Mathematics of the Rubik's Cube

Alex Wein

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Abstract

We present an introduction to the mathematical theory behind the popular puzzle known as the Rubik's cube. We first introduce the cube in a formal setting. We define moves of the cube and show that the moves form a group $\mathbb{G}$. We then discuss configurations of the cube and present a form in which they can be conveniently specified. Finally we characterize valid configurations, i.e. configurations from which the cube can be solved. This includes showing how to solve the cube from any valid configuration.

1 Introduction to the Cube

The 3×3 Rubik's cube has 26 "cubies", with each visible face colored one of 6 colors. There is a unique (up to rotations of the entire cube) "solved" configuration in which each face is a different color. We fix some orientation of the solved cube in space and name the faces r (right), l (left), u (up), d (down), f (front), and b (back). Now we can name each cubie by listing (clockwise) the faces it intersects. Examples include urf, ur, and r.

We now make a distinction between cubies and cubicles. Cubicles are named the same way as cubies but represent the positions in space where the cubies reside rather than the physical blocks. If the cube is solved then each cubie resides in the cubicle that has the same name as it. When moves are applied to the cube, cubies can move and occupy other cubicles but the cubicles are fixed in space.

Cubies and cubicles are either "corner", "edge", or "center". Note that (for instance) a corner cubie can only occupy a corner cubicle.

2 Moves

There are 6 basic moves that can be applied to the cube: clockwise rotation of a single face. We denote the basic moves R, L, U, D, F, B, corresponding to the 6 faces. We define a move to be any sequence of 0 or more of these basic moves. In other words, the 6 basic moves generate a group $(\mathbb{G}, *)$ of all moves. The group operation $*$ is simply defined as follows: $A * B$ is the move that performs move A followed by move B. 2 moves are considered equal in $\mathbb{G}$ iff (when applied to a solved cube) they leave the cube in identical states. It is easy to verify that $\mathbb{G}$ is indeed a group.

- $\mathbb{G}$ is clearly closed under $*$
- The identity element is the move that leaves the cube unchanged
- Each move has an inverse move that simply reverses the move
- $*$ is clearly associative

Moves can be written in cycle notation, e.g. D = (dlf dfr drb dbl)(df dr db dl). We can also write D(df) = dr for instance. Note that the ordering of the letters in df and dr is important here because the "front" face of the cubie df maps to the "right" face of the cubie dr.

Note that center cubies are unchanged by all moves.

3 Configurations

A configuration is any state that the cube could conceivably be in, i.e. corner cubies occupy corner cubicles with some orientation, etc. A valid configuration is one that is reachable from the solved configuration.

A configuration is fully specified by the following:

(1) The positions of the corner cubies

(2) The positions of the edge cubies

(3) The orientations of the corner cubies

(4) The orientations of the edge cubies

(1) is described by a permutation $sigma \in S_8$. (2) is described by a permutation $\tau \in S_{12}$. In order to describe (3) and (4) we need to number the faces of some cubicles and cubies. Label the following faces of corner cubicles:

1 on the u face of the ufl cubicle
2 on the u face of the urf cubicle
3 on the u face of the ubr cubicle
4 on the u face of the ulb cubicle
5 on the d face of the dbl cubicle
6 on the d face of the dlf cubicle
7 on the d face of the dfr cubicle
8 on the d face of the drb cubicle

We now label corner cubies. For each corner cubie we will label the 3 visible faces with 0, 1, and 2. In the solved configuration, one face of each corner cubie lies on one of the labeled cubicle faces 1-8. Label this cubie face 0. Moving clockwise from 0, label the next face of the cubie 1 and the remaining one 2. We can now describe item (3), the orientations of the corner cubies, by a tuple $x = (x_1, x_2, ..., x_8) \in (\mathbb{Z}/3\mathbb{Z})^8$ where x_i is the label on the cubie face that is lying on the cubicle face labeled i.

We label edge cubicles and cubies similarly in order to define (4), the orientations of the edge cubies. Label some faces of the edge cubicles:

1 on the u face of the ub cubicle
2 on the u face of the ur cubicle
3 on the u face of the uf cubicle
4 on the u face of the ul cubicle
5 on the b face of the lb cubicle
6 on the b face of the rb cubicle
7 on the f face of the rf cubicle
8 on the f face of the lf cubicle
9 on the d face of the db cubicle
10 on the d face of the dr cubicle
11 on the d face of the df cubicle
12 on the d face of the dl cubicle

Each edge cubie has 2 visible faces which we will label 0 and 1 such that

0 lies on a labeled cubicle face in the solved configuration. Item (4), the orientations of the edge cubies can be described by $y = (y_1, y_2, ..., y_{12}) \in (\mathbb{Z}/2\mathbb{Z})^{12}$ where y_i is the label on the cubie face that is lying on the edge cubicle face labeled i.

We can now specify a configuration with (σ, τ, x, y). The solved configuration is $(1, 1, 0, 0)$.

4 Moves as a Group Action

The group $\mathbb{G}$ of moves acts on the set of configurations $\mathcal{C}$. If $m \in \mathbb{G}$ and $c \in \mathcal{C}$ then $m \cdot c$ is the configuration that results when move m is applied to configuration c. It is simple to verify the group action axioms:

- Associativity: $(m * n) \cdot c = m * (n \cdot c)$
- Identity: $1 \cdot c = c$

We can similarly let $\mathbb{G}$ act on the following sets:

If $\mathbb{G}$ acts on the set of corner cubies there is an induced homomorphism $\phi_{corner} : \mathbb{G} \to S_8$. $\phi_{corner}(m)$ is the permutation induced on the corner cubies when move m is applied to the cube.

Similarly, if $\mathbb{G}$ acts on the set of edge cubies there is an induced homomorphism $\phi_{edge} : \mathbb{G} \to S_{12}$. $\phi_{edge}(m)$ is the permutation induced on the edge cubies when move m is applied to the cube.

5 Valid Configurations of the Cube

Now we characterize all valid configurations. Valid configurations are configurations in the orbit of the solved configuration (under the group action of $\mathbb{G}$ on $\mathcal{C}$).

Theorem 1: A configuration (σ, τ, x, y) is valid if and only if the following 3 conditions hold:

(1) $sgn\, \sigma = sgn\, \tau$ where sgn gives the sign (even/odd) of a permutation

(2) $\sum x_i \equiv 0 \, (mod \, 3)$

(3) $\sum y_i \equiv 0 \, (mod \, 2)$

Proof sketch

$\Rightarrow$) To prove that a valid configuration must satisfy the 3 conditions we notice that the solved cube satisfies the conditions and prove that the conditions are invariant under a basic move.

$\Leftarrow$) To prove that any configuration satisfying the 3 conditions can be used as a starting point to solve the cube, we simple write down the steps to solve the cube from any such configuration. The cube is solved in 4 steps:

(1) Put the corner cubies in the correct position

(2) Put the corner cubies in the correct position with correct orientation

(3) Put the edge cubies in the correct position, leaving the corner cubies unchanged

(4) Put the edge cubies in the correct position with correct orientation, leaving the corner cubies unchanged

References

[1] Chen, Janet. "Group Theory and the Rubik's Cube."