Tuesday Jan 12 ①

Irrational rotation algebra / noncommutative torus / quantum torus

$\theta \in \mathbb{R} \setminus \mathbb{Q}$ fixed $\quad \mathcal{H} = L^2(S^1) = L^2(\mathbb{R}/\mathbb{Z})$ uniform Lebesgue measure

$z(t) = \exp(2\pi i t)$

$(Uf)(t) = z(t) f(t)$ multiplication operator

$(Vf)(t) = f(t-\theta)$ phase translation

both unitary operators on $\mathcal{H}$

$$(VUf)(t) = (Uf)(t-\theta) = z(t-\theta)\, f(t-\theta)$$
$$(UVf)(t) = z(t)\, f(t-\theta)$$

$$\Rightarrow \quad UV = e^{2\pi i \theta} VU$$

$\mathcal{A}_\theta$ = C^*-algebra generated by two unitaries U, V with commutation relation

$$UV = e^{2\pi i \theta} VU$$

(if fact can define $\mathcal{A}_\theta$ as universal C^*-alg w/ relations
$U^*U = UU^* = 1 \quad V^*V = VV^* = 1 \quad UV = e^{2\pi i\theta} VU$)

$\mathcal{A}_\theta \cong C(S^1) \rtimes_\alpha \mathbb{Z}$ crossed product algebra description

where U = generator of algebra $C(S^1)$

enough to see $\sigma(U) = S^1$ but $\alpha(U) = VUV^* = e^{-2\pi i\theta} U$

so $\sigma(U)$ invariant under rotation by $2\pi i\theta$ θ-irrational (dense orbit in S^1)

$\Rightarrow \sigma(U) = S^1$

$C(S^1)$ $f(z) = \sum a_n z^n$ Laurent series

or Fourier series as

$f(t) = \sum a_n \exp(2\pi i n t)$

(π, V) rep $\pi: C(S^1) \to B(L^2(S^1))$ multipl. operators

$U = \pi(z)$ and $V: \mathbb{Z} \to B(L^2(S^1))$

generator $V = V(1)$ $V(1)(f)(z) = f(e^{2\pi i\theta} z)$

$V \pi(f) V^* = \pi(f \circ R_\theta^{-1})$

$R_\theta(z) = e^{2\pi i\theta} z$ $\alpha(f) = f \circ R_\theta^{-1}$

$C(S^1) \rtimes_\alpha \mathbb{Z} \cong \mathcal{A}_\theta$

Interpretation as quotient space:

Action of $\mathbb{Z}$ on S^1 by irrational translation

$z \mapsto e^{2\pi i\theta} z$

$e^{2\pi i t} \mapsto e^{2\pi i(t+\theta)}$

has dense orbits: usual quotient $Y = S^1/\mathbb{Z}$ $S^1/e^{2\pi i\theta\mathbb{Z}}$

Would have $C(Y) = \mathbb{C}$ only constant functions

But $\mathcal{A}_\theta$ is NC quotient : nontrivial algebra of functions

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Relation to elliptic curves

(why NC torus?)

a 1-dim complex torus (or elliptic curve)

a quotient $\mathbb{C}/\Lambda$ where $\Lambda \subset \mathbb{R}^2$ a lattice

i.e. rank 2 abelian group $\Lambda \cong \mathbb{Z}^2$

$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ $\quad \omega_1, \omega_2 \in \mathbb{C}$ lin indep over $\mathbb{R}$

i.e. cocompact $\mathbb{C}/\Lambda$ compact (from this cond.)

up to a change of coords:

$\Lambda = \mathbb{Z} + \mathbb{Z}\tau \qquad \tau \in \mathbb{H}$ = upper half plane

when $\mathbb{C}/\Lambda_1 = E_{\tau_1}(\mathbb{C})$ and $\mathbb{C}/\Lambda_2 = E_{\tau_2}(\mathbb{C})$ are isomorphic complex varieties?

When $\exists\, g \in SL(2,\mathbb{Z})$ $\quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\tau_1 = g(\tau_2) = \frac{a\tau_2 + b}{c\tau_2 + d}$$

in fact if $\varphi : E_{\tau_1} \to E_{\tau_2}$ biholom. map

$\exists\, \Phi : \mathbb{C} \to \mathbb{C}$ biholom s.t. $\pi_2 \phi = \phi \pi_1$

