Noncommutative Geometry models for Particle Physics and Cosmology, Lecture II

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This lecture based on

- Ali Chamseddine, Alain Connes, Matilde Marcolli, *Gravity and the Standard Model with neutrino mixing*, ATMP 11 (2007) 991–1090, arXiv:hep-th/0610241

Symmetries and NCG

Symmetries of gravity coupled to matter:
$G = U(1) \times SU(2) \times SU(3)$

$$\mathcal{G} = \mathrm{Map}(M, G) \rtimes \mathrm{Diff}(M)$$

Is it $\mathcal{G} = \mathrm{Diff}(X)$? Not for a manifold, yes for an NC space

Example: $\mathcal{A} = C^\infty(M, M_n(\mathbb{C}))$ $\quad G = PSU(n)$

$$1 \to \mathrm{Inn}(\mathcal{A}) \to \mathrm{Aut}(\mathcal{A}) \to \mathrm{Out}(\mathcal{A}) \to 1$$

$$1 \to \mathrm{Map}(M, G) \to \mathcal{G} \to \mathrm{Diff}(M) \to 1.$$

- Symmetries viewpoint: can think of $X = M \times F$ noncommutative with $\mathcal{G} = \mathrm{Diff}(X)$ pure gravity symmetries for X combining gravity and gauge symmetries together (no a priori distinction between "base" and "fiber" directions)

- Want same with action functional for pure gravity on NC space $X = M \times F$ giving gravity coupled to matter on M

Product geometry $M \times F$

Two spectral triples $(\mathcal{A}_i, \mathcal{H}_i, D_i, \gamma_i, J_i)$ of KO-dim 4 and 6:

$$\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2 \quad \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

$$D = D_1 \otimes 1 + \gamma_1 \otimes D_2$$

$$\gamma = \gamma_1 \otimes \gamma_2 \quad J = J_1 \otimes J_2$$

Case of 4-dimensional spin manifold M and finite NC geometry F:

$$\mathcal{A} = C^\infty(M) \otimes \mathcal{A}_F = C^\infty(M, \mathcal{A}_F)$$

$$\mathcal{H} = L^2(M, S) \otimes \mathcal{H}_F = L^2(M, S \otimes \mathcal{H}_F)$$

$$D = \partial\!\!\!/_M \otimes 1 + \gamma_5 \otimes D_F$$

D_F chosen in the moduli space described last time

Dimension of NC spaces: different notions of dimension for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$

- Metric dimension: growth of eigenvalues of Dirac operator
- KO-dimension (mod 8): sign commutation relations of J, γ, D
- Dimension spectrum: poles of zeta functions $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$

For manifolds first two agree and third contains usual dim; for NC spaces not same: DimSp $\subset \mathbb{C}$ can have non-integer and non-real points, KO not always metric dim mod 8, see F case

$X = M \times F$ metrically four dim $4 = 4 + 0$; KO-dim is $10 = 4 + 6$ (equal 2 mod 8); DimSp $k \in \mathbb{Z}_{\geq 0}$ with $k \leq 4$

Variant: almost commutative geometries

$$(C^\infty(M,\mathcal{E}), L^2(M,\mathcal{E}\otimes S), \mathcal{D}_{\mathcal{E}})$$

- M smooth manifold, $\mathcal{E}$ algebra bundle: fiber $\mathcal{E}_x$ finite dimensional algebra $\mathcal{A}_F$
- $C^\infty(M,\mathcal{E})$ smooth sections of a algebra bundle $\mathcal{E}$
- Dirac operator $\mathcal{D}_{\mathcal{E}} = c\circ(\nabla^{\mathcal{E}}\otimes 1 + 1\otimes\nabla^S)$ with spin connection ∇^S and hermitian connection on bundle
- Compatible grading and real structure

An equivalent intrinsic (abstract) characterization in:

- Branimir Ćaćić, *A reconstruction theorem for almost-commutative spectral triples*, arXiv:1101.5908

Here on assume for simplicity product $M\times F$

Inner fluctuations and gauge fields

Setup:

- Right $\mathcal{A}$-module structure on $\mathcal{H}$

$$\xi\, b = b^0\, \xi, \quad \xi \in \mathcal{H}, \quad b \in \mathcal{A}$$

- Unitary group, adjoint representation:

$$\xi \in \mathcal{H} \to \mathrm{Ad}(u)\,\xi = u\,\xi\, u^* \quad \xi \in \mathcal{H}$$

