Noncommutative Geometry models for Particle Physics and Cosmology, Lecture IV

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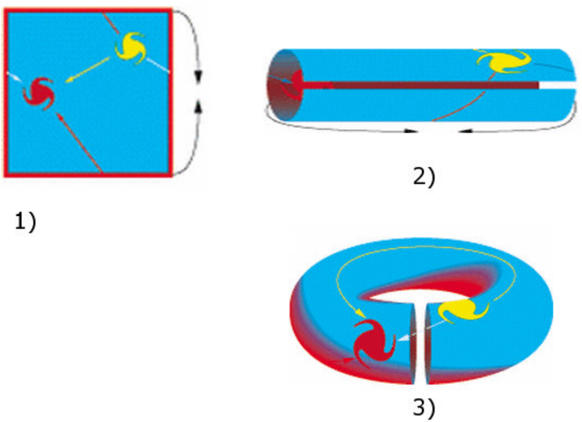
Villa de Leyva school, July 2011

This lecture based on

- Matilde Marcolli, Elena Pierpaoli, Kevin Teh, *The spectral action and cosmic topology*, Commun.Math.Phys.304 (2011) 125–174, arXiv:1005.2256
- Matilde Marcolli, Elena Pierpaoli, Kevin Teh, *The coupling of topology and inflation in noncommutative cosmology*, arXiv:1012.0780
- Branimir Ćaćić, Matilde Marcolli, Kevin Teh, *Coupling of gravity to matter, spectral action and cosmic topology*, arXiv:1106.5473

The question of Cosmic Topology:

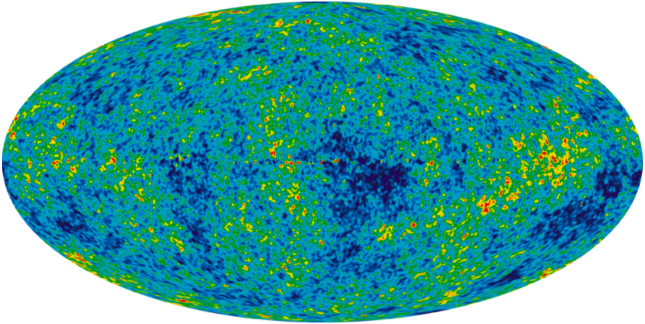


Nontrivial (non-simply-connected) spatial sections of spacetime, homogeneous spherical or flat spaces: how can this be detected from cosmological observations?

Our approach:

- NCG provides a modified gravity model through the spectral action
- The nonperturbative form of the spectral action determines a slow-roll inflation potential
- The underlying geometry (spherical/flat) affects the shape of the potential
- Different inflation scenarios depending on geometry and topology of the cosmos
- Shape of the inflation potential readable from cosmological data (CMB)

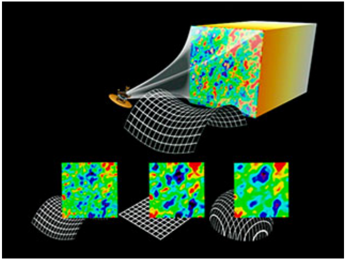
Cosmic Microwave Background best source of cosmological data on which to test theoretical models (modified gravity models, cosmic topology hypothesis, particle physics models)



- COBE satellite (1989)
- WMAP satellite (2001)
- Planck satellite (2009): new data available now!

Cosmic topology and the CMB

- Einstein equations determine geometry not topology (don't distinguish S^3 from S^3/Γ with round metric)
- Cosmological data (BOOMERanG experiment 1998, WMAP data 2003): spatial geometry of the universe is flat or slightly positively curved
- Homogeneous and isotropic compact case: spherical space forms S^3/Γ or Bieberbach manifolds T^3/Γ



Is cosmic topology detected by the Cosmic Microwave Background (CMB)? Search for signatures of multiconnected topologies

CMB sky and spherical harmonics temperature fluctuations

$$\frac{\Delta T}{T} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} a_{\ell m} Y_{\ell m}$$

