Noncommutative Geometry models for Particle Physics and Cosmology, Lecture I

Matilde Marcolli

Villa de Leyva school, July 2011

Plan of lectures

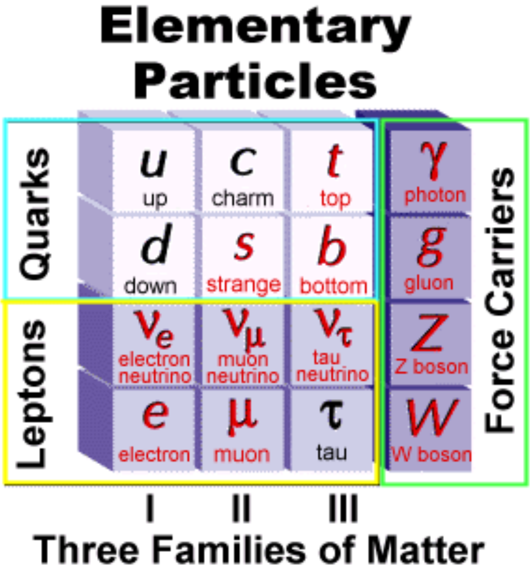
1. Noncommutative geometry approach to elementary particle physics; Noncommutative Riemannian geometry; finite noncommutative geometries; moduli spaces; the finite geometry of the Standard Model
2. The product geometry; the spectral action functional and its asymptotic expansion; bosons and fermions; the Standard Model Lagrangian; renormalization group flow, geometric constraints and low energy limits
3. Parameters: relations at unification and running; running of the gravitational terms; the RGE flow with right handed neutrinos; cosmological timeline and the inflation epoch; effective gravitational and cosmological constants and models of inflation
4. The spectral action and the problem of cosmic topology; cosmic topology and the CMB; slow-roll inflation; Poisson summation formula and the nonperturbative spectral action; spherical and flat space forms

Geometrization of physics

- Kaluza-Klein theory: electromagnetism described by circle bundle over spacetime manifold, connection = EM potential;
- Yang–Mills gauge theories: bundle geometry over spacetime, connections = gauge potentials, sections = fermions;
- String theory: 6 extra dimensions (Calabi-Yau) over spacetime, strings vibrations = types of particles
- NCG models: extra dimensions are NC spaces, pure gravity on product space becomes gravity + matter on spacetime

Why a geometrization of physics?

The standard model of elementary particles

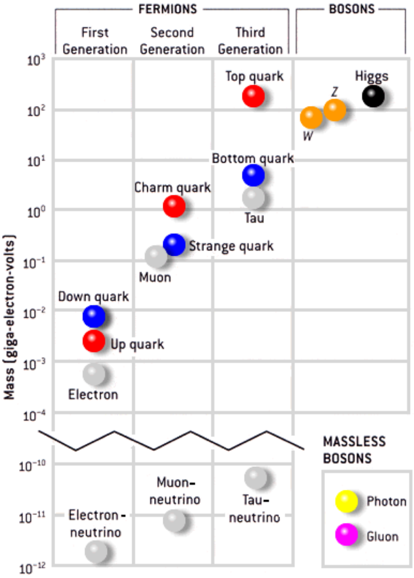


Standard model Lagrangian: can compute from *simpler* data?

