, 2^{1-i(I)}] \to U_{i(I)}$ between $p|_{\bar{I}}$ and the constant loop at y which is constant at y over the ends of $\bar{I}$. Then we define H: $[0, 1] \times [0, 1] \to X$ by

(1) if $p(s) = y$, then $H(s, t) = y$ for all $t \in [0, 1]$,
(2) if $s \in I \in W_p$ and $i(I) = 0$, then $H(s, t) = p(s)$ for all $t \in [0, 1]$,
(3) if $s \in I \in W_p$ and $i(I) > 0$, then

$$H(s, t) = \begin{cases} p(s) & \text{if } t \leq 2^{-i(I)}, \\ h_I(s, t) & \text{if } 2^{-i(I)} \leq t \leq 2^{1-i(I)}, \\ y & \text{if } t \geq 2^{1-i(I)}. \end{cases}$$

We claim this H is continuous. Given this, H clearly defines a based homotopy between p and a loop p' for which $W_{p'} = \{I \in W_p \mid i(I) = 0\}$. Our initial observation then shows that p' is geometrically finite. Since the requirement in the last sentence of the lemma is guaranteed by (1), H will be the required homotopy.

The continuity of H is clear except at points $(s, 0)$ for which $p(s) = y$. Fix such a point. We must show that for each $n > 0$, $H^{-1}(U_n)$ is a neighbourhood of $(s, 0)$ in $[0, 1] \times [0, 1]$. Let $J = \bigcup_{i(I)<n} I$. Then $H^{-1}(U_n)$ contains $([0, 1] \backslash J) \times [0, 1]$. On the other hand, since p is continuous, $p^{-1}(U_n)$ contains a neighbourhood in $[0, 1]$, of the endpoints of each $\bar{I}$ for which $I \in W_p$. Since J is the union of finitely many such I, there is a closed $J' \subset J$ for which $p^{-1}(U_n) \supseteq ([0, 1] \backslash J')$. By construction, if $s \in J$, then $H_p(s, t) = p(s)$ for $t \leq 2^{-n}$. Therefore, $([0, 1] \backslash J') \times [0, 2^{-n}]$ is a neighbourhood of $(s, 0)$ contained in $H^{-1}(U_n)$.

In § 4, we will need certain explicit homotopies to the constant loop of geometrically finite, null-homotopic loops. These are based on the following construction. Fix a finite set W of disjoint open intervals in $(0, 1)$. The *arch*, A_W

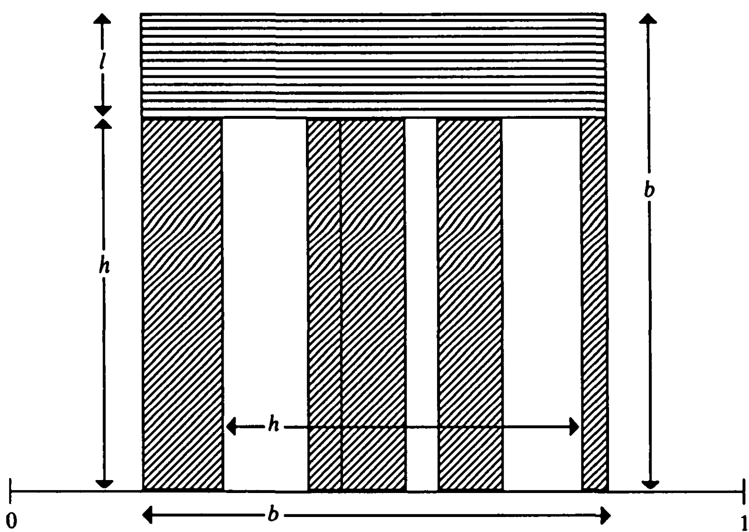

FIG. 1.

of W consists of a set of open rectangular columns supporting an open rectangular lintel as shown in Fig. 1. There is one column based on each interval in W and all of these have the same height h which is the distance between the left-most and right-most intervals in W. The height of the lintel is therefore the sum of the length(s) of the extremal intervals (or interval) of W. The arch A_W therefore lies inside the smallest square whose base contains all the intervals in W and its boundary contains the sides and top of this square, that is, in Fig. 1, $b = l + h$.