Inner fluctuations:

$$D \to D_A = D + A + \varepsilon'\, J\, A\, J^{-1}$$

with $A = A^*$ self-adjoint operator of the form

$$A = \sum a_j [D, b_j]\,, \quad a_j, b_j \in \mathcal{A}$$

Note: not an equivalence relation (finite geometry, can fluctuate D to zero) but like "self Morita equivalences"

Properties of inner fluctuations $(\mathcal{A}, \mathcal{H}, D, J)$

- Gauge potential $A \in \Omega^1_D$, $A = A^*$
- Unitary $u \in \mathcal{A}$, then

$$\mathrm{Ad}(u)(D + A + \varepsilon' J A J^{-1})\mathrm{Ad}(u^*) =$$

$$D + \gamma_u(A) + \varepsilon' J \gamma_u(A) J^{-1}$$

where $\gamma_u(A) = u[D, u^*] + u A u^*$

- $D' = D + A$ (with $A \in \Omega^1_D$, $A = A^*$) then

$$D' + B = D + A', \quad A' = A + B \in \Omega^1_D$$

$\forall B \in \Omega^1_{D'}$, $B = B^*$

- $D' = D + A + \varepsilon' J A J^{-1}$ then

$$D' + B + \varepsilon' J B J^{-1} = D + A' + \varepsilon' J A' J^{-1} \quad A' = A + B \in \Omega^1_D$$

$\forall B \in \Omega^1_{D'}$, $B = B^*$

Gauge bosons and Higgs boson

- Unitary $\mathrm{U}(\mathcal{A}) = \{u \in \mathcal{A} \mid uu^* = u^* u = 1\}$
- Special unitary

$$\mathrm{SU}(\mathcal{A}_F) = \{u \in \mathrm{U}(\mathcal{A}_F) \mid \det(u) = 1\}$$

det of action of u on $\mathcal{H}_F$

- Up to a finite abelian group

$$\mathrm{SU}(\mathcal{A}_F) \sim \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$$

- Unimod subgr of $\mathcal{U}(\mathcal{A})$ adjoint rep $\mathrm{Ad}(u)$ on $\mathcal{H}$ is gauge group of SM
- Unimodular inner fluctuations (in M directions) $\Rightarrow$ gauge bosons of SM: $U(1)$, $SU(2)$ and $SU(3)$ gauge bosons
- Inner fluctuations in F direction $\Rightarrow$ Higgs field

More on Gauge bosons

Inner fluctuations $A^{(1,0)} = \sum_i a_i[\partial\!\!\!/_M \otimes 1, a_i']$ with with $a_i = (\lambda_i, q_i, m_i)$, $a_i' = (\lambda_i', q_i', m_i')$ in $\mathcal{A} = C^\infty(M, \mathcal{A}_F)$

- $U(1)$ gauge field $\Lambda = \sum_i \lambda_i \, d\lambda_i' = \sum_i \lambda_i[\partial\!\!\!/_M \otimes 1, \lambda_i']$
- $SU(2)$ gauge field $Q = \sum_i q_i \, dq_i'$, with $q = f_0 + \sum_\alpha i f_\alpha \sigma^\alpha$ and $Q = \sum_\alpha f_\alpha[\partial\!\!\!/_M \otimes 1, i f_\alpha' \sigma^\alpha]$
- $U(3)$ gauge field $V' = \sum_i m_i \, dm_i' = \sum_i m_i[\partial\!\!\!/_M \otimes 1, m_i']$
- reduce the gauge field V' to $SU(3)$ passing to unimodular subgroup $\mathrm{SU}(\mathcal{A}_F)$ and unimodular gauge potential $\mathrm{Tr}(A) = 0$

$$V' = -V - \frac{1}{3}\begin{pmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{pmatrix} = -V - \frac{1}{3}\Lambda 1_3$$

Gauge bosons and hypercharges

The $(1,0)$ part of $A + JAJ^{-1}$ acts on quarks and leptons by

$$\begin{pmatrix} \frac{4}{3}\Lambda + V & 0 & 0 & 0 \\ 0 & -\frac{2}{3}\Lambda + V & 0 & 0 \\ 0 & 0 & Q_{11} + \frac{1}{3}\Lambda + V & Q_{12} \\ 0 & 0 & Q_{21} & Q_{22} + \frac{1}{3}\Lambda + V \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -2\Lambda & 0 & 0 \\ 0 & 0 & Q_{11} - \Lambda & Q_{12} \\ 0 & 0 & Q_{21} & Q_{22} - \Lambda \end{pmatrix}$$

$\Rightarrow$ correct hypercharges!