$\pi_2 : \mathbb{C} \to \mathbb{C}/\Lambda_2 \qquad \pi_1 : \mathbb{C} \to \mathbb{C}/\Lambda_1$ quotient maps

$$\pi_2 \circ \phi\,(z+1) = \phi \circ \pi_1 (z+1) = \phi \circ \pi(z) = \pi_2 \circ \phi\,(z)$$

$$\Rightarrow \quad \phi(z+1) = \phi(z) + a\tau_2 + b \qquad \text{some integers } a, b$$

same: $\phi(z+\tau_1) = \phi(z) + c\tau_2 + d$

Same reasoning applied to ϕ^{-1} shows $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

We'll see analogous statement for NC tori: Morita equivalence for $\theta_1 = g(\theta_2)$ $\quad g \in SL(2,\mathbb{Z})$

See $\mathcal{A}_\theta$ as a "limiting case of an elliptic curve"

$$\mathbb{C}/\Lambda = \left(\mathbb{C}/\mathbb{Z}\right)/\mathbb{Z}$$

mod out by two commuting actions of $\mathbb{Z}$ and $\mathbb{Z}\tau$ one after the other instead of simultaneously

$\mathbb{C}/\mathbb{Z}$: $z \mapsto z+1$ translations

gives a cylinder
of width 1

equivalent description : changing coordinates via exponential map

$$\exp : \mathbb{C}/\mathbb{Z} \longrightarrow \mathbb{C}^* = \mathbb{C} \setminus \{0\}$$
$$z \longmapsto \exp(2\pi i z)$$

$z \in [0,1) \subset \mathbb{R} \subset \mathbb{C}$ maps to S^1

$\operatorname{Im} z < 0$ maps to $D \setminus \{0\} = \{\lambda \in \mathbb{C}^* : |\lambda| < 1\}$

$\operatorname{Im} z > 0$ maps to $\{\lambda \in \mathbb{C}^* : |\lambda| > 1\}$

Then mod out by second action $\mathbb{Z}\tau$

in new coordinates action becomes

$$z \in \mathbb{C}^* \longmapsto q^n \cdot z$$

$z = \exp(2\pi i w)$, $w \in \mathbb{C}/\mathbb{Z}$ $\qquad q = \exp(2\pi i \tau) \qquad |q| \neq 1$

$$\mathbb{C}^*/q^{\mathbb{Z}} \cong \mathbb{C}/\mathbb{Z}+\mathbb{Z}\tau$$

Jacobi uniformization
of elliptic curve
$E_q(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}}$

what happens when $|q| \to 1$

two cases: $q \to \exp(2\pi i \frac{n}{m})$ a root of unity in S^1 $\xi^m = 1$

then $\mathbb{C}^* / q^{\mathbb{Z}} \cong \mathbb{C}^*$

it is another cylinder of smaller width mapped to isomorphic ~~cylinder~~ $\mathbb{C}^*$ under exp map

(isom. given by multipl by q) in terms of lattices $(\mathbb{Z} + \mathbb{Z}\tau \rightsquigarrow \mathbb{Z} + \mathbb{Z}\frac{n}{m})$ not lin indep over $\mathbb{Q}$

More interesting case when $q \to \exp(2\pi i \theta)$ $\theta \in \mathbb{R} \setminus \mathbb{Q}$

then $\mathbb{C}^* / q^{\mathbb{Z}} \cong (S^1 / q^{\mathbb{Z}}) \times \mathbb{R}^*_+$

"bad quotient" NC torus

"invisible degeneration of elliptic curve"

Semigroups

S associative product
unit element
(not necessarily inverses)

notation:
monoid w/ unit

R commutative ring with unit

$R[S]$ semigroup ring (discrete case)

$f : S \to R$ finite support $(f_1 * f_2)(s) = \sum_{s = s_1 s_2} f_1(s_1) f_2(s_2)$

$f = \sum_{s \in S} a_s \delta_s$ $f_1 * f_2 = \sum a_{s_1} b_{s_2} \delta_{s_1 s_2}$

What about C^*-algebraic version?

$R = \mathbb{C}$ but <u>no</u> involution on $\mathbb{C}[S]$

(because no inverses in general)

$\ell^2(S) = \mathcal{H}$ Hilbert space (for S discrete)

Can represent $\mathbb{C}[S]$ as bounded operators on $\ell^2(S)$

by $$f = \sum_{s \in S} a_s \delta_s$$

$\{\varepsilon_r\}_{r \in S}$ o.n. basis of $\mathbb{C}[S]$

$$\pi(f)\,\varepsilon_r = \sum_{s \in S} a_s\, \delta_s(\varepsilon_r)$$ where $\delta_s(\varepsilon_r) = \varepsilon_{sr}$

multiplicative shift of indices

$$\pi(f)\,\varepsilon_r = \sum_{s \in S} a_s\, \varepsilon_{sr}$$

<u>Adjoint</u>: $\xi \in \ell^2(S)$ $\quad \xi = \sum_{r \in S} c_r \varepsilon_r$

$$\delta_s^* \varepsilon_r = \begin{cases} 0 & \text{if } r \neq su \\ \varepsilon_u & \text{if } r = su \end{cases}$$

$$\langle \xi_1, \xi_2 \rangle = \sum_{s \in S} \overline{\xi_1(s)}\, \xi_2(s)$$

$$\langle \xi_1, \pi(f)\xi_2 \rangle = \sum_{s \in S} \overline{\xi_1(s)}\, (f * \xi_2)(s)$$