By analogy with (2.8)(iii), we say that $W' < W$ if either

(3.2) (a) every interval in W' lies to the left of every interval in W or
(b) every interval in W' lies between two consecutive intervals in W.

The following lemma justifies our careful choice of the proportions of A_W.

LEMMA 3.3. *If $W' < W$, then $A_{W'}$ and A_W are disjoint.*

Proof. In Case (a), this is clear as $A_{W'}$ lies entirely to the left of A_W. If W' lies between consecutive intervals (a_1, b_1) and (a_2, b_2) in W, then $A_{W'}$ lies in the square with base $(b_1, a_2) \times \{0\}$. But the height of the columns of A_W is at least $(a_2 - b_1)$ so $A_{W'} \cap A_W = \emptyset$.

We conclude this section by reviewing a variant of Van Kampen's theorem. Fix a finite or countable collection of spaces (X_i, y_i) with X_i locally simply connected and first countable at y_i, and let $\mathbf{X} = \bigvee_{i \in I} X_i$ be their join. It is easy to check that $\mathbf{X}$ is locally simply connected and first countable at its base-point $\mathbf{y}$, so that every element in $\pi_1(\mathbf{X}, \mathbf{y})$ is represented by a geometrically finite loop. The natural inclusions $X_i \hookrightarrow \mathbf{X}$ induce maps $\sigma^i\colon \pi_1(X_i) \to \pi_1(\mathbf{X})$. There are also natural inclusions $\rho^i\colon \pi_1(X_i) \to *_{i \in I}\, \pi_1(X_i)$ and the universal property of free products implies that there is a unique homomorphism $\tau\colon *_{i \in I}\, \pi_1(X) \to \pi_1(\mathbf{X})$ satisfying $\sigma^i = \tau \circ \rho^i$ for every $i \in I$. The fact that every element in $\pi_1(\mathbf{X}, y)$ is represented by a geometrically finite loop means that τ is surjective. Van Kampen's theorem asserts that τ is an isomorphism.

In fact, τ fits into a more geometric framework which we now describe. Given a word $\mathbf{q} = ((q_1, i_1), \ldots, (q_l, i_l))$ with $q_j \in \pi_1(X_{i_j})$, define $t(\mathbf{q}) \in \pi_1(\mathbf{X})$ as follows: pick loops p_j in X_{i_j} representing each q_j, let $\mathbf{p}$ be the concatenation of the loops p_j in the natural order, and let $t(\mathbf{q}) = \hat{\mathbf{p}}$, the class of $\mathbf{p}$ in $\pi_1(\mathbf{X})$. Easy calculations show that $t(\mathbf{q} \circ \mathbf{r}) = t(\mathbf{q}) \circ t(\mathbf{r})$, that $t(\bar{\mathbf{q}}) = t(\mathbf{q})^{-1}$, and that t is constant on simplification classes. The map t therefore induces a homomorphism $\bar{t}\colon\ *_{i \in I} \pi_1(X_i) \to \pi_1(\mathbf{X})$. Since $t \circ \rho^i = \sigma^i$, the uniqueness mentioned in the definition of τ implies $\tau = \bar{t}$.

Conversely, given a geometrically finite loop $\mathbf{p}$ in $\mathbf{X}$, write $W_{\mathbf{p}} = \{I_1, \ldots, I_k\}$ with I_j to the left of I_k if $j < k$. Suppose the loop p_j traced by $\mathbf{p}$ on I_j lies in X_{i_j} and has class q_j in $\pi_1(X_{i_j})$. Then we let $a(\mathbf{p}) = ((q_1, i_1), \ldots, (q_k, i_k))$, so a takes geometrically finite loops in $\mathbf{X}$ to finite words. The key point, for the proof of which we refer to [**2**, Lemma 3.4] or [**5**, § 3.4], is that $\hat{a} = \hat{\ } \circ a$ is constant on homotopy classes and hence yields a map $\alpha\colon \pi_1(\mathbf{X}) \to *_{i \in I} \pi_1(X_i)$. It is then quickly checked from the definitions that α and τ are inverse to each other.