More on Higgs boson

Inner fluctuations $A^{(0,1)}$ in the F-space direction

$$\sum_i a_i[\gamma_5 \otimes D_F, a_i'](x)|_{\mathcal{H}_f} = \gamma_5 \otimes (A_q^{(0,1)} + A_\ell^{(0,1)})$$

$$A_q^{(0,1)} = \begin{pmatrix} 0 & X \\ X' & 0 \end{pmatrix} \otimes 1_3 \qquad A_1^{(0,1)} = \begin{pmatrix} 0 & Y \\ Y' & 0 \end{pmatrix}$$

$$X = \begin{pmatrix} \Upsilon_u^* \varphi_1 & \Upsilon_u^* \varphi_2 \\ -\Upsilon_d^* \bar{\varphi}_2 & \Upsilon_d^* \bar{\varphi}_1 \end{pmatrix} \quad \text{and} \quad X' = \begin{pmatrix} \Upsilon_u \varphi_1' & \Upsilon_d \varphi_2' \\ -\Upsilon_u \bar{\varphi}_2' & \Upsilon_d \bar{\varphi}_1' \end{pmatrix}$$

$$Y = \begin{pmatrix} \Upsilon_\nu^* \varphi_1 & \Upsilon_\nu^* \varphi_2 \\ -\Upsilon_e^* \bar{\varphi}_2 & \Upsilon_e^* \bar{\varphi}_1 \end{pmatrix} \quad \text{and} \quad Y' = \begin{pmatrix} \Upsilon_\nu \varphi_1' & \Upsilon_e \varphi_2' \\ -\Upsilon_\nu \bar{\varphi}_2' & \Upsilon_e \bar{\varphi}_1' \end{pmatrix}$$

$\varphi_1 = \sum \lambda_i(\alpha_i' - \lambda_i')$, $\varphi_2 = \sum \lambda_i \beta_i'$ $\varphi_1' = \sum \alpha_i(\lambda_i' - \alpha_i') + \beta_i \bar{\beta}_i'$ and $\varphi_2' = \sum(-\alpha_i \beta_i' + \beta_i(\bar{\lambda}_i' - \bar{\alpha}_i'))$, for $a_i(x) = (\lambda_i, q_i, m_i)$ and $a_i'(x) = (\lambda_i', q_i', m_i')$ and $q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$

More on Higgs boson

Discrete part of inner fluctuations: quaternion valued function $H = \varphi_1 + \varphi_2 j$ or $\varphi = (\varphi_1, \varphi_2)$

$$D_A^2 = (D^{1,0})^2 + 1_4 \otimes (D^{0,1})^2 - \gamma_5 \, [D^{1,0}, 1_4 \otimes D^{0,1}]$$

$$[D^{1,0}, 1_4 \otimes D^{0,1}] = \sqrt{-1} \, \gamma^\mu \, [(\nabla^s_\mu + \mathbb{A}_\mu), 1_4 \otimes D^{0,1}]$$

This gives $D_A^2 = \nabla^* \nabla - E$ where $\nabla^* \nabla$ Laplacian of $\nabla = \nabla^s + \mathbb{A}$

$$-E = \frac{1}{4} \, s \otimes \mathrm{id} + \sum_{\mu < \nu} \gamma^\mu \gamma^\nu \otimes \mathbb{F}_{\mu\nu} - i \, \gamma_5 \, \gamma^\mu \otimes \mathbb{M}(D_\mu \varphi) + 1_4 \otimes (D^{0,1})^2$$

with $s = -R$ scalar curvature and $\mathbb{F}_{\mu\nu}$ curvature of $\mathbb{A}$

$$D_\mu \varphi = \partial_\mu \varphi + \frac{i}{2} g_2 W_\mu^\alpha \varphi \, \sigma^\alpha - \frac{i}{2} g_1 B_\mu \, \varphi$$

$SU(2)$ and $U(1)$ gauge potentials

The spectral action functional

- Ali Chamseddine, Alain Connes, *The spectral action principle*, Comm. Math. Phys. 186 (1997), no. 3, 731–750.