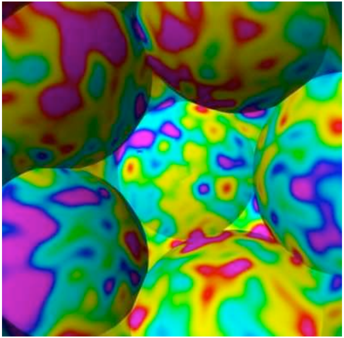
$Y_{\ell m}$ spherical harmonics

Methods to address cosmic topology problem

- Statistical search for matching circles in the CMB sky: identify a nontrivial fundamental domain
- Anomalies of the CMB: quadrupole suppression, the small value of the two- point temperature correlation function at angles above 60 degrees, and the anomalous alignment of the quadrupole and octupole
- Residual gravity acceleration: gravitational effects from other fundamental domains
- Bayesian analysis of different models of CMB sky for different candidate topologies

Results: no conclusive evidence of a non-simply connected topology

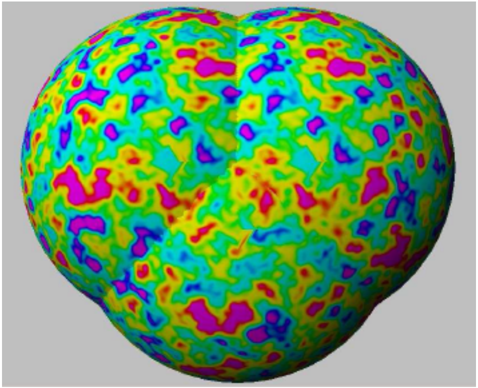
Simulated CMB sky: Laplace spectrum on spherical space forms



(Luminet, Lehoucq, Riazuelo, Weeks, et al.)

Best spherical candidate: Poincaré homology 3-sphere (dodecahedral cosmology)

Simulated CMB sky for a flat Bieberbach G6-cosmology



(from Riazuelo, Weeks, Uzan, Lehoucq, Luminet, 2003)

Slow-roll models of inflation in the early universe

Minkowskian Friedmann metric on $Y \times \mathbb{R}$

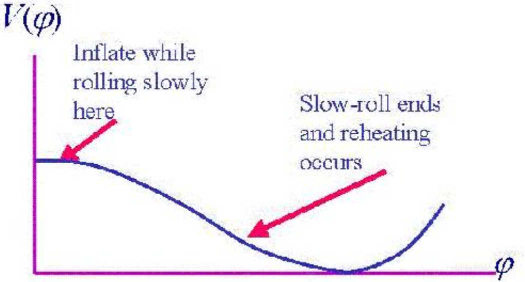
$$ds^2 = -dt^2 + a(t)^2 ds_Y^2$$

accelerated expansion $\frac{\ddot{a}}{a} = H^2(1-\epsilon)$ Hubble parameter

$$H^2(\phi)\left(1 - \frac{1}{3}\epsilon(\phi)\right) = \frac{8\pi}{3m_{Pl}^2} V(\phi)$$

m_{Pl} Planck mass, inflation phase $\epsilon(\phi) < 1$

A potential $V(\phi)$ for a scalar field ϕ that runs the inflation



Slow roll parameters

$$\epsilon(\phi) = \frac{m_{Pl}^2}{16\pi} \left(\frac{V'(\phi)}{V(\phi)} \right)^2$$

$$\eta(\phi) = \frac{m_{Pl}^2}{8\pi} \frac{V''(\phi)}{V(\phi)}$$

$$\xi(\phi) = \frac{m_{Pl}^4}{64\pi^2} \frac{V'(\phi)V'''(\phi)}{V^2(\phi)}$$

$\Rightarrow$ measurable quantities

$$n_s \simeq 1 - 6\epsilon + 2\eta, \quad n_t \simeq -2\epsilon, \quad r = 16\epsilon,$$

$$\alpha_s \simeq 16\epsilon\eta - 24\epsilon^2 - 2\xi, \quad \alpha_t \simeq 4\epsilon\eta - 8\epsilon^2$$

spectral index n_s, tensor-to-scalar ratio r, etc.