Let $G_i = \pi_1(X_i)$. Let $\mathbf{G} = *_{i \in I} \pi_1(X_i)$, and let $\mathbf{Q}$ be the associated semi-group of words. Let $\mathbf{P}$ be the set of geometrically finite loops in $\mathbf{X}$ and let $\hat{\ }\colon\ \mathbf{P} \to \pi_1(\mathbf{X})$ be the map of taking homotopy classes. Our discussion may be summarized in

THEOREM 3.4 (Van Kampen's theorem for joins). *There is a commutative diagram*

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{a} & \mathbf{Q} \\ {\scriptstyle \hat{}}\downarrow & \swarrow{\scriptstyle t} & \downarrow{\scriptstyle \hat{}} \\ \pi_1(\mathbf{X}) & \underset{\tau}{\overset{\alpha}{\rightleftarrows}} & \mathbf{G} \end{array}$$

In particular, α and τ are inverse isomorphisms.

REMARK 3.5. The definition of τ shows that τ is natural with respect to joins of based maps between joins of based spaces.

4. *A Van Kampen theorem for weak joins*

We now address directly, for the first time, our original question: what is the fundamental group of the infinite Hawaiian earring? The key observation is that this space is a sort of weak infinite join—more precisely, it is an inverse limit of finite joins. The main result here is a form of Van Kampen's theorem applicable to such weak joins.

We begin by fixing some notation. Let (X_i, y_i), with $i \in \mathbb{N}$, be a sequence of based spaces. For each $i \in \mathbb{N}$, we have the finite join $\mathbf{X}_i = \bigvee_{j \leq i} X_j$. The space $\mathbf{X}_\infty$ is defined to be the direct limit of these spaces. It is also the infinite join of the spaces X_i. On the other hand, if $j \leq i$ then there is a collapsing map $\pi^i_j\colon\ \mathbf{X}_i \to \mathbf{X}_j$ which is the identity on X_k for $k \leq j$ and which collapses X_k to the distinguished point if $j < k \leq i$. These spaces and maps together form an inverse system whose inverse limit we denote by $\mathscr{X}$. By definition, this is the set of compatible sequences in the product $\prod_{i\in\mathbb{N}}\mathbf{X}_i$, with the subspace topology. Hence, giving a function $f\colon Z\to\mathbf{X}$ is equivalent to giving a compatible sequence of functions $f_i\colon Z\to\mathbf{X}_i$ and f is continuous if and only if each f_i is. We denote by $\pi_i\colon \mathscr{X}\to\mathbf{X}_i$ the natural projections and call $\mathscr{X}$ the *weak join* of the X_i.

Let us compare $\mathbf{X}$ and $\mathscr{X}$. The inclusions $\mathbf{X}_j\hookrightarrow\mathbf{X}_i$ for $j\leq i$ are compatible with the projections π^i_j and hence define an embedding of $\mathbf{X}_k$ into $\mathscr{X}$. This in turn is compatible with the inclusions $\mathbf{X}_k\hookrightarrow\mathbf{X}_\infty$ and hence defines a map $\varphi\colon\mathbf{X}_\infty\to\mathscr{X}$. It is easily checked that φ is a continuous bijection which is a local homeomorphism everywhere except at the distinguished point $\mathscr{y}\in\mathscr{X}$. This allows us to view $\mathbf{X}_\infty$ and $\mathscr{X}$ as topologies on the same set, that of $\mathscr{X}$ being coarser. The two differ only at the distinguished point $\mathscr{y}$. A neighbourhood of $\mathbf{y}$ in $\mathbf{X}_\infty$ is any set of the form $\bigvee_{i\in\mathbb{N}} U_i$ where U_i is a neighbourhood of y_i in X_i. Neighbourhoods of $\mathscr{y}$ in $\mathscr{X}$ have the same form but must also satisfy $U_i=X_i$ for almost all i. In particular, if X_i is the ith circle in the infinite Hawaiian earring, we obtain immediately a homeomorphism between the weak join of the X_i and the infinite Hawaiian earring.