A good action functional for noncommutative geometries

$$\mathrm{Tr}(f(D/\Lambda))$$

D Dirac, Λ mass scale, $f > 0$ even smooth function (cutoff approx)
Simple dimension spectrum $\Rightarrow$ expansion for $\Lambda \to \infty$

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_k f_k \Lambda^k \int\!\!\!\!- |D|^{-k} + f(0)\,\zeta_D(0) + o(1),$$

with $f_k = \int_0^\infty f(v)\, v^{k-1}\, dv$ momenta of f
where $\mathrm{DimSp}(\mathcal{A}, \mathcal{H}, D) =$ poles of $\zeta_{b,D}(s) = \mathrm{Tr}(b|D|^{-s})$

Asymptotic expansion of the spectral action

$$\mathrm{Tr}(e^{-t\Delta}) \sim \sum a_\alpha \, t^\alpha \qquad (t \to 0)$$

and the ζ function

$$\zeta_D(s) = \mathrm{Tr}(\Delta^{-s/2})$$

- Non-zero term a_α with $\alpha < 0 \Rightarrow$ *pole* of ζ_D at -2α with

$$\mathrm{Res}_{s=-2\alpha}\, \zeta_D(s) = \frac{2\, a_\alpha}{\Gamma(-\alpha)}$$

- No log t terms $\Rightarrow$ regularity at 0 for ζ_D with $\zeta_D(0) = a_0$

- Get first statement from

$$|D|^{-s} = \Delta^{-s/2} = \frac{1}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty e^{-t\Delta}\, t^{s/2-1}\, dt$$

with $\int_0^1 t^{\alpha+s/2-1}\, dt = (\alpha + s/2)^{-1}$.

- Second statement from

$$\frac{1}{\Gamma\left(\frac{s}{2}\right)} \sim \frac{s}{2} \qquad \text{as} \quad s \to 0$$

contrib to $\zeta_D(0)$ from pole part at $s = 0$ of

$$\int_0^\infty \mathrm{Tr}(e^{-t\Delta})\, t^{s/2-1}\, dt$$

given by $a_0 \int_0^1 t^{s/2-1}\, dt = a_0 \frac{2}{s}$

Spectral action with fermionic terms

$$S = \mathrm{Tr}(f(D_A/\Lambda)) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle, \quad \tilde{\xi} \in \mathcal{H}_{cl}^{+},$$

D_A = Dirac with unimodular inner fluctuations, J = real structure, $\mathcal{H}_{cl}^{+}$ = classical spinors, Grassmann variables

Fermionic terms

$$\frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

antisymmetric bilinear form $\mathfrak{A}(\tilde{\xi})$ on

$$\mathcal{H}_{cl}^{+} = \{\xi \in \mathcal{H}_{cl} \,|\, \gamma\xi = \xi\}$$

$\Rightarrow$ nonzero on Grassmann variables

Euclidean functional integral $\Rightarrow$ Pfaffian

$$Pf(\mathfrak{A}) = \int e^{-\frac{1}{2}\mathfrak{A}(\tilde{\xi})} D[\tilde{\xi}]$$

avoids Fermion doubling problem of previous models based on symmetric $\langle \xi, D_A\xi\rangle$ for NC space with KO-dim=0

Grassmann variables

Anticommuting variables with basic integration rule

$$\int \xi \, d\xi = 1$$

An antisymmetric bilinear form $\mathfrak{A}(\xi_1, \xi_2)$: if ordinary commuting variables $\mathfrak{A}(\xi, \xi) = 0$ but not on Grassmann variables
Example: 2-dim case $\mathfrak{A}(\xi', \xi) = a(\xi'_1 \xi_2 - \xi'_2 \xi_1)$, if ξ_1 and ξ_2 anticommute, with integration rule as above

$$\int e^{-\frac{1}{2}\mathfrak{A}(\xi,\xi)} D[\xi] = \int e^{-a\xi_1\xi_2} d\xi_1 d\xi_2 = a$$

Pfaffian as functional integral: antisymmetric quadratic form

$$Pf(\mathfrak{A}) = \int e^{-\frac{1}{2}\mathfrak{A}(\xi,\xi)} \, D[\xi]$$

Method to treat Majorana fermions in the Euclidean setting

Fermionic part of SM Lagrangian

Explicit computation of

$$\frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$$

gives part of SM Larangian with

- $\mathcal{L}_{Hf}$ = coupling of Higgs to fermions
- $\mathcal{L}_{gf}$ = coupling of gauge bosons to fermions
- $\mathcal{L}_{f}$ = fermion terms