Slow roll parameters and the CMB

Friedmann metric (expanding universe)

$$ds^2 = -dt^2 + a(t)^2 ds_Y^2$$

Separate tensor and scalar perturbation h_{ij} of metric (traceless and trace part) $\Rightarrow$ Fourier modes: power spectra for scalar and tensor fluctuations, $\mathcal{P}_s(k)$ and $\mathcal{P}_t(k)$ satisfy power law

$$\mathcal{P}_s(k) \sim \mathcal{P}_s(k_0) \left(\frac{k}{k_0} \right)^{1 - n_s + \frac{\alpha_s}{2} \log(k/k_0)}$$

$$\mathcal{P}_t(k) \sim \mathcal{P}_t(k_0) \left(\frac{k}{k_0} \right)^{n_t + \frac{\alpha_t}{2} \log(k/k_0)}$$

Amplitudes and exponents: constrained by observational parameters and predicted by models of *slow roll inflation* (slow roll potential)

Poisson summation formula: $h \in \mathcal{S}(\mathbb{R})$ rapidly decaying function

$$\sum_{k\in\mathbb{Z}} h(x+2\pi k) = \frac{1}{2\pi} \sum_{n\in\mathbb{Z}} \hat{h}(n) e^{inx}$$

function $f(x) = \sum_{k\in\mathbb{Z}} h(x+2\pi k)$ is 2π-periodic with Fourier coefficients

$$\hat{f}_n = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_0^{2\pi} h(x+2\pi k) e^{-inx} dx$$

$$= \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \int_{2\pi k}^{2\pi(k+1)} g(x) e^{-inx} dx = \frac{1}{2\pi} \int_{\mathbb{R}} h(x) e^{-inx} dx = \frac{1}{2\pi} \hat{h}(n)$$

Spectral action and Poisson summation formula

$$\sum_{n\in\mathbb{Z}} h(x+\lambda n) = \frac{1}{\lambda}\sum_{n\in\mathbb{Z}} \exp\left(\frac{2\pi i n x}{\lambda}\right)\ \widehat{h}(\frac{n}{\lambda})$$

$\lambda \in \mathbb{R}_+^*$ and $x \in \mathbb{R}$ with

$$\widehat{h}(x) = \int_{\mathbb{R}} h(u)\, e^{-2\pi i u x}\, du$$

Idea: write $\mathrm{Tr}(f(D/\Lambda))$ as sums over lattices

- Need explicit spectrum of D with multiplicities
- Need to write as a union of arithmetic progressions $\lambda_{n,i}$, $n \in \mathbb{Z}$
- Multiplicities polynomial functions $m_{\lambda_{n,i}} = P_i(\lambda_{n,i})$

$$\mathrm{Tr}(f(D/\Lambda)) = \sum_i \sum_{n\in\mathbb{Z}} P_i(\lambda_{n,i}) f(\lambda_{n,i}/\Lambda)$$

The standard topology S^3 Dirac spectrum $\pm a^{-1}(\frac{1}{2}+n)$ for $n \in \mathbb{Z}$, with multiplicity $n(n+1)$

$$\mathrm{Tr}(f(D/\Lambda)) = (\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{4}(\Lambda a)\widehat{f}(0) + O((\Lambda a)^{-k})$$

with $\widehat{f}^{(2)}$ Fourier transform of $v^2 f(v)$ 4-dimensional Euclidean $S^3 \times S^1$

$$\mathrm{Tr}(h(D^2/\Lambda^2)) = \pi \Lambda^4 a^3 \beta \int_0^\infty u\, h(u)\, du - \frac{1}{2}\pi \Lambda a \beta \int_0^\infty h(u)\, du + O(\Lambda^{-k})$$

$$g(u,v) = 2P(u)\, h(u^2(\Lambda a)^{-2} + v^2(\Lambda\beta)^{-2})$$

$$\widehat{g}(n,m) = \int_{\mathbb{R}^2} g(u,v) e^{-2\pi i(xu+yv)}\, du\, dv$$

Spectral action in this case computed in

- Ali Chamseddine, Alain Connes, *The uncanny precision of the spectral action*, arXiv:0812.0165

A slow roll potential: perturbation $D^2 \mapsto D^2 + \phi^2$ gives potential $V(\phi)$ scalar field coupled to gravity

$$\mathrm{Tr}(h((D^2+\phi^2)/\Lambda^2))) = \pi\Lambda^4\beta a^3 \int_0^\infty uh(u)du - \frac{\pi}{2}\Lambda^2\beta a \int_0^\infty h(u)du$$

$$+\pi\Lambda^4\beta a^3\, \mathcal{V}(\phi^2/\Lambda^2) + \frac{1}{2}\Lambda^2\beta a\, \mathcal{W}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u(h(u+x) - h(u))du, \qquad \mathcal{W}(x) = \int_0^x h(u)du$$