For the rest of this paper, we let (X_i, y_i) be a sequence of topological spaces with each X_i locally simply connected and first countable at y_i. We let $G_i=\pi_1(X_i, y_i)$, $\mathbf{G}_i = *_{j\leq i} G_j$, and $\mathscr{G}=\varprojlim \mathbf{G}_i$. The projections $\pi^i_j\colon\mathbf{X}_i\to\mathbf{X}_j$ induce maps $(\pi^i_j)_\#\colon \pi_1(\mathbf{X}_i)\to\pi_1(\mathbf{X}_j)$. The corresponding maps $(\pi_i)_\#\colon\pi_1(\mathscr{X})\to\pi_1(\mathbf{X}_i)$ are compatible with the $(\pi^i_j)_\#$ and hence induce a map $\beta\colon\pi_1(\mathscr{X})\to\mathscr{G}=\varprojlim \pi_1(\mathbf{X}_i)$. Our main result is:

THEOREM 4.1. *The map* $\beta\colon\pi_1(\mathscr{X})\to\mathscr{G}$ *is an injection whose image is the subgroup* $\mathscr{E}$ *of* $\mathscr{G}$ *of locally eventually constant sequences. In particular,* $\pi_1(\mathscr{X})\cong\mathscr{E}$.

We will need first a technical lemma which is the analogue of Lemma 3.1. A loop $\mathscr{p}$ in $\mathscr{X}$ determines a sequence of loops $\mathbf{p}_i$ in $\mathbf{X}_i$ by $\mathbf{p}_i=\pi_i\circ\mathscr{p}$. We say $\mathscr{p}$ is *locally geometrically finite* if each $\mathbf{p}_i$ is geometrically finite.

LEMMA 4.2 (local finiteness lemma). *If each* X_i *is locally simply connected and first countable at* y_i, *then every class in* $\pi_1(\mathscr{X})$ *is represented by a locally geometrically finite loop.*

Proof. Fix a loop $\mathscr{p}$ in $\mathscr{X}$. We shall proceed by inductively constructing homotopies $H_i\colon [0,1]\times[0,1]\to\mathbf{X}_i$ satisfying

(i) $\mathbf{p}_i(s)=H_i(s,0)$,
(ii) the loop $\mathbf{p}'_i(s)=H_i(s,1)$ is geometrically finite,
(iii) $\pi^i_j\circ H_i=H_j$.

Such homotopies give, in the limit, a homotopy H of $\mathscr{p}$ to a loop $\mathscr{p}'$. Since $\pi_i\circ H=H_i$, $\mathscr{p}'$ is a locally geometrically finite loop.

For the convenience of the induction, we pick an increasing sequence $t_1<t_2<\ldots<t_i<\ldots$ in $(0,1)$ and add the following requirements:

(iv) $H_i(s,t)=H_i(s,t_i)$ if $t\geq t_i$,
(v) if $\mathbf{p}_i(s)=\mathbf{y}_i$, then $H_i(s,t)=\mathbf{y}_i$ for all t.

Suppose that $H_1, \ldots, H_{i-1}$ have been constructed. Let p^*: $[0, 1] \to X_i$ be the loop obtained from $\mathscr{p}$ under the natural collapsing of $\mathscr{X}$ to X_i. By Lemma 3.1, there is a homotopy H^* from p^* to a geometrically finite loop with $H^*(s, t) = y_i$ for all t if $p^*(s) = y_i$. Clearly we can suppose that, in addition $H^*(s, t)$ is equal to $H^*(s, t_{i-1})$ if $t \leqslant t_{i-1}$, and equal to $H^*(s, t_i)$ if $t \geqslant t_i$.

Now define $H_i(s, t)$ to equal $H_{i-1}(s, t)$ if $p^*(s) = y_i$ and to equal $H^*(s, t)$ if $p^*(s) \neq y_i$. (We have suppressed the inclusion of $\mathbf{X}_{i-1}$ and X_i into $\mathbf{X}_i$.) An easy variation on the argument of Lemma 3.1 shows that H_i is continuous. Since it clearly verifies (i)–(v), the lemma is proved.

Proof of Theorem 4.1. There are three things to prove:

(i) $\beta(\pi_1(\mathscr{X})) \subset \mathscr{E}$;
(ii) $\beta(\pi_1(\mathscr{X})) \supset \mathscr{E}$;
(iii) β is injective.