Bosonic part of the spectral action

$$
\begin{aligned}
S = \;& \frac{1}{\pi^2}(48\, f_4\, \Lambda^4 - f_2\, \Lambda^2\, \mathfrak{c} + \frac{f_0}{4}\, \mathfrak{d}) \int \sqrt{g}\, d^4x \\
+ \;& \frac{96\, f_2\, \Lambda^2 - f_0\, \mathfrak{c}}{24\pi^2} \int R\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0}{10\, \pi^2} \int (\frac{11}{6}\, R^* R^* - 3\, C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma})\, \sqrt{g}\, d^4x \\
+ \;& \frac{(-2\, \mathfrak{a}\, f_2\, \Lambda^2 + \mathfrak{e}\, f_0)}{\pi^2} \int |\varphi|^2\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0 \mathfrak{a}}{2\, \pi^2} \int |D_\mu \varphi|^2\, \sqrt{g}\, d^4x \\
- \;& \frac{f_0 \mathfrak{a}}{12\, \pi^2} \int R\, |\varphi|^2\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0 \mathfrak{b}}{2\, \pi^2} \int |\varphi|^4\, \sqrt{g}\, d^4x \\
+ \;& \frac{f_0}{2\, \pi^2} \int (g_3^2\, G^i_{\mu\nu}\, G^{\mu\nu i} + g_2^2\, F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha} + \frac{5}{3}\, g_1^2\, B_{\mu\nu}\, B^{\mu\nu})\, \sqrt{g}\, d^4x,
\end{aligned}
$$

Parameters:

- f_0, f_2, f_4 free parameters, $f_0 = f(0)$ and, for $k > 0$,

$$f_k = \int_0^\infty f(v) v^{k-1} dv.$$

- $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ functions of Yukawa parameters of νMSM

$$\begin{aligned}
\mathfrak{a} &= \mathrm{Tr}(Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e + 3(Y_u^\dagger Y_u + Y_d^\dagger Y_d)) \\
\mathfrak{b} &= \mathrm{Tr}((Y_\nu^\dagger Y_\nu)^2 + (Y_e^\dagger Y_e)^2 + 3(Y_u^\dagger Y_u)^2 + 3(Y_d^\dagger Y_d)^2) \\
\mathfrak{c} &= \mathrm{Tr}(MM^\dagger) \\
\mathfrak{d} &= \mathrm{Tr}((MM^\dagger)^2) \\
\mathfrak{e} &= \mathrm{Tr}(MM^\dagger Y_\nu^\dagger Y_\nu).
\end{aligned}$$

Gilkey's theorem using $D_A^2 = \nabla^* \nabla - E$
Differential operator $P = -(g^{\mu\nu} I \, \partial_\mu \partial_\nu + A^\mu \partial_\mu + B)$ with A, B bundle endomorphisms, $m = \dim M$

$$\mathrm{Tr}\, e^{-tP} \sim \sum_{n \geq 0} t^{\frac{n-m}{2}} \int_M a_n(x, P)\, dv(x)$$

$P = \nabla^* \nabla - E$ and $E_{;\mu}{}^{\mu} := \nabla_\mu \nabla^\mu E$

$$\nabla_\mu = \partial_\mu + \omega'_\mu, \qquad \omega'_\mu = \frac{1}{2} g_{\mu\nu} (A^\nu + \Gamma^\nu \cdot \mathrm{id})$$

$$E = B - g^{\mu\nu} (\partial_\mu \omega'_\nu + \omega'_\mu \omega'_\nu - \Gamma^\rho_{\mu\nu} \omega'_\rho)$$

$$\Omega_{\mu\nu} = \partial_\mu \omega'_\nu - \partial_\nu \omega'_\mu + [\omega'_\mu, \omega'_\nu]$$

Seeley-DeWitt coefficients

$$\begin{aligned}
a_0(x, P) &= (4\pi)^{-m/2} \mathrm{Tr}(\mathrm{id}) \\
a_2(x, P) &= (4\pi)^{-m/2} \mathrm{Tr}\left(-\frac{R}{6}\, \mathrm{id} + E\right) \\
a_4(x, P) &= (4\pi)^{-m/2} \frac{1}{360} \mathrm{Tr}(-12 R_{;\mu}{}^{\mu} + 5 R^2 - 2 R_{\mu\nu} R^{\mu\nu} \\
&+ 2 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 60\, R\, E + 180\, E^2 + 60\, E_{;\mu}{}^{\mu} \\
&+ 30\, \Omega_{\mu\nu} \Omega^{\mu\nu})
\end{aligned}$$

Normalization and coefficients

- Rescale Higgs field $H = \frac{\sqrt{\mathfrak{a}\, f_0}}{\pi} \varphi$ to normalize kinetic term $\int \frac{1}{2} |D_\mu \mathbf{H}|^2 \sqrt{g}\, d^4x$
- Normalize Yang-Mills terms $\frac{1}{4} G^i_{\mu\nu} \overline{G}^{\mu\nu i} + \frac{1}{4} F^\alpha_{\mu\nu} \overline{F}^{\mu\nu\alpha} + \frac{1}{4} B_{\mu\nu} \overline{B}^{\mu\nu}$