$$
\begin{gathered}
\mathcal{L}_{SM} = -\tfrac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \tfrac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - \\
M^2 W^+_\mu W^-_\mu - \tfrac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \tfrac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \tfrac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - igc_w(\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - \\
igs_w(\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - \\
W^-_\nu \partial_\nu W^+_\mu)) - \tfrac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \tfrac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - \\
Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w (A_\mu Z^0_\nu (W^+_\mu W^-_\nu - \\
W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \tfrac{1}{2}\partial_\mu H \partial_\mu H - 2M^2 \alpha_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - \tfrac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \\
\beta_h \left(\tfrac{2M^2}{g^2} + \tfrac{2M}{g}H + \tfrac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \tfrac{2M^4}{g^2}\alpha_h - \\
g\alpha_h M\left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) - \\
\tfrac{1}{8}g^2\alpha_h\left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2\right) - \\
gMW^+_\mu W^-_\mu H - \tfrac{1}{2}g\tfrac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \\
\tfrac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right) + \\
\tfrac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) + W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right) + \tfrac{1}{2}g\tfrac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) + \\
M\left(\tfrac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+\right) - ig\tfrac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_wMA_\mu(W^+_\mu\phi^- - \\
W^-_\mu\phi^+) - ig\tfrac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_wA_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \\
\tfrac{1}{4}g^2W^+_\mu W^-_\mu\left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right) - \tfrac{1}{8}g^2\tfrac{1}{c_w^2}Z^0_\mu Z^0_\mu\left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-\right) - \\
\tfrac{1}{2}g^2\tfrac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- + W^-_\mu\phi^+) - \tfrac{1}{2}ig^2\tfrac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \tfrac{1}{2}g^2s_wA_\mu\phi^0(W^+_\mu\phi^- + \\
W^-_\mu\phi^+) + \tfrac{1}{2}ig^2s_wA_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\tfrac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- - \\
g^2s_w^2A_\mu A_\mu\phi^+\phi^- + \tfrac{1}{2}ig_s\,\lambda^a_{ij}(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda(\gamma\partial + m^\lambda_\nu)\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + \\
m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j + igs_wA_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \tfrac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \tfrac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right) + \\
\tfrac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{e}^\lambda\gamma^\mu(4s_w^2 - 1 - \gamma^5)e^\lambda) + (\bar{d}^\lambda_j\gamma^\mu(\tfrac{4}{3}s_w^2 - 1 - \gamma^5)d^\lambda_j) + \\
(\bar{u}^\lambda_j\gamma^\mu(1 - \tfrac{8}{3}s_w^2 + \gamma^5)u^\lambda_j)\} + \tfrac{ig}{2\sqrt{2}}W^+_\mu\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)U^{lep}{}_{\lambda\kappa}e^\kappa) + (\bar{u}^\lambda_j\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d^\kappa_j)\right) + \\
\tfrac{ig}{2\sqrt{2}}W^-_\mu\left((\bar{e}^\kappa U^{lep\dagger}{}_{\kappa\lambda}\gamma^\mu(1+\gamma^5)\nu^\lambda) + (\bar{d}^\kappa_j C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u^\lambda_j)\right) + \\
\tfrac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_e(\bar{\nu}^\lambda U^{lep}{}_{\lambda\kappa}(1-\gamma^5)e^\kappa) + m^\lambda_\nu(\bar{\nu}^\lambda U^{lep}{}_{\lambda\kappa}(1+\gamma^5)e^\kappa\right) + \\
\tfrac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_e(\bar{e}^\lambda U^{lep\dagger}{}_{\lambda\kappa}(1+\gamma^5)\nu^\kappa) - m^\kappa_\nu(\bar{e}^\lambda U^{lep\dagger}{}_{\lambda\kappa}(1-\gamma^5)\nu^\kappa\right) - \tfrac{g}{2}\tfrac{m^\lambda_\nu}{M}H(\bar{\nu}^\lambda\nu^\lambda) - \\
\tfrac{g}{2}\tfrac{m^\lambda_e}{M}H(\bar{e}^\lambda e^\lambda) + \tfrac{ig}{2}\tfrac{m^\lambda_\nu}{M}\phi^0(\bar{\nu}^\lambda\gamma^5\nu^\lambda) - \tfrac{ig}{2}\tfrac{m^\lambda_e}{M}\phi^0(\bar{e}^\lambda\gamma^5 e^\lambda) - \tfrac{1}{4}\,\bar{\nu}_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa - \\
\tfrac{1}{4}\,\overline{\bar{\nu}_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa} + \tfrac{ig}{2M\sqrt{2}}\phi^+\left(-m^\kappa_d(\bar{u}^\lambda_j C_{\lambda\kappa}(1-\gamma^5)d^\kappa_j) + m^\lambda_u(\bar{u}^\lambda_j C_{\lambda\kappa}(1+\gamma^5)d^\kappa_j\right) + \\
\tfrac{ig}{2M\sqrt{2}}\phi^-\left(m^\lambda_d(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1+\gamma^5)u^\kappa_j) - m^\kappa_u(\bar{d}^\lambda_j C^\dagger_{\lambda\kappa}(1-\gamma^5)u^\kappa_j\right) - \tfrac{g}{2}\tfrac{m^\lambda_u}{M}H(\bar{u}^\lambda_j u^\lambda_j) - \\
\tfrac{g}{2}\tfrac{m^\lambda_d}{M}H(\bar{d}^\lambda_j d^\lambda_j) + \tfrac{ig}{2}\tfrac{m^\lambda_u}{M}\phi^0(\bar{u}^\lambda_j\gamma^5 u^\lambda_j) - \tfrac{ig}{2}\tfrac{m^\lambda_d}{M}\phi^0(\bar{d}^\lambda_j\gamma^5 d^\lambda_j) + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu\bar{G}^a G^b g^c_\mu + \\
\bar{X}^+(\partial^2 - M^2)X^+ + \bar{X}^-(\partial^2 - M^2)X^- + \bar{X}^0(\partial^2 - \tfrac{M^2}{c_w^2})X^0 + \bar{Y}\partial^2 Y + igc_wW^+_\mu(\partial_\mu\bar{X}^0X^- - \\
\partial_\mu\bar{X}^+X^0) + igs_wW^+_\mu(\partial_\mu\bar{Y}X^- - \partial_\mu\bar{X}^+Y) + igc_wW^-_\mu(\partial_\mu\bar{X}^-X^0 - \\
\partial_\mu\bar{X}^0X^+) + igs_wW^-_\mu(\partial_\mu\bar{X}^-Y - \partial_\mu\bar{Y}X^+) + igc_wZ^0_\mu(\partial_\mu\bar{X}^+X^+ - \\
\partial_\mu\bar{X}^-X^-) + igs_wA_\mu(\partial_\mu\bar{X}^+X^+ - \\
\partial_\mu\bar{X}^-X^-) - \tfrac{1}{2}gM\left(\bar{X}^+X^+H + \bar{X}^-X^-H + \tfrac{1}{c_w^2}\bar{X}^0X^0H\right) + \tfrac{1-2c_w^2}{2c_w}igM\left(\bar{X}^+X^0\phi^+ - \bar{X}^-X^0\phi^-\right) + \\
\tfrac{1}{2c_w}igM\left(\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-\right) + igMs_w\left(\bar{X}^0X^-\phi^+ - \bar{X}^0X^+\phi^-\right) + \\
\tfrac{1}{2}igM\left(\bar{X}^+X^+\phi^0 - \bar{X}^-X^-\phi^0\right).
\end{gathered}
$$