Let a_i: $\mathbf{P}_i \to \mathbf{Q}_i$ denote the map of (3.5) between the space $\mathbf{p}_i$ of geometrically finite loops in $\mathbf{X}_i$ and the semi-group $\mathbf{Q}_i$ associated to $\mathbf{G}_i$. If $\mathscr{p}$ is a locally geometrically finite loop in $\mathscr{X}$ and $\mathbf{p}_i$ is its (geometrically finite) projection to $\mathbf{X}_i$, let $\mathbf{q}_i = a_i(\mathbf{p}_i)$. These words are readily checked to be compatible and hence yield an element $\mathscr{q} = (\mathbf{q}_i) \in \mathscr{Q}$. Setting $b(\mathscr{p}) = \mathscr{q}$ defines a map b from the space $\mathscr{p}$ of locally geometrically finite loops in $\mathscr{X}$ to the inverse limit $\mathscr{Q}$ which covers the map β: $\pi_1(\mathscr{X}) \to \mathscr{G}$ in the sense that $\beta(\hat{\mathscr{p}}) = (b(\hat{\mathscr{p}}))\hat{}$. Now, by Lemma 4.2, any class $\ell \in \pi_1(\mathscr{X})$ is represented by a loop $\mathscr{p} \in \mathscr{P}$. Hence $\beta(\ell) = \beta(\hat{\mathscr{p}}) = (b(\mathscr{p}))\hat{} \in \hat{\mathscr{Q}} = \mathscr{E}$, the last equality being Corollary 2.7.

This already proves (i). Moreover (ii) will follow if we show that given $\mathscr{q} \in \mathscr{Q}$ there is a $\mathscr{p} \in \mathscr{P}$ with $b(\mathscr{p}) = \mathscr{q}$. Fix $\mathscr{q} = (\mathbf{q}_i) \in \mathscr{Q}$. Define $\mathbf{q}_\infty$ as in § 2 and use Lemma 2.4 to choose a collection W_∞ of disjoint open subintervals of $(0, 1)$ of the same order type as $\mathbf{q}_\infty$. For each letter r in $\mathbf{q}_\infty$ of type i, choose a based loop p_r in X_i representing r in $\pi_1(X_i)$ and parametrized on the closure of the interval I_r in W_∞ corresponding to r. Then let $\mathscr{p}$ be the loop in $\mathscr{X}$ which equals p_r on I_r for each $r \in \mathbf{q}_\infty$, and which lies at $\mathscr{y}$ elsewhere on $[0, 1]$. Then $\mathbf{p}_i = \pi_i \circ \mathscr{p}$ equals p_r on the intervals I_r in $W_i \subset W_\infty$ and lies at y_i elsewhere on $[0, 1]$. Therefore, $\mathbf{p}_i$ is continuous and geometrically finite and since W_i is order isomorphic to $\mathbf{q}_i$, $a_i(\mathbf{p}_i) = \mathbf{q}_i$. This implies that $\mathscr{p}$ is continuous and locally geometrically finite and that $b(\mathscr{p}) = \mathscr{q}$.

To show the injectivity of β, we must show that given a locally geometrically finite loop $\mathscr{p}$ in $\mathscr{X}$ such that $(b(\mathscr{p}))\hat{} = e_{\mathscr{G}}$, that is, such that *each* $\mathbf{p}_i$ is null-homotopic in $\mathbf{X}_i$, there is a based homotopy between $\mathscr{p}$ and the constant loop. Let $\mathscr{q} = b(\mathscr{p}) \in \mathscr{Q}$ and form $\mathbf{q}_\infty$ as in § 2. Then $\mathbf{q}_\infty$ is naturally order isomorphic to the set of intervals $W = W_p$. We let I_r be the interval in W corresponding to a letter r in $\mathbf{q}_\infty$ and if r has type i, let p_r: $I_r \to X_i$ be the restriction of $\mathscr{p}$ to I_r. Since $\hat{\mathscr{q}} = \mathscr{Q}$, $\mathbf{q}_\infty$ has a cancellation C_∞. For each equivalence class c in C_∞, let $W(c)$ be the set of intervals in W corresponding to the letters of $\mathbf{q}_\infty$ in c, and let $d(c)$ be the data $\{W(c), \{p_r\colon r \in c\}\}$. Finally choose a continuous map $H_{d(c)}$: $\overline{A_{W(c)}} \to X_{i(c)}$ satisfying

(i) $H_{d(c)|\bar{I}_r \times \{0\}} = p_r$,
(ii) $H_{d(c)}$ sends the remainder of the boundary of $\overline{A_{W(c)}}$ to the distinguished point $y_i \in X_{i(c)}$.