Parameters: a = radius of 3-sphere, β = auxiliary inverse temperature parameter (choice of Euclidean S^1-compactification), Λ = energy scale

Slow-roll parameters from spectral action: case $S = S^3$

$$\epsilon(x) = \frac{m_{Pl}^2}{16\pi} \left(\frac{h(x) - 2\pi(\Lambda a)^2 \int_x^\infty h(u)du}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

$$\eta(x) = \frac{m_{Pl}^2}{8\pi} \frac{h'(x) + 2\pi(\Lambda a)^2 h(x)}{\int_0^x h(u)du + 2\pi(\Lambda a)^2 \int_0^\infty u(h(u+x) - h(u))du}$$

- In Minkowskian Friedmann metric $\Lambda(t) \sim 1/a(t)$
- Also independent of β (artificial Euclidean compactification)

Slow-roll potential, cases of spherical and flat topologies:

- Matilde Marcolli, Elena Pierpaoli, Kevin Teh, *The spectral action and cosmic topology*, arXiv:1005.2256
- Matilde Marcolli, Elena Pierpaoli, Kevin Teh, *The coupling of topology and inflation in noncommutative cosmology*, arXiv:1012.0780

The quaternionic space $SU(2)/Q8$ (quaternion units $\pm 1, \pm\sigma_k$)
Dirac spectrum (Ginoux)

$$\frac{3}{2} + 4k \quad \text{with multiplicity} \quad 2(k+1)(2k+1)$$

$$\frac{3}{2} + 4k + 2 \quad \text{with multiplicity} \quad 4k(k+1)$$

Polynomial interpolation of multiplicities

$$P_1(u) = \frac{1}{4}u^2 + \frac{3}{4}u + \frac{5}{16}$$

$$P_2(u) = \frac{1}{4}u^2 - \frac{3}{4}u - \frac{7}{16}$$

Spectral action

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{8}(\Lambda a)^3 \widehat{f}^{(2)}(0) - \frac{1}{32}(\Lambda a)\widehat{f}(0) + O(\Lambda^{-k})$$

(1/8 of action for S^3) with $g_i(u) = P_i(u) f(u/\Lambda)$:

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{4}\left(\widehat{g}_1(0) + \widehat{g}_2(0)\right) + O(\Lambda^{-k})$$

from Poisson summation $\Rightarrow$ Same slow-roll parameters

Other spherical space forms: method of generating functions to compute multiplicities (C. Bär)

- Spin structures on S^3/Γ: homomorphisms $\epsilon : \Gamma \to \mathrm{Spin}(4) \cong SU(2) \times SU(2)$ lifting inclusion $\Gamma \hookrightarrow SO(4)$ under double cover $\mathrm{Spin}(4) \to SO(4)$, $(A, B) \mapsto AB$
- Dirac spectrum for S^3/Γ subset of spectrum of S^3
- Multiplicities given by a generating function: ρ^+ and ρ^- two half-spin irreducible reps, $\chi^\pm$ their characters

$$F_+(z) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{\chi^-(\epsilon(\gamma)) - z\chi^+(\epsilon(\gamma))}{\det(1 - z\gamma)}$$

$$F_-(z) = \frac{1}{|\Gamma|} \sum_{\gamma\in\Gamma} \frac{\chi^+(\epsilon(\gamma)) - z\chi^-(\epsilon(\gamma))}{\det(1 - z\gamma)}$$

Then $F_+(z)$ and $F_-(z)$ generating functions of spectral multiplicities

$$F_+(z) = \sum_{k=0}^\infty m(\frac{3}{2} + k, D) z^k \quad F_-(z) = \sum_{k=0}^\infty m(-(\frac{3}{2} + k), D) z^k$$

The dodecahedral space Poincaré homology sphere S^3/Γ
binary icosahedral group 120 elements
using generating function method (Bär):

$$F_+(z) = -\frac{16(710647 + 317811\sqrt{5})G^+(z)}{(7+3\sqrt{5})^3(2207+987\sqrt{5})H^+(z)}$$

$$G^+(z) = 6z^{11}+18z^{13}+24z^{15}+12z^{17}-2z^{19}-6z^{21}-2z^{23}+2z^{25}+4z^{27}+3z^{29}+z^{31}$$