1

Standard model parameters: are there relations? why these values?



Building mathematical models: essential requirements

- Conceptualize: complicated things (SM Lagrangian) should follow from simple things (geometry)
- Enrich: extend existing models with new features
- Predict: Higgs mass? new particles? new parameter relations? new phenomena?

The elephant in the room: Gravity!

Coupling gravity to matter: **the good:** better conceptual structure, links particle physics to cosmology; **the bad:** worse chances of passing from classical to quantum theory

In NCG models: "all forces become gravity" (on an NC space)

What the NCG model provides:

- Einstein–Hilbert action:

$$S_{EH}(g_{\mu\nu}) = \frac{1}{16\pi G} \int_M R \sqrt{g}\; d^4x$$

- Gravity minimally coupled to matter: $S = S_{EH} + S_{SM}$ with S_{SM} = particle physics ⇒ *Standard Model Lagrangian*
- Right handed neutrinos with Majorana masses
- Modified gravity model $f(R, R^{\mu\nu}, C_{\lambda\mu\nu\kappa})$ with conformal gravity: Weyl curvature tensor

$$C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \frac{1}{2}(g_{\lambda\nu}R_{\mu\kappa} - g_{\lambda\kappa}R_{\mu\nu} - g_{\mu\nu}R_{\lambda\kappa} + g_{\mu\kappa}R_{\lambda\nu})$$

$$+\frac{1}{6}R^{\alpha}_{\alpha}(g_{\lambda\nu}g_{\mu\kappa} - g_{\lambda\kappa}g_{\mu\nu})$$

- Non-minimal coupling of Higgs to gravity

$$\int_M \left(\frac{1}{2}|D_\mu \mathbf{H}|^2 - \mu_0^2|\mathbf{H}|^2 - \xi_0\, R\,|\mathbf{H}|^2 + \lambda_0|\mathbf{H}|^4 \right) \sqrt{g}\, d^4x$$

What is Noncommutative Geometry?