Normalized form:

$$
\begin{aligned}
S = \;& \frac{1}{2\kappa_0^2} \int R \sqrt{g}\, d^4x + \gamma_0 \int \sqrt{g}\, d^4x \\
+ \;& \alpha_0 \int C_{\mu\nu\rho\sigma}\, C^{\mu\nu\rho\sigma} \sqrt{g}\, d^4x + \tau_0 \int R^* R^* \sqrt{g}\, d^4x \\
+ \;& \frac{1}{2} \int |DH|^2 \sqrt{g}\, d^4x - \mu_0^2 \int |H|^2 \sqrt{g}\, d^4x \\
- \;& \xi_0 \int R\, |H|^2 \sqrt{g}\, d^4x + \lambda_0 \int |H|^4 \sqrt{g}\, d^4x \\
+ \;& \frac{1}{4} \int (G^i_{\mu\nu}\, G^{\mu\nu i} + F^\alpha_{\mu\nu}\, F^{\mu\nu\alpha} + B_{\mu\nu}\, B^{\mu\nu}) \sqrt{g}\, d^4x
\end{aligned}
$$

where $R^* R^* = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}_{\rho\sigma}$ integrates to the Euler characteristic $\chi(M)$ and $C^{\mu\nu\rho\sigma}$ Weyl curvature

Coefficients

$$\frac{1}{2\kappa_0^2} = \frac{96 f_2 \Lambda^2 - f_0 \mathfrak{c}}{24\pi^2} \qquad \gamma_0 = \frac{1}{\pi^2}\left(48 f_4 \Lambda^4 - f_2 \Lambda^2 \mathfrak{c} + \frac{f_0}{4}\mathfrak{d}\right)$$

$$\alpha_0 = -\frac{3 f_0}{10\pi^2} \qquad \tau_0 = \frac{11 f_0}{60\pi^2}$$

$$\mu_0^2 = 2\frac{f_2 \Lambda^2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}} \qquad \xi_0 = \frac{1}{12}$$

$$\lambda_0 = \frac{\pi^2 \mathfrak{b}}{2 f_0 \mathfrak{a}^2}$$

Energy scale: Unification (10^{15} – 10^{17} GeV)

$$\frac{g^2 f_0}{2\pi^2} = \frac{1}{4}$$

Preferred energy scale, unification of coupling constants

Renormalization group flow

- The coefficients $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ (depend on Yukawa parameters) run with the RGE flow
- Initial conditions at unification energy: compatibility with physics at low energies

RGE in the MSM case

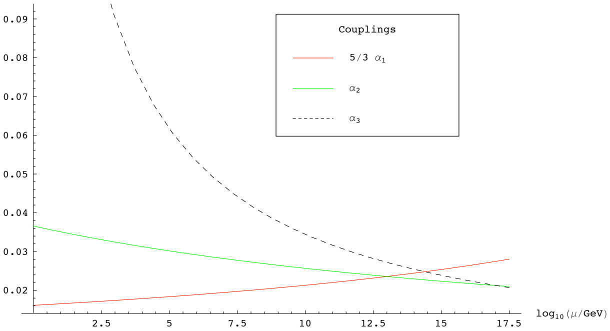
Running of coupling constants at one loop: $\alpha_i = g_i^2/(4\pi)$

$$\beta_{g_i} = (4\pi)^{-2}\, b_i\, g_i^3, \qquad \text{with} \quad b_i = (\frac{41}{6}, -\frac{19}{6}, -7),$$

$$\begin{aligned}
\alpha_1^{-1}(\Lambda) &= \alpha_1^{-1}(M_Z) - \frac{41}{12\pi} \log \frac{\Lambda}{M_Z} \\
\alpha_2^{-1}(\Lambda) &= \alpha_2^{-1}(M_Z) + \frac{19}{12\pi} \log \frac{\Lambda}{M_Z} \\
\alpha_3^{-1}(\Lambda) &= \alpha_3^{-1}(M_Z) + \frac{42}{12\pi} \log \frac{\Lambda}{M_Z}
\end{aligned}$$

$M_Z \sim 91.188$ GeV mass of Z^0 boson

At one loop RGE for coupling constants decouples from Yukawa parameters (not at 2 loops!)