$$H^+(z) = -1-3z^2-4z^4-2z^6+2z^8+6z^{10}+9z^{12}+9z^{14}+4z^{16}-4z^{18}-9z^{20}$$
$$-9z^{22}-6z^{24}-2z^{26}+2z^{28}+4z^{30}+3z^{32}+z^{34}$$

$$F_-(z) = -\frac{1024(5374978561 + 2403763488\sqrt{5})G^-(z)}{(7+3\sqrt{5})^8(2207+987\sqrt{5})H^-(z)}$$

$$G^-(z) = 1+3z^2+4z^4+2z^6-2z^8-6z^{10}-2z^{12}+12z^{14}+24z^{16}+18z^{18}+6z^{20}$$

$$H^-(z) = -1-3z^2-4z^4-2z^6+2z^8+6z^{10}+9z^{12}+9z^{14}+4z^{16}-4z^{18}-9z^{20}$$
$$-9z^{22}-6z^{24}-2z^{26}+2z^{28}+4z^{30}+3z^{32}+z^{34}$$

Polynomial interpolation of multiplicities: 60 polynomials $P_i(u)$

$$\sum_{j=0}^{59} P_j(u) = \frac{1}{2}u^2 - \frac{1}{8}$$

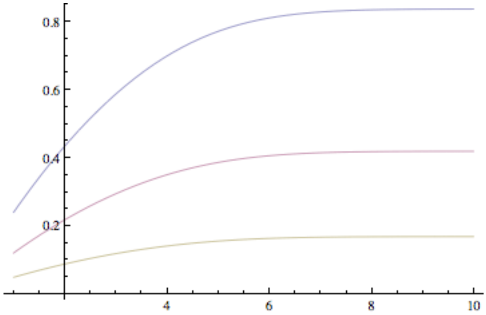
Spectral action: functions $g_j(u) = P_j(u)f(u/\Lambda)$

$$\mathrm{Tr}(f(D/\Lambda)) = \frac{1}{60}\sum_{j=0}^{59} \widehat{g}_j(0) + O(\Lambda^{-k})$$

$$= \frac{1}{60}\int_{\mathbb{R}} \sum_j P_j(u)f(u/\Lambda)du + O(\Lambda^{-k})$$

by Poisson summation $\Rightarrow 1/120$ of action for S^3
Same slow-roll parameters

But ... different amplitudes of power spectra:
multiplicative factor of potential $V(\phi)$



$$\mathcal{P}_s(k) \sim \frac{V^3}{(V')^2}, \quad \mathcal{P}_t(k) \sim V$$

$$V \mapsto \lambda V \quad \Rightarrow \mathcal{P}_s(k_0) \mapsto \lambda \mathcal{P}_s(k_0), \quad \mathcal{P}_t(k_0) \mapsto \lambda \mathcal{P}_t(k_0)$$

$\Rightarrow$ distinguish different spherical topologies

Topological factors (spherical cases):

Theorem (K.Teh): spherical forms $Y = S^3/\Gamma$, up to $O(\Lambda^{-\infty})$:

$$\mathrm{Tr}(f(D_Y/\Lambda)) = \frac{1}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right) = \frac{1}{\#\Gamma}\mathrm{Tr}(f(D_{S^3}/\Lambda))$$

Y spherical	λ_Y
sphere	1
lens N	$1/N$
binary dihedral $4N$	$1/(4N)$
binary tetrahedral	$1/24$
binary octahedral	$1/48$
binary icosahedral	$1/120$

Note: λ_Y does not distinguish all of them

- Kevin Teh, *Nonperturbative Spectral Action of Round Coset Spaces of SU(2)*, arXiv:1010.1827.

The flat tori: Dirac spectrum (Bär)

$$\pm 2\pi \parallel (m,n,p) + (m_0,n_0,p_0) \parallel, \tag{1}$$

$(m,n,p) \in \mathbb{Z}^3$ multiplicity 1 and constant vector (m_0,n_0,p_0) depending on spin structure

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \sum_{(m,n,p)\in\mathbb{Z}^3} 2f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right)$$

Poisson summation

$$\sum_{\mathbb{Z}^3} g(m,n,p) = \sum_{\mathbb{Z}^3} \widehat{g}(m,n,p)$$

$$\widehat{g}(m,n,p) = \int_{\mathbb{R}^3} g(u,v,w)e^{-2\pi i(mu+nv+pw)}dudvdw$$

$$g(m,n,p) = f\left(\frac{4\pi^2((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)}{\Lambda^2}\right)$$