- X compact Hausdorff topological space $\Leftrightarrow$ $C(X)$ abelian C^*-algebra of continuous functions (Gelfand–Naimark)
- Noncommutative C^*-algebra $\mathcal{A}$: think of as "continuous functions on an NC space"
- Describe geometry in terms of algebra $C(X)$ and dense subalgebras like $C^\infty(X)$ if X manifold
- Differential forms, bundles, connections, cohomology: all continue to make sense without commutativity (NC geometry)
- Can describe "bad quotients" as good spaces: functions on the graph of the equivalence relation with convolution product

To do physics: need the analog of *Riemannian geometry*

Spectral triples and NC Riemannian manifolds $(\mathcal{A}, \mathcal{H}, D)$

- involutive algebra $\mathcal{A}$
- representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$
- self adjoint operator D on $\mathcal{H}$, dense domain
- compact resolvent $(1 + D^2)^{-1/2} \in \mathcal{K}$
- $[a, D]$ bounded $\forall a \in \mathcal{A}$
- *even* if $\mathbb{Z}/2$- grading γ on $\mathcal{H}$

$$[\gamma, a] = 0, \ \forall a \in \mathcal{A}, \qquad D\gamma = -\gamma D$$

Main example $(C^\infty(M), L^2(M, S), \partial\!\!\!/_M)$ with chirality γ_5 in 4-dim

• Alain Connes, *Geometry from the spectral point of view*, Lett. Math. Phys. 34 (1995), no. 3, 203–238.

Note: to apply spectral triples methods need to work with M compact and Euclidean signature (Euclidean gravity)

Real structure KO-dimension $n \in \mathbb{Z}/8\mathbb{Z}$
antilinear isometry $J : \mathcal{H} \to \mathcal{H}$

$$J^2 = \varepsilon, \qquad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon'' \gamma J$$

n	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Commutation: $[a, b^0] = 0 \quad \forall\, a, b \in \mathcal{A}$
where $b^0 = Jb^* J^{-1} \qquad \forall b \in \mathcal{A}$
Order one condition:

$$[[D, a], b^0] = 0 \qquad \forall\, a, b \in \mathcal{A}$$

Spectral triples in NCG need not be manifolds:

- Quantum groups
- Fractals
- NC tori

For particle physics models $M \times F$, product of a 4-dimensional spacetime manifold M by a "finite NC space" F (extra dimensions)

- Alain Connes, *Gravity coupled with matter and the foundation of non-commutative geometry*, Comm. Math. Phys. 182 (1996), no. 1, 155–176.

Almost commutative geometries: more general form, fibration (not product) over manifold, fiber finite NC space

- Branimir Ćaćić, *A reconstruction theorem for almost-commutative spectral triples*, arXiv:1101.5908
- Jord Boeijink, Walter D. van Suijlekom, *The noncommutative geometry of Yang-Mills fields*, arXiv:1008.5101.

Finite real spectral triples: $F = (\mathcal{A}_F, \mathcal{H}_F, D_F)$

- $\mathcal{A}$ finite dimensional (real) C^*-algebra

$$\mathcal{A} = \oplus_{i=1}^N M_{n_i}(\mathbb{K}_i)$$

$\mathbb{K}_i = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ quaternions (Wedderburn)

- Representation on finite dimensional Hilbert space $\mathcal{H}$, with bimodule structure given by J (condition $[a, b^0] = 0$)
- $D^* = D$ with order one condition

$$[[D, a], b^0] = 0$$

$\Rightarrow$ Moduli spaces (under unitary equivalence)

- Ali Chamseddine, Alain Connes, Matilde Marcolli, *Gravity and the standard model with neutrino mixing*, arXiv:hep-th/0610241
- Branimir Ćaćić, *Moduli spaces of Dirac operators for finite spectral triples*, arXiv:0902.2068

Building a particle physics model ...this lecture based on:

- Ali Chamseddine, Alain Connes, Matilde Marcolli, *Gravity and the standard model with neutrino mixing*, arXiv:hep-th/0610241

Minimal input ansatz:

- left-right symmetric algebra

$$\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$$

- involution $(\lambda, q_L, q_R, m) \mapsto (\bar{\lambda}, \bar{q}_L, \bar{q}_R, m^*)$
- subalgebra $\mathbb{C} \oplus M_3(\mathbb{C})$ integer spin $\mathbb{C}$-alg
- subalgebra $\mathbb{H}_L \oplus \mathbb{H}_R$ half-integer spin $\mathbb{R}$-alg

More general choices of initial ansatz:

- A.Chamseddine, A.Connes, *Why the Standard Model*, J.Geom.Phys. 58 (2008) 38–47

Slogan: algebras better than Lie algebras, more constraints on reps

Comment: associative algebras versus Lie algebras

- In geometry of gauge theories: bundle over spacetime, connections and sections, automorphisms gauge group: Lie group
- Decomposing composite particles into elementary particles: Lie group representations (hadrons and quarks)
- If want only elementary particles: associative algebras have very few representations (very constrained choice)
- Get gauge groups later from inner automorphisms

Adjoint action:
$\mathcal{M}$ bimodule over $\mathcal{A}$, $u \in \mathcal{U}(\mathcal{A})$ unitary

$$\mathrm{Ad}(u)\xi = u\xi u^* \quad \forall \xi \in \mathcal{M}$$

Odd bimodule: $\mathcal{M}$ bimodule for $\mathcal{A}_{LR}$ *odd* iff
$s = (1, -1, -1, 1)$ acts by $Ad(s) = -1$

$\Leftrightarrow$ Rep of $\mathcal{B} = (\mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}_{LR}^{op})_p$ as $\mathbb{C}$-algebra
$p = \frac{1}{2}(1 - s \otimes s^0)$, with $\mathcal{A}^0 = \mathcal{A}^{op}$

$$\mathcal{B} = \oplus^{4-times} M_2(\mathbb{C}) \oplus M_6(\mathbb{C})$$

Contragredient bimodule of $\mathcal{M}$

$$\mathcal{M}^0 = \{\bar{\xi} \,;\, \xi \in \mathcal{M}\}, \quad a\,\bar{\xi}\,b = \overline{b^* \xi\, a^*}$$

The bimodule $\mathcal{M}_F$

$\mathcal{M}_F =$ sum of all inequivalent irreducible odd $\mathcal{A}_{LR}$-bimodules

- $\dim_{\mathbb{C}} \mathcal{M}_F = 32$
- $\mathcal{M}_F = \mathcal{E} \oplus \mathcal{E}^0$

$$\mathcal{E} = \mathbf{2}_L \otimes \mathbf{1}^0 \oplus \mathbf{2}_R \otimes \mathbf{1}^0 \oplus \mathbf{2}_L \otimes \mathbf{3}^0 \oplus \mathbf{2}_R \otimes \mathbf{3}^0$$

- $\mathcal{M}_F \cong \mathcal{M}_F^0$ by antilinear $J_F(\xi, \bar{\eta}) = (\eta, \bar{\xi})$ for $\xi\,,\ \eta \in \mathcal{E}$

$$J_F^2 = 1\,, \quad \xi\, b = J_F\, b^* J_F\, \xi \qquad \xi \in \mathcal{M}_F\,,\ b \in \mathcal{A}_{LR}$$

- Sum irreducible representations of $\mathcal{B}$

$$\mathbf{2}_L \otimes \mathbf{1}^0 \oplus \mathbf{2}_R \otimes \mathbf{1}^0 \oplus \mathbf{2}_L \otimes \mathbf{3}^0 \oplus \mathbf{2}_R \otimes \mathbf{3}^0$$

$$\oplus \mathbf{1} \otimes \mathbf{2}_L^0 \oplus \mathbf{1} \otimes \mathbf{2}_R^0 \oplus \mathbf{3} \otimes \mathbf{2}_L^0 \oplus \mathbf{3} \otimes \mathbf{2}_R^0$$