Such an $H_{d(c)}$ exists. To see this, we must check that the loop specified by (i) and (ii) as the image of the boundary of the arch $\overline{A_{W(c)}}$ is nullhomotopic. This in turn follows from the definition of a cancellation: the product of the loops p_r, with $r \in c$, has homotopy class $\prod_{r \in C} r$ in $\pi_1(X_i)$, and this is trivial by (2.8)(ii).

If $c \neq c'$, we may suppose that $c' < c$. Then by (2.8)(iii) and (3.2), $W'_c < W_c$. Finally, Lemma 3.3 asserts that the arches $A_{W(c')}$ and $A_{W(c)}$ are disjoint. We may therefore define a map H: $[0, 1] \times [0, 1] \to \mathscr{X}$ by the requirements that $H = H_{d(c)}$ on $\overline{A_{W(c)}}$ and that H send the remainder of the square to the base point y. Clearly $H(s, 0) = p(s)$ and $H(s, 1) = y$ for all $s \in [0, 1]$, so we have only to check the continuity of H, or equivalently that of all the maps $H_i = \pi_i \circ H$. Let B_i be the complement of the arches $A_{W(c)}$, for which c has type at most i. Then H_i sends all of the closed set B_i to the base-point $\mathbf{y}_i$ and so is continuous there. By construction H_i is continuous on each of the finitely many closed arches $\overline{A_{W(c)}}$ for which c has type at most i. Since H_i is continuous when restricted to each of the sets in a *finite* closed cover of the square, H_i is continuous. This completes the proof of Theorem 4.1.

We wish to conclude by briefly sketching one refinement. The space $\Omega(X)$ of based loops in X with base point $\bar{y}$ carries the usual compact open topology and there is a canonical identification of $\pi_1(X)$ with the quotient $\pi_0(\Omega X)$ of ΩX by the equivalence relation of homotopy. Alternatively, if U is an open neighbourhood of y in X, let $\bar{U}$ be the set of loops lying inside U and let $V = V(U) = (\bar{U})\hat{}$. Then as U runs through a basis for the topology of X at y, $\bar{U}$ and V run through bases for the topology of ΩX at $\bar{y}$ and of $\pi_1(X)$ at the identity, respectively. Unfortunately this description depends on the fact that $\bar{y}$ is a constant map so that it fails, in general, to define a topological group structure on $\pi_1(X)$. This defect can be remedied in a canonical way. Indeed, if T is any topology on a group G, there is a finest topology $\bar{T}$ coarser than T with respect to which G becomes a topological group: if $\bar{V}$ denotes the normal closure of V in G, then as V runs through a basis for the topology T at e_G, $\bar{V}$ runs through a basis for $\bar{T}$ at e_G. Usually (for example, whenever X is locally simply connected at y) the resulting topology is the discrete topology. For weak joins, however, this is not generally the case. Indeed, if $X = \mathscr{X}$ and W_i is a simply connected neighbourhood of y_i in X_i, then a convenient basis is provided by the sets

$$\mathscr{U}_k = \Big(\bigcup_{i \leq k} W_i\Big) \cup \Big(\bigcup_{i \geq k} X_i\Big).$$

Then V_k consists of all classes in $\pi_1(\mathscr{X})$ which have a representative p for which $\mathbf{p}_k$ is the constant loop at $\mathbf{y}_k$.