$\xi_1 = \sum c_r \varepsilon_r$

$f = \delta_s$

$\Downarrow$

$c_{ru}\ \delta_u$

$$= \sum_{s \in S} \overline{\xi_1(s)} \sum_{s = s_1 s_2} f(s_1)\, \xi_2(s_2)$$

$$= \sum_{\substack{s:\ \exists r\ r \mid s \\ s = ru}} \overline{\xi_1(ru)}\, f(r)\, \xi_2(u) = \sum_{\substack{s:\ \exists r \\ s = ru}} (\pi(f)^* \xi_1)(u)\, \xi_2(u)$$

$$\delta_s^* \delta_s \, \varepsilon_r = \varepsilon_r \quad \forall r \in S \qquad \delta_s^* \delta_s = 1$$

$$\delta_s \delta_s^* \, \varepsilon_r = \begin{cases} 0 & r \neq su \\ \varepsilon_r & r = su \end{cases} \qquad \delta_s \delta_s^* = P_s$$

P_s = orthog proj. on subspace of $\mathcal{H}$ spanned by ε_r w/ $r = su$ for some $u \in S$.

δ_s no longer unitaries but isometries

(not normal operators : do not commute with adjoints)

Creation-annihilation operators

$\mathcal{H} = \ell^2(\mathbb{N} \cup \{0\})$ basis $\{\varepsilon_n\}_{n \geq 0}$

$$S \varepsilon_n = \varepsilon_{n+1} \qquad \text{range}(S) = \mathcal{H}_1 \subset \mathcal{H}$$

$\mathcal{H}_1 = \text{Span}\{\varepsilon_n : n \geq 1\}$

$$S^* \varepsilon_n = \begin{cases} 0 & n = 0 \\ \varepsilon_{n-1} & n \geq 1 \end{cases}$$

unilateral shift "phase part" of creation operator of QM

$a^\dagger, a$ $\quad a^* = a^\dagger$ acting on $\ell^2(\mathbb{N} \cup \{0\})$

$$a^\dagger \varepsilon_n = \sqrt{n+1}\, \varepsilon_{n+1} \qquad a\, \varepsilon_n = \begin{cases} \sqrt{n}\, \varepsilon_{n-1} \\ 0 & n = 0 \end{cases}$$

$$[a^\dagger, a] = 1 \qquad a^\dagger = N^{1/2} S \qquad a = S^* N^{1/2}$$

polar decomposition

$N \varepsilon_n = n\, \varepsilon_n$ occupation number (grading operator)

$$[N, S] = -S \qquad \text{Hamiltonian } H = a^\dagger a$$

$$Sp(H) = \mathbb{N} \cup \{0\}$$

C^*-algebras generated by isometries

Toeplitz algebras and Cuntz algebras

$\mathcal{B}$ = finite set $\quad T_a \quad a \in \mathcal{B}$ family of isometries

$$T_a^* T_a = 1 \qquad T_a T_a^* = P_a$$

with mutually orthogonal range projections P_a

$$P_a P_b = \delta_{ab}$$

$\mathcal{T}_{\mathcal{B}}$ = universal unital C^*-algebra generated by these T_a

$$P_{\mathcal{B}} = \sum_{a \in \mathcal{B}} P_a \quad \text{projector} \quad P_{\mathcal{B}}^* = P_{\mathcal{B}} = P_{\mathcal{B}}^2$$

(but do not necessarily require $P_{\mathcal{B}} = id$)

Cuntz algebra $\mathcal{O}_{\mathcal{B}}$ quotient

$$1 \to \mathcal{I}_B \to \mathcal{T}_B \xrightarrow{\pi} \mathcal{O}_B \to 1$$

$\mathcal{I}_B$ = two sided ideal of $\mathcal{T}_B$ generated by $1 - P_B$

So $\mathcal{O}_B$ = universal C^*-alg. generated by isometries $S_a = \pi(T_a)$

with $S_a^* S_a = 1$ and $\sum_{a \in B} S_a S_a^* = 1$

<u>Functorialities</u>: $\mathcal{T}_B$ with respect to <u>injective</u> maps

$$\varphi : B_1 \hookrightarrow B_2 \qquad \mathcal{T}(\varphi) : \mathcal{T}_{B_1} \to \mathcal{T}_{B_2}$$