- Grading: $\gamma_F = c - J_F\, c\, J_F$ with $c = (0, 1, -1, 0) \in \mathcal{A}_{LR}$

$$J_F^2 = 1\,, \quad J_F\, \gamma_F = -\,\gamma_F\, J_F$$

Grading and KO-dimension: commutations $\Rightarrow$ KO-dim $6 \mod 8$

Interpretation as particles (Fermions)

$$q(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix} \qquad q(\lambda)|\uparrow\rangle = \lambda|\uparrow\rangle, \qquad q(\lambda)|\downarrow\rangle = \bar{\lambda}|\downarrow\rangle$$

- $\mathbf{2}_L \otimes \mathbf{1}^0$: neutrinos $\nu_L \in |\uparrow\rangle_L \otimes \mathbf{1}^0$ and charged leptons $e_L \in |\downarrow\rangle_L \otimes \mathbf{1}^0$
- $\mathbf{2}_R \otimes \mathbf{1}^0$: right-handed neutrinos $\nu_R \in |\uparrow\rangle_R \otimes \mathbf{1}^0$ and charged leptons $e_R \in |\downarrow\rangle_R \otimes \mathbf{1}^0$
- $\mathbf{2}_L \otimes \mathbf{3}^0$ (color indices): u/c/t quarks $u_L \in |\uparrow\rangle_L \otimes \mathbf{3}^0$ and d/s/b quarks $d_L \in |\downarrow\rangle_L \otimes \mathbf{3}^0$
- $\mathbf{2}_R \otimes \mathbf{3}^0$ (color indices): u/c/t quarks $u_R \in |\uparrow\rangle_R \otimes \mathbf{3}^0$ and d/s/b quarks $d_R \in |\downarrow\rangle_R \otimes \mathbf{3}^0$
- $\mathbf{1} \otimes \mathbf{2}^0_{L,R}$: antineutrinos $\bar{\nu}_{L,R} \in \mathbf{1} \otimes \uparrow\rangle^0_{L,R}$, and charged antileptons $\bar{e}_{L,R} \in \mathbf{1} \otimes |\downarrow\rangle^0_{L,R}$
- $\mathbf{3} \otimes \mathbf{2}^0_{L,R}$ (color indices): antiquarks $\bar{u}_{L,R} \in \mathbf{3} \otimes |\uparrow\rangle^0_{L,R}$ and $\bar{d}_{L,R} \in \mathbf{3} \otimes |\downarrow\rangle^0_{L,R}$

Subalgebra and order one condition:

$N = 3$ generations (input): $\mathcal{H}_F = \mathcal{M}_F \oplus \mathcal{M}_F \oplus \mathcal{M}_F$

Left action of $\mathcal{A}_{LR}$ sum of representations $\pi|_{\mathcal{H}_f} \oplus \pi'|_{\mathcal{H}_{\bar{f}}}$ with $\mathcal{H}_f = \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{E}$ and $\mathcal{H}_{\bar{f}} = \mathcal{E}^0 \oplus \mathcal{E}^0 \oplus \mathcal{E}^0$ and with no equivalent subrepresentations (disjoint)

If D mixes $\mathcal{H}_f$ and $\mathcal{H}_{\bar{f}}$ $\Rightarrow$ no order one condition for $\mathcal{A}_{LR}$

Problem for coupled pair: $\mathcal{A} \subset \mathcal{A}_{LR}$ and D with off diagonal terms

maximal $\mathcal{A}$ where order one condition holds

$$\mathcal{A}_F = \{(\lambda, q_L, \lambda, m) \mid \lambda \in \mathbb{C}, \; q_L \in \mathbb{H}, \; m \in M_3(\mathbb{C})\}$$

$$\sim \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}).$$

unique up to $\mathrm{Aut}(\mathcal{A}_{LR})$
$\Rightarrow$ spontaneous breaking of LR symmetry

Subalgebras with off diagonal Dirac and order one condition

Operator $T \,:\, \mathcal{H}_f \to \mathcal{H}_{\bar f}$

$$\mathcal{A}(T) = \{ b \in \mathcal{A}_{LR} \mid \pi'(b) T = T \pi(b) \, ,$$

$$\pi'(b^*) T = T \pi(b^*) \}$$

involutive unital subalgebra of $\mathcal{A}_{LR}$

$\mathcal{A} \subset \mathcal{A}_{LR}$ involutive unital subalgebra of $\mathcal{A}_{LR}$