From the Van Kampen isomorphism τ_i: $\mathbf{G}_i \to \pi_1(\mathbf{X}_i, \mathbf{y}_i)$ each $\mathbf{G}_i$ inherits a topology—the discrete topology, since we continue to assume that each $\mathbf{X}_i$ is locally simply connected at $\mathbf{y}_i$. The group $\mathscr{G} = \lim \mathbf{G}_i$, as the inverse limit of topological spaces, has a natural inverse limit topology. A basis of neighbourhoods of the identity in $\mathscr{G}$ is given by the subgroups $\mathscr{A}_k = \{g = (\mathbf{g}_i) \in \mathscr{G} \mid \mathbf{g}_k = \mathbf{e}\}$ for $k \in \mathbb{N}$. Likewise $\mathscr{E}$ has the subspace topology from $\mathscr{G}$ with the subgroups $\mathscr{B}_k = \mathscr{E} \cap \mathscr{A}_k$ forming a basis at the identity. We let $\mathscr{G}_k$ be the subgroup of sequences in $\mathscr{G}$ whose words contain no letters of type less than or equal to k, and we let $\mathscr{E}_k$ be $\mathscr{G}_k \cap \mathscr{E}$ and $\bar{\mathscr{E}}_k$ be the normal closure of $\mathscr{E}_k$ in $\mathscr{E}$. Clearly $\mathscr{E}_k \subset \mathscr{B}_k$ and since $\mathscr{B}_k \lhd \mathscr{E}$, $\bar{\mathscr{E}}_k \subset \mathscr{B}_k$. In fact,

LEMMA 4.3. $\mathscr{B}_k = \bar{\mathscr{E}}_k$.

Proof. To get the other inclusion $\mathcal{B}_k \subset \bar{\mathcal{E}}_k$, we must, given $g \in \mathcal{E}$ such that $\mathbf{g}_k = \mathbf{e}$, find an equation $g = \prod_{i=1}^{n} (g^i)^{h_i}$ (the exponential notation denotes conjugation) with each $g_i \in \mathcal{E}_k$ and each $h_i \in \mathcal{E}$. To obtain this choose q such that $\hat{q} = g$ and write $\mathbf{q}_k = r^1 \ldots r^n$. We suppress the types here using r to denote letters of type less than or equal to k and s to denote possibly empty strings containing *no* such letters. Then for $i > k$ we can write $\mathbf{q}_i = \mathbf{s}_i^0 r^1 \mathbf{s}_i^1 r^2 \ldots r^n \mathbf{s}_i^n$ and it is easily checked that all these equations are induced by an equation in $\mathcal{Q}$:

$$\begin{aligned} q &= s^0 r^1 s^1 r^2 \ldots s^{n-1} r^n s^n \\ &= (s^0)^e (s^1)^{r_1} (s^2)^{r_1 r_2} \ldots (s^n)^{r_1 \ldots r_n} (r_1 \ldots r_n). \end{aligned}$$

But then since $(r_1, \ldots, r_n)\hat{} = \hat{\mathbf{q}}_k = \mathbf{g}_k = \mathbf{e}$, setting $g^i = (s^i)\hat{}$ and $h^i = (\prod_{j \leq i} r_j)\hat{}$ yields the desired equation.

With these preparations in hand, our refinement may be stated as:

PROPOSITION 4.4 (Griffiths). $\beta(\tilde{\mathcal{V}}_k) = \mathcal{B}_k$.

Since these sequences form bases for the respective topologies on $\pi_1(\mathcal{X})$ and $\mathcal{E}$ at the identity we find

COROLLARY 4.5. *The map* $\beta \colon \pi_1(\mathcal{X}) \to \mathcal{E}$ *is a homeomorphism, and therefore an isomorphism of topological groups.*

Proof of Proposition 4. In view of Lemma 4.3, the proposition will follow by taking normal closures if we show that $\beta(\mathcal{V}_k) = \mathcal{E}_k$. Let ℓ be a class in $\mathcal{V}_k$ and p be a locally geometrically finite loop representing ℓ with $\mathbf{p}_k$ constant. If $b(p) = q$ and $\hat{q} = g$, then $\mathbf{q}_k = \mathbf{e}$ and a fortiori so does $\mathbf{g}_k$. Therefore $\beta(\ell) = g \in \mathcal{E}_k$. Conversely, if $g \in \mathcal{E}_k$ and $q \in \mathcal{Q}$ is the sequence satisfying $\hat{q} = g$ constructed in Corollary 2.7 then for $i \gg 0$, $\mathbf{q}_k = T_k^i(\mathbf{g}_i) = \mathbf{e}$. Therefore $\mathbf{q}_\infty$ contains no letters of type less than or equal to k. If p is the loop satisfying $b(p) = q$ constructed in the proof of Theorem 4.1, then $\mathbf{p}_k$ is constant. Therefore $\beta^{-1}(g) = \hat{p} \in \mathcal{V}_k$.

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