Spectral action for the flat tori

$$\mathrm{Tr}(f(D_3^2/\Lambda^2)) = \frac{\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2+v^2+w^2)du\,dv\,dw + O(\Lambda^{-k})$$

$X = T^3 \times S^1_\beta$:

$$\mathrm{Tr}(h(D_X^2/\Lambda^2)) = \frac{\Lambda^4\beta\ell^3}{4\pi}\int_0^\infty uh(u)du + O(\Lambda^{-k})$$

using

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} 2\ h\left(\frac{4\pi^2}{(\Lambda\ell)^2}((m+m_0)^2+(n+n_0)^2+(p+p_0)^2)+\frac{1}{(\Lambda\beta)^2}(r+\frac{1}{2})^2\right)$$

$$g(u,v,w,y) = 2\ h\left(\frac{4\pi^2}{\Lambda^2}(u^2+v^2+w^2)+\frac{y^2}{(\Lambda\beta)^2}\right)$$

$$\sum_{(m,n,p,r)\in\mathbb{Z}^4} g(m+m_0,n+n_0,p+p_0,r+\frac{1}{2}) = \sum_{(m,n,p,r)\in\mathbb{Z}^4}(-1)^r\,\widehat{g}(m,n,p,r)$$

Different slow-roll potential and parameters Introducing the perturbation $D^2 \mapsto D^2 + \phi^2$:

$$\mathrm{Tr}(h((D_X^2 + \phi^2)/\Lambda^2)) = \mathrm{Tr}(h(D_X^2/\Lambda^2)) + \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

slow-roll potential

$$V(\phi) = \frac{\Lambda^4 \beta \ell^3}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

$$\mathcal{V}(x) = \int_0^\infty u\,(h(u+x) - h(u))\,du$$

Slow-roll parameters (different from spherical cases)

$$\epsilon = \frac{m_{Pl}^2}{16\pi} \left(\frac{\int_x^\infty h(u)du}{\int_0^\infty u(h(u+x) - h(u))du} \right)^2$$

$$\eta = \frac{m_{Pl}^2}{8\pi} \left(\frac{h(x)}{\int_0^\infty u(h(u+x) - h(u))du} \right)$$

Bieberbach manifolds

Quotients of T^3 by group actions: $G2$, $G3$, $G4$, $G5$, $G6$

spin structures

	δ_1	δ_2	δ_3
(a)	± 1	1	1
(b)	± 1	-1	1
(c)	± 1	1	-1
(d)	± 1	-1	-1

$G2(a)$, $G2(b)$, $G2(c)$, $G2(d)$, etc.

Dirac spectra known (Pfäffle)

Note: spectra often different for different spin structures

... but spectral action same!

Bieberbach cosmic topologies (t_i = translations by a_i)

- $G2$ = half turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$, and $a_3 = (T,S,0)$, with $H, L, S \in \mathbb{R}_+^*$ and $T \in \mathbb{R}$

$$\alpha^2 = t_1, \quad \alpha t_2 \alpha^{-1} = t_2^{-1}, \quad \alpha t_3 \alpha^{-1} = t_3^{-1}$$

- $G3$ = third turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$ and $a_3 = (-\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, for H and L in $\mathbb{R}_+^*$

$$\alpha^3 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1} t_3^{-1}$$

- $G4$ = quarter turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$, and $a_3 = (0,L,0)$, with $H, L > 0$

$$\alpha^4 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1}$$

- $G5$ = sixth turn space

lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$ and $a_3 = (\frac{1}{2}L, \frac{\sqrt{3}}{2}L, 0)$, $H, L > 0$

$$\alpha^6 = t_1, \quad \alpha t_2 \alpha^{-1} = t_3, \quad \alpha t_3 \alpha^{-1} = t_2^{-1} t_3$$

- $G6$ = Hantzsche–Wendt space (π-twist along each coordinate axis)