- restriction of π and π' to $\mathcal{A}$ disjoint $\Rightarrow$ no off diag D for $\mathcal{A}$
- $\exists$ off diag D for $\mathcal{A} \Rightarrow$ pair e, e' min proj in commutants of $\pi(\mathcal{A}_{LR})$ and $\pi'(\mathcal{A}_{LR})$ and operator T

$$e' T e = T \neq 0 \quad \text{and} \quad \mathcal{A} \subset \mathcal{A}(T)$$

- Then case by case analysis to identify max dimensional

Symmetries

Up to a finite abelian group

$$\mathrm{SU}(\mathcal{A}_F) \sim \mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$$

Adjoint action of $\mathrm{U}(1)$ (in powers of $\lambda \in \mathrm{U}(1)$)

	$\uparrow \otimes \mathbf{1}^0$	$\downarrow \otimes \mathbf{1}^0$	$\uparrow \otimes \mathbf{3}^0$	$\downarrow \otimes \mathbf{3}^0$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	0	-2	$\frac{4}{3}$	$-\frac{2}{3}$

$\Rightarrow$ correct hypercharges of fermions (confirms identification of $\mathcal{H}_F$ basis with fermions)

Classifying Dirac operators for $(\mathcal{A}_F, \mathcal{H}_F, \gamma_F, J_F)$ all possible D_F self adjoint on $\mathcal{H}_F$, commuting with J_F, anticommuting with γ_F and $[[D,a],b^0]=0$, $\forall a,b\in\mathcal{A}_F$

Input conditions (massless photon): commuting with subalgebra

$$\mathbb{C}_F \subset \mathcal{A}_F\,, \quad \mathbb{C}_F = \{(\lambda,\lambda,0)\,,\lambda\in\mathbb{C}\}$$

then $D(Y) = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}$ with $S = S_1 \oplus (S_3 \otimes 1_3)$

$$S_1 = \begin{pmatrix} 0 & 0 & Y^*_{(\uparrow 1)} & 0 \\ 0 & 0 & 0 & Y^*_{(\downarrow 1)} \\ Y_{(\uparrow 1)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 1)} & 0 & 0 \end{pmatrix}$$

same for S_3, with $Y_{(\downarrow 1)}$, $Y_{(\uparrow 1)}$, $Y_{(\downarrow 3)}$, $Y_{(\uparrow 3)} \in \mathrm{GL}_3(\mathbb{C})$ and Y_R symmetric:

$$T : E_R = \uparrow_R \otimes \mathbf{1}^0 \to J_F\, E_R$$

Moduli space $\mathcal{C}_3 \times \mathcal{C}_1$ $\mathcal{C}_3 =$ pairs $(Y_{(\downarrow 3)}, Y_{(\uparrow 3)})$ modulo

$$Y'_{(\downarrow 3)} = W_1\, Y_{(\downarrow 3)}\, W_3^*, \qquad Y'_{(\uparrow 3)} = W_2\, Y_{(\uparrow 3)}\, W_3^*$$

W_j unitary matrices

$$\mathcal{C}_3 = (K \times K)\backslash(G \times G)/K$$

$G = \mathrm{GL}_3(\mathbb{C})$ and $K = U(3)$ $\dim_{\mathbb{R}} \mathcal{C}_3 = 10 = 3 + 3 + 4$
$(3 + 3$ eigenvalues, 3 angles, 1 phase$)$

$\mathcal{C}_1 =$ triplets $(Y_{(\downarrow 1)}, Y_{(\uparrow 1)}, Y_R)$ with Y_R symmetric modulo

$$Y'_{(\downarrow 1)} = V_1\, Y_{(\downarrow 1)} V_3^*, \quad Y'_{(\uparrow 1)} = V_2\, Y_{(\uparrow 1)}\, V_3^*, \quad Y'_R = V_2\, Y_R\, \bar{V}_2^*$$

$\pi : \mathcal{C}_1 \to \mathcal{C}_3$ surjection forgets Y_R fiber symmetric matrices mod
$Y_R \mapsto \lambda^2 Y_R$ $\dim_{\mathbb{R}}(\mathcal{C}_3 \times \mathcal{C}_1) = 31$ (dim fiber 12-1=11)