Well known triangle problem: with known low energy values constants don't meet at unification $g_3^2 = g_2^2 = 5g_1^2/3$

Geometry point of view

- At one loop coupling constants decouple from Yukawa parameters
- Solving for coupling constants, RGE flow defines a vector field on moduli space $\mathcal{C}_3 \times \mathcal{C}_1$ of Dirac operators on the finite NC space F
- Subvarieties invariant under flow are relations between the SM parameters that hold at all energies
- At two loops or higher, RGE flow on a rank three vector bundle (fiber = coupling constants) over the moduli space $\mathcal{C}_3 \times \mathcal{C}_1$
- Geometric problem: studying the flow and the geometry of invariant subvarieties on the moduli space

Constraints at unification

The geometry of the model imposes conditions at unification energy: specific to this NCG model

- λ parameter constraint

$$\lambda(\Lambda_{unif}) = \frac{\pi^2}{2f_0} \frac{\mathfrak{b}(\Lambda_{unif})}{\mathfrak{a}(\Lambda_{unif})^2}$$

- Higgs vacuum constraint

$$\frac{\sqrt{\mathfrak{a} f_0}}{\pi} = \frac{2M_W}{g}$$

- See-saw mechanism and $\mathfrak{c}$ constraint

$$\frac{2f_2\Lambda_{unif}^2}{f_0} \leq \mathfrak{c}(\Lambda_{unif}) \leq \frac{6f_2\Lambda_{unif}^2}{f_0}$$

- Mass relation at unification

$$\sum_{generations} (m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2)|_{\Lambda=\Lambda_{unif}} = 8M_W^2|_{\Lambda=\Lambda_{unif}}$$

Need to have compatibility with low energy behavior

Mass relation at unification $Y_2(S) = 4g^2$

$$Y_2 = \sum_\sigma (y_\nu^\sigma)^2 + (y_e^\sigma)^2 + 3\,(y_u^\sigma)^2 + 3\,(y_d^\sigma)^2$$

$$\begin{aligned}
(k_{(\uparrow 3)})_{\sigma\kappa} &= \frac{g}{2M}\, m_u^\sigma\, \delta_\sigma^\kappa \\
(k_{(\downarrow 3)})_{\sigma\kappa} &= \frac{g}{2M}\, m_d^\mu\, C_{\sigma\mu}\delta_\mu^\rho C_{\rho\kappa}^\dagger \\
(k_{(\uparrow 1)})_{\sigma\kappa} &= \frac{g}{2M}\, m_\nu^\sigma\, \delta_\sigma^\kappa \\
(k_{(\downarrow 1)})_{\sigma\kappa} &= \frac{g}{2M}\, m_e^\mu\, {U^{lep}}_{\sigma\mu}\delta_\mu^\rho {U^{lep}}^\dagger_{\rho\kappa}
\end{aligned}$$

δ_i^j = Kronecker delta, then constraint:

$$\mathrm{Tr}(k_{(\uparrow 1)}^* k_{(\uparrow 1)} + k_{(\downarrow 1)}^* k_{(\downarrow 1)} + 3(k_{(\uparrow 3)}^* k_{(\uparrow 3)} + k_{(\downarrow 3)}^* k_{(\downarrow 3)})) = 2\,g^2$$

$\Rightarrow$ mass matrices satisfy

$$\sum_\sigma (m_\nu^\sigma)^2 + (m_e^\sigma)^2 + 3\,(m_u^\sigma)^2 + 3\,(m_d^\sigma)^2 = 8\,M^2$$

See-saw mechanism: $D = D(Y)$ Dirac

$$\begin{pmatrix} 0 & M_\nu^* & M_R^* & 0 \\ M_\nu & 0 & 0 & 0 \\ M_R & 0 & 0 & \bar{M}_\nu^* \\ 0 & 0 & \bar{M}_\nu & 0 \end{pmatrix}$$

on subspace $(\nu_R, \nu_L, \bar{\nu}_R, \bar{\nu}_L)$: largest eigenvalue of $M_R \sim \Lambda$ unification scale. Take $M_R = x\, k_R$ in flat space, Higgs vacuum v small (w/resp to unif scale) $\partial_u \mathrm{Tr}(f(D_A/\Lambda)) = 0 \qquad u = x^2$

$$x^2 = \frac{2\, f_2\, \Lambda^2\, \mathrm{Tr}(k_R^* k_R)}{f_0\, \mathrm{Tr}((k_R^* k_R)^2)}$$