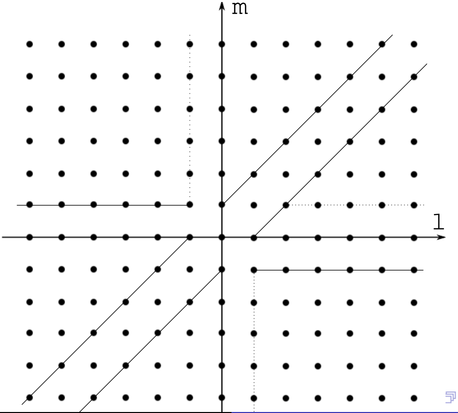
lattice $a_1 = (0,0,H)$, $a_2 = (L,0,0)$, and $a_3 = (0,S,0)$, with $H, L, S > 0$

$$\begin{array}{lll} \alpha^2 = t_1, & \alpha t_2 \alpha^{-1} = t_2^{-1}, & \alpha t_3 \alpha^{-1} = t_3^{-1}, \\ \beta^2 = t_2, & \beta t_1 \beta^{-1} = t_1^{-1}, & \beta t_3 \beta^{-1} = t_3^{-1}, \\ \gamma^2 = t_3, & \gamma t_1 \gamma^{-1} = t_1^{-1}, & \gamma t_2 \gamma^{-1} = t_2^{-1}, \\ & \gamma\beta\alpha = t_1 t_3. & \end{array}$$

Lattice summation technique for Bieberbach manifolds:
Example $G3$ case: $\lambda^{\pm}_{klm}$ symmetries $R : l \mapsto -l, m \mapsto -m$,
$S : l \mapsto m, m \mapsto l$, $T : l \mapsto l - m, m \mapsto -m$

$$\mathbb{Z}^3 = I \cup R(I) \cup S(I) \cup RS(I) \cup T(\tilde{I}) \cup RT(\tilde{I}) \cup \{l = m\}$$

$I = \{(k, l, m) \in \mathbb{Z}^3 : l \geq 1, m = 0, \dots, l - 1\}$ and
$\tilde{I} = \{(k, l, m) \in \mathbb{Z}^3 : l \geq 2, m = 1, \dots, l - 1\}$



Topological factors (flat cases):

Theorem [MPT2]: Bieberbach manifolds spectral action

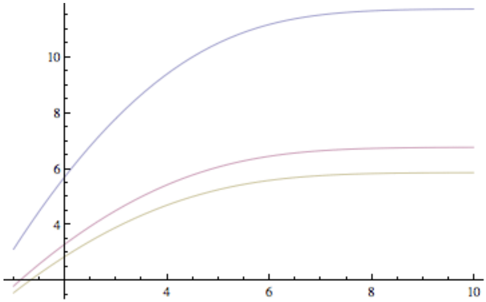
$$\mathrm{Tr}(f(D_Y^2/\Lambda^2)) = \frac{\lambda_Y \Lambda^3}{4\pi^3} \int_{\mathbb{R}^3} f(u^2+v^2+w^2)\, du dv dw$$

up to oder $O(\Lambda^{-\infty})$ with factors

$$\lambda_Y = \begin{cases} \frac{HSL}{2} & G2 \\ \frac{HL^2}{2\sqrt{3}} & G3 \\ \frac{HL^2}{4} & G4 \\ \frac{HLS}{4} & G6 \end{cases}$$

Note lattice summation technique not immediately suitable for $G5$, but expect like $G3$ up to factor of 2

Topological factors and inflation slow-roll potential



$\Rightarrow$ Multiplicative factor in amplitude of power spectra

Adding the coupling to matter $Y \times F$

Not only product but nontrivial fibration
Vector bundle V over 3-manifold Y, fiber $\mathcal{H}_F$ (fermion content)
Dirac operator D_Y twisted with connection on V (bosons)

Spectra of twisted Dirac operators on spherical manifolds (Cisneros–Molina)

Similar computation with Poisson summation formula [CMT]

$$\mathrm{Tr}(f(D_Y^2/\Lambda^2)) = \frac{N}{\#\Gamma}\left(\Lambda^3 \widehat{f}^{(2)}(0) - \frac{1}{4}\Lambda \widehat{f}(0)\right)$$

up to order $O(\Lambda^{-\infty})$
representation V dimension N; spherical form $Y = S^3/\Gamma$
$\Rightarrow$ topological factor $\lambda_Y \mapsto N\lambda_Y$

Conclusion (for now)

A modified gravity model based on the spectral action can distinguish between the different cosmic topology in terms of the slow-roll parameters (distinguish spherical and flat cases) and the amplitudes of the power spectral (distinguish different spherical space forms and different Bieberbach manifolds).
Different inflation scenarios in different topologies