Physical interpretation: Yukawa parameters and Majorana masses
Representatives in $\mathcal{C}_3 \times \mathcal{C}_1$:

$$Y_{(\uparrow 3)} = \delta_{(\uparrow 3)} \qquad Y_{(\downarrow 3)} = U_{CKM}\, \delta_{(\downarrow 3)}\, U^*_{CKM}$$

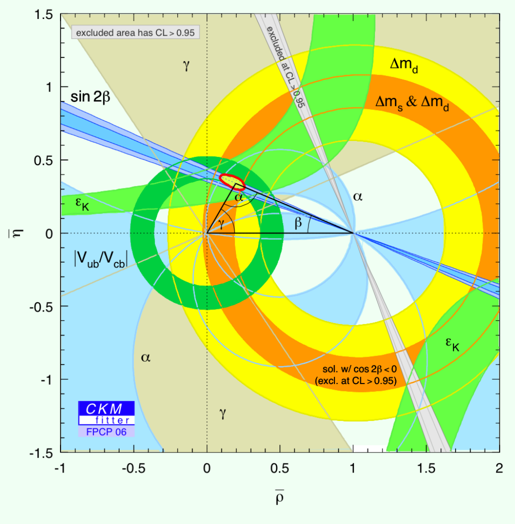
$$Y_{(\uparrow 1)} = U^*_{PMNS}\, \delta_{(\uparrow 1)}\, U_{PMNS} \qquad Y_{(\downarrow 1)} = \delta_{(\downarrow 1)}$$

$\delta_\uparrow$, $\delta_\downarrow$ diagonal: Dirac masses

$$U = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e_\delta & c_1 c_2 s_3 + s_2 c_3 e_\delta \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e_\delta & c_1 s_2 s_3 - c_2 c_3 e_\delta \end{pmatrix}$$

angles and phase $c_i = \cos\theta_i$, $s_i = \sin\theta_i$, $e_\delta = \exp(i\delta)$
U_{CKM} = Cabibbo–Kobayashi–Maskawa
U_{PMNS} = Pontecorvo–Maki–Nakagawa–Sakata
$\Rightarrow$ neutrino mixing
Y_R = Majorana mass terms for right-handed neutrinos

CKM matrix very strict experimental constraints (SM compatible)



unitarity triangle (offdiag elements of V^*V adding to 0)

constraints also on neutrino mixing matrix

Geometric point of view:

- CKM and PMNS matrices data: coordinates on moduli space of Dirac operators
- Experimental constraints define subvarieties in the moduli space
- Symmetric spaces $(K \times K)\backslash(G \times G)/K$ interesting geometry
- Get parameter relations from "interesting subvarieties"?

Summary: matter content of the NCG model

νMSM: Minimal Standard Model with additional right handed neutrinos with Majorana mass terms

Free parameters in the model:

- 3 coupling constants
- 6 quark masses, 3 mixing angles, 1 complex phase
- 3 charged lepton masses, 3 lepton mixing angles, 1 complex phase
- 3 neutrino masses
- 11 Majorana mass matrix parameters
- 1 QCD vacuum angle

Moduli space of Dirac operators on the finite NC space F: all masses, mixing angles, phases, Majorana mass terms

Other parameters:

- coupling constants: product geometry and action functional
- vacuum angle not there (but quantum corrections...?)

Other particle models

Changing the finite geometry produces other particle models:

- Minimal Standard Model
- Supersymmetric QCD
- QED

Respecively:

- Alain Connes, *Gravity coupled with matter and the foundation of non-commutative geometry*, Comm. Math. Phys. 182 (1996), no. 1, 155–176
- Thijs van den Broek, Walter D. van Suijlekom, *Supersymmetric QCD and noncommutative geometry*, arXiv:1003.3788
- Koen van den Dungen, Walter D. van Suijlekom, *Electrodynamics from Noncommutative Geometry*, arXiv:1103.2928

Next episode:

- Product geometries
- Almost commutative geometries
- The spectral action
- Asymptotic expansion at large energies
- The Lagrangian
- Renormalization group equations and low energy limit