$\mathcal{O}_B$ only w/ respect to ~~[illegible]~~ bijections $\varphi : B_1 \xrightarrow{\simeq} B_2$

$\mathcal{O}(\varphi) : \mathcal{O}_{B_1} \xrightarrow{\simeq} \mathcal{O}_{B_2}$ <u>up to C^* isom</u> $\mathcal{O}_n \qquad n = \# B$

Variant: Cuntz-Krieger algebras $\mathcal{O}_A$

A is an $N\times N$ - matrix with entries in $\{0,1\}$

$\mathcal{O}_A$ = universal C^*-algebra generated by (partial isometries

S_i $i=1,\dots,N$ w/ relations

$$\sum_{i=1}^N S_i S_i^* = 1 \qquad S_i^* S_i = \sum_{j=1}^N A_{ij} \, S_j S_j^*$$

range projections P_i add to 1 as in Cuntz alg $\mathcal{O}_N$

this replaces $S_i^* S_i = 1$ of $\mathcal{O}_N$ ($\mathcal{O}_N$ case where all entries $A_{ij}=1$)

$$\begin{cases} S_i^* S_i = Q_i & \text{domain proj.} \\ S_i S_i^* = P_i & \text{range projection} \end{cases}$$

Graphs as Noncommutative Spaces

Γ = finite oriented graphs; no sinks & sources (no valence 1 vertices)

algebra associated to Γ

$E^{(1)}$ = set of oriented edges of Γ

$E^{(0)}$ = set of vertices $\qquad s,t : E^{(1)} \to E^{(0)}$ source & target of edge

S_e $e\in E^{(1)}$ partial isometries

P_v $v\in E^{(0)}$ projections $\qquad$ w/ relations

$$S_e^* S_e = P_{r(e)} \qquad P_v = \sum_{s(e)=v} S_e S_e^*$$

- Assuming finite graph one requires $\sum_v P_v = 1$ (corresponds to first CK relation)
- $S_e^* S_e = \sum_{s(e')=r(e)} S_{e'} S_{e'}^*$ from combin. of two rel'ts

$$A_{ee'} = \begin{cases} 0 & r(e) \neq s(e') \\ 1 & r(e) = s(e') \end{cases}$$

matrix assoc. to graph which determines when two arrows can be composed

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We'll give later reinterpretation of these algebras in terms of dynamical systems

Note: later will see various ways to associate algebras to a category (small)

One is to interpret the category as a graph w/ objects as vertices and morphisms as arrows and take associated graph algebra

but:

(Above can be generalized to infinite graphs: good properties under row-finite assumption and loc. finite graphs)

Semigroupoids (= small categories)

Groupoids
(invertible arrows only)

Semigroups
~~invertible arrows~~
(one object only)

Groups
(one object; only invertible arrows)

Can further extend group ring idea to <u>semigroupoid</u> ring

$$R[\mathcal{S}]$$

a way to associate an associative ring (algebra) to a small category

(discrete top. on sets of Morphisms)

$$f: \mathcal{S}^{(1)} \longrightarrow R$$

<u>finite</u> support

$$\mathcal{S}^{(1)} = \bigcup_{X,Y \in \mathrm{Obj}(\mathcal{S}) = \mathcal{S}^{(0)}} \mathrm{Mor}_{\mathcal{S}}(X,Y)$$

$$f = \sum_{\varphi \in \mathcal{S}^{(1)}} a_\varphi \, \delta_\varphi$$

$$(f_1 * f_2)(\varphi) = \sum_{\varphi = \varphi_1 \circ \varphi_2} f_1(\varphi_1) f_2(\varphi_2)$$

split a morphism in all possible ways in which it can be a composition

composition of morphisms

i.e. if

$\varphi \in \mathrm{Mor}_{\mathcal{S}}(X,Y)$ $\quad \exists Z \in \mathcal{S}^{(0)} = \mathrm{Obj}(\mathcal{S})$ $\quad \exists \varphi_1 \in \mathrm{Mor}_{\mathcal{S}}(Z,Y)$

$\exists \varphi_2 \in \mathrm{Mor}_{\mathcal{S}}(X,Z)$

s.t. $\varphi = \varphi_1 \circ \varphi_2$

$\varphi \in \mathcal{S}^{(1)}$ $\quad s(\varphi) = X \quad t(\varphi) = Y \quad \in \mathrm{Obj}(\mathcal{S})$

if $\varphi \in \mathrm{Mor}_{\mathcal{S}}(X,Y) \subset \mathcal{S}^{(1)}$

source and target as for groupoids: can compose iff target of first morphism is source of second

