Noncommutative Geometry models for Particle Physics and Cosmology, Lecture I

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Plan of lectures

1. Noncommutative geometry approach to elementary particle physics; Noncommutative Riemannian geometry; finite noncommutative geometries; moduli spaces; the finite geometry of the Standard Model

2. The product geometry; the spectral action functional and its asymptotic expansion; bosons and fermions; the Standard Model Lagrangian; renormalization group flow, geometric constraints and low energy limits

3. Parameters: relations at unification and running; running of the gravitational terms; the RGE flow with right handed neutrinos; cosmological timeline and the inflation epoch; effective gravitational and cosmological constants and models of inflation

4. The spectral action and the problem of cosmic topology; cosmic topology and the CMB; slow-roll inflation; Poisson summation formula and the nonperturbative spectral action; spherical and flat space forms
Geometrization of physics

- Kaluza-Klein theory: electromagnetism described by circle bundle over spacetime manifold, connection = EM potential;
- Yang–Mills gauge theories: bundle geometry over spacetime, connections = gauge potentials, sections = fermions;
- String theory: 6 extra dimensions (Calabi-Yau) over spacetime, strings vibrations = types of particles
- NCG models: extra dimensions are NC spaces, pure gravity on product space becomes gravity + matter on spacetime
Why a geometrization of physics?
The standard model of elementary particles
Standard model Lagrangian: can compute from simpler data?

\[ \mathcal{L}_{SM} = \cdots \]

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NCG models for particles and cosmology, I
Standard model parameters: are there relations? why these values?
Building mathematical models: essential requirements

- Conceptualize: complicated things (SM Lagrangian) should follow from simple things (geometry)
- Enrich: extend existing models with new features
- Predict: Higgs mass? new particles? new parameter relations? new phenomena?

The elephant in the room: Gravity!

Coupling gravity to matter: the good: better conceptual structure, links particle physics to cosmology; the bad: worse chances of passing from classical to quantum theory

In NCG models: “all forces become gravity” (on an NC space)
What the NCG model provides:

- Einstein–Hilbert action:

\[
S_{EH}(g_{\mu\nu}) = \frac{1}{16\pi G} \int_M R \sqrt{g} \, d^4x
\]

- Gravity minimally coupled to matter: \( S = S_{EH} + S_{SM} \) with \( S_{SM} = \) particle physics \( \Rightarrow \) Standard Model Lagrangian

- Right handed neutrinos with Majorana masses

- Modified gravity model \( f(R, R^{\mu\nu}, C_{\lambda\mu\nu\kappa}) \) with conformal gravity: Weyl curvature tensor

\[
C_{\lambda\mu\nu\kappa} = R_{\lambda\mu\nu\kappa} - \frac{1}{2}(g_{\lambda\nu} R_{\mu\kappa} - g_{\lambda\kappa} R_{\mu\nu} - g_{\mu\nu} R_{\lambda\kappa} + g_{\mu\kappa} R_{\lambda\nu})
\]

\[
+ \frac{1}{6} R^{\alpha}_\alpha (g_{\lambda\nu} g_{\mu\kappa} - g_{\lambda\kappa} g_{\mu\nu})
\]

- Non-minimal coupling of Higgs to gravity

\[
\int_M \left( \frac{1}{2} |D_\mu \mathbf{H}|^2 - \mu_0^2 |\mathbf{H}|^2 - \xi_0 R |\mathbf{H}|^2 + \lambda_0 |\mathbf{H}|^4 \right) \sqrt{g} \, d^4x
\]
What is Noncommutative Geometry?