Dirac mass $M_\nu \sim$ Fermi energy v

$$\frac{1}{2}(\pm m_R \pm \sqrt{m_R^2 + 4\, v^2})$$

two eigenvalues $\sim \pm m_R$ and two $\sim \pm \frac{v^2}{m_R}$
Compare with estimates

$$(m_R)_1 \geq 10^7\, GeV\,, \quad (m_R)_2 \geq 10^{12}\, GeV\,, \quad (m_R)_3 \geq 10^{16}\, GeV$$

Low energy limit: compatibilities and predictions
Running of top Yukawa coupling (dominant term):

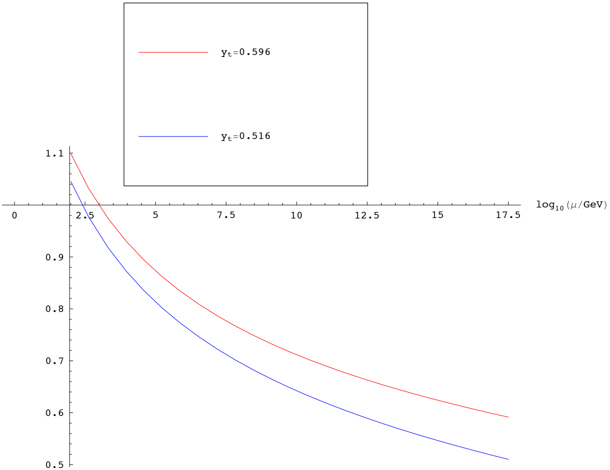
$$\frac{v}{\sqrt{2}}(y^{\sigma}_{\cdot}) = (m^{\sigma}_{\cdot}),$$

$$\frac{dy_t}{dt} = \frac{1}{16\pi^2}\left[\frac{9}{2}y_t^3 - \left(a\, g_1^2 + b\, g_2^2 + c\, g_3^2\right) y_t\right],$$

$$(a, b, c) = \left(\frac{17}{12}, \frac{9}{4}, 8\right)$$

$\Rightarrow$ value of top quark mass agrees with known (1.04 times if neglect other Yukawa couplings)

Top quark running using mass relation at unification



correction to MSM flow by y_ν^σ for τ neutrino (allowed to be comparably large by see-saw) lowers value

Higgs mass prediction using RGE for MSM

Higgs scattering parameter:

$$\frac{f_0}{2\,\pi^2}\int b\,|\varphi|^4\,\sqrt{g}\,d^4x = \frac{\pi^2}{2\,f_0}\frac{b}{a^2}\int |\mathbf{H}|^4\,\sqrt{g}\,d^4x$$

$\Rightarrow$ relation at unification ($\tilde{\lambda}$ is $|\mathbf{H}|^4$ coupling)

$$\tilde{\lambda}(\Lambda) = g_3^2\,\frac{b}{a^2}$$

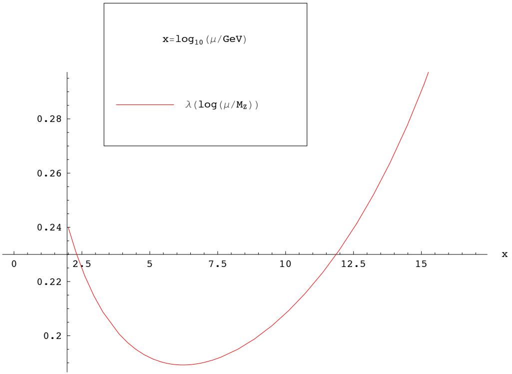
Running of Higgs scattering parameter:

$$\frac{d\lambda}{dt} = \lambda\gamma + \frac{1}{8\pi^2}(12\lambda^2 + B)$$

$$\gamma = \frac{1}{16\pi^2}(12y_t^2 - 9g_2^2 - 3g_1^2) \quad B = \frac{3}{16}(3g_2^4 + 2g_1^2g_2^2 + g_1^4) - 3y_t^4$$

Higgs estimate (in MSM approximation for RGE flow)

$$m_H^2 = 8\lambda \frac{M^2}{g^2}, \quad m_H = \sqrt{2\lambda}\,\frac{2M}{g}$$



$\lambda(M_Z) \sim 0.241$ and Higgs mass ~ 170 GeV (w/correction from see-saw ~ 168 GeV) ... Heavy Higgs! ... exclusion zones

Next time

- RGE flow for νMSM and see-saw scales
- Sensitive dependence on initial condition and constraints
- RGE scales and the cosmology timeline
- Gravitational terms and RGE running
- Models of the Very Early Universe