$*$-convolution product associative (from associativity of composition of morphisms in categories)

but not commutative

$R = \mathbb{C}$: <u>not</u> an involutive algebra (no inverses for $\varphi \in \mathcal{S}^{(1)}$ unlike groupoids)

Units of semigroupoid $\mathcal{S}^{(0)} \subset \mathcal{S}^{(1)}$ viewed as subset by $X \mapsto 1_X$ the identity morphism in $\mathrm{Mor}_{\mathcal{S}}(X,X)$ which always exists

$\mathcal{S}_X^{(1)} = \{ \varphi \in \mathcal{S}^{(1)} : s(\varphi) = X \}$ sub-semigroupoid

(enough know there $\mathcal{S}_X^{(1)}$ discrete not nec $\mathcal{S}^{(1)}$)

$\mathcal{H}_X = \ell^2(\mathcal{S}_X^{(1)})$

representation $\pi_X : \mathbb{C}[\mathcal{S}] \to \mathcal{B}(\mathcal{H}_X)$

$$\pi_X(f)\,\xi(\varphi) = (f * \xi)(\varphi) = \sum_{\varphi = \varphi_1 \circ \varphi_2} f(\varphi_1)\, \xi(\varphi_2)$$

to obtain a C^*-algebra: algebra generated by the $\pi_X(f)$, their adjoints $\pi_X(f)^*$ in inner product $\langle \xi_1, \xi_2 \rangle = \sum_{\varphi \in \mathcal{S}^{(1)}} \overline{\xi_1(\varphi)}\, \xi_2(\varphi)$

and $\|f\| = \sup_{X \in \mathcal{S}^{(0)}} \| \pi_X(f) \|_{\mathcal{B}(\mathcal{H}_X)}$ (assume $\mathcal{S}^{(0)}$ compact / finite or anyway this sup finite)

$\pi_X(f)^*$: $\langle \xi_1, \pi(f)\xi_2\rangle = \sum_{\varphi=\varphi_1\varphi_2} \overline{\xi_1(\varphi)}\, f(\varphi_1)\,\xi_2(\varphi_2)$

say $f = \delta_\varphi$ $\quad \delta_\varphi\, \varepsilon_\psi = \varepsilon_{\varphi\circ\psi}$

$$\delta_\varphi^* \varepsilon_\psi = \begin{cases} 0 & \psi \neq \varphi\circ\eta \\ \varepsilon_\eta & \psi = \varphi\circ\eta \end{cases}$$

$\pi(f)^* = \sum \bar{a}_\varphi \delta_\varphi^*$ for $f = \sum a_\varphi \delta_\varphi$

$\delta_\varphi^*\delta_\varphi = 1$ $\quad \delta_\varphi\delta_\varphi^* = P_\varphi$ = proj. on subspace of $\mathcal{H}_X$ of all $\psi = \varphi\circ\psi_1$; $\psi_1 \in H_X$

Categorical version of creation & annihilation operators

Algebras from categories : another way

Hall – Ringel algebras

Requires <u>more</u> structure on the category

$\mathcal{C}$ = <u>abelian category</u>

$\mathrm{Iso}(\mathcal{C})$ = set of isomorphism classes of objects the abelian category $\mathcal{C}$

$\mathcal{A}_{\mathcal{C}}$ Hall algebra generated by elements $\delta_{[X]}$ for $[X] \in \mathrm{Iso}(\mathcal{C})$

with convolution product

$$\delta_{[X]} * \delta_{[Y]} = \sum_{[Z] \in \mathrm{Iso}(\mathcal{C})} c^{[Z]}_{[X][Y]} \, \delta_{[Z]}$$

where the coefficient

$$c^{[Z]}_{[X],[Y]} = \# \{\text{subobjects } X' \subset Z \text{ with } X' \cong X \text{ and } Z/X' \cong Y\}$$

i.e. # of exact sequences

$$0 \to X \to Z \to Y \to 0$$

mod action of $\mathrm{Aut}(X)$ and $\mathrm{Aut}(Y)$

It is an associative product

Equivalent description:

$f : \mathrm{Iso}(\mathcal{C}) \to \mathbb{C}$ with finite support

$$(f_1 * f_2)(X) = \sum_{X' \subset X} f_1(X') \, f_2(X/X')$$

splitting X according to: $0 \to X' \to X \to X/X' \to 0$