- $X$ compact Hausdorff topological space $\iff C(X)$ abelian $C^*$-algebra of continuous functions (Gelfand–Naimark)
- Noncommutative $C^*$-algebra $\mathcal{A}$: think of as “continuous functions on an NC space”
- Describe geometry in terms of algebra $C(X)$ and dense subalgebras like $C^\infty(X)$ if $X$ manifold
- Differential forms, bundles, connections, cohomology: all continue to make sense without commutativity (NC geometry)
- Can describe “bad quotients” as good spaces: functions on the graph of the equivalence relation with convolution product

To do physics: need the analog of *Riemannian geometry*
Spectral triples and NC Riemannian manifolds \((\mathcal{A}, \mathcal{H}, D)\)

- involutive algebra \(\mathcal{A}\)
- representation \(\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})\)
- self adjoint operator \(D\) on \(\mathcal{H}\), dense domain
- compact resolvent \((1 + D^2)^{-1/2} \in \mathcal{K}\)
- \([a, D]\) bounded \(\forall a \in \mathcal{A}\)
- even if \(\mathbb{Z}/2\)- grading \(\gamma\) on \(\mathcal{H}\)

\[ [\gamma, a] = 0, \ \forall a \in \mathcal{A}, \quad D\gamma = -\gamma D \]

Main example \((C^\infty(M), L^2(M, S), \hat{\phi}_M)\) with chirality \(\gamma_5\) in 4-dim

Note: to apply spectral triples methods need to work with \(M\) compact and Euclidean signature (Euclidean gravity)
Real structure \( KO \)-dimension \( n \in \mathbb{Z}/8\mathbb{Z} \)

antilinear isometry \( J : \mathcal{H} \to \mathcal{H} \)

\[
J^2 = \epsilon, \quad JD = \epsilon' DJ, \quad \text{and} \quad J\gamma = \epsilon'' \gamma J
\]

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Commutation: \([a, b^0] = 0 \quad \forall a, b \in \mathcal{A}\)

where \( b^0 = Jb^* J^{-1} \quad \forall b \in \mathcal{A} \)

Order one condition:

\([D, a], b^0] = 0 \quad \forall a, b \in \mathcal{A}\)
Spectral triples in NCG need not be manifolds:

- Quantum groups
- Fractals
- NC tori

For particle physics models $M \times F$, product of a 4-dimensional spacetime manifold $M$ by a “finite NC space” $F$ (extra dimensions)


**Almost commutative geometries**: more general form, fibration (not product) over manifold, fiber finite NC space

Finite real spectral triples: \( F = (\mathcal{A}_F, \mathcal{H}_F, D_F) \)

- \( \mathcal{A} \) finite dimensional (real) \( C^* \)-algebra

\[
\mathcal{A} = \bigoplus_{i=1}^{N} M_{n_i}(\mathbb{K}_i)
\]

\( \mathbb{K}_i = \mathbb{R} \) or \( \mathbb{C} \) or \( \mathbb{H} \) quaternions (Wedderburn)

- Representation on finite dimensional Hilbert space \( \mathcal{H} \), with bimodule structure given by \( J \) (condition \( [a, b^0] = 0 \))

- \( D^* = D \) with order one condition

\[
[[D, a], b^0] = 0
\]

\( \Rightarrow \) Moduli spaces (under unitary equivalence)


Building a particle physics model ...this lecture based on:


Minimal input ansatz:

- left-right symmetric algebra

\[ \mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C}) \]

- involution \((\lambda, q_L, q_R, m) \mapsto (\bar{\lambda}, \bar{q}_L, \bar{q}_R, m^*)\)

- subalgebra \(\mathbb{C} \oplus M_3(\mathbb{C})\) integer spin \(\mathbb{C}\)-alg

- subalgebra \(\mathbb{H}_L \oplus \mathbb{H}_R\) half-integer spin \(\mathbb{R}\)-alg

More general choices of initial ansatz:


Slogan: algebras better than Lie algebras, more constraints on reps
Comment: **associative algebras versus Lie algebras**

- In geometry of gauge theories: bundle over spacetime, connections and sections, automorphisms gauge group: Lie group
- Decomposing composite particles into elementary particles: Lie group representations (hadrons and quarks)
- If want only elementary particles: associative algebras have very few representations (very constrained choice)
- Get gauge groups later from inner automorphisms
Adjoint action: 
\( \mathcal{M} \) bimodule over \( \mathcal{A} \), \( u \in U(\mathcal{A}) \) unitary

\[
\text{Ad}(u)\xi = u\xi u^* \quad \forall \xi \in \mathcal{M}
\]

Odd bimodule: \( \mathcal{M} \) bimodule for \( \mathcal{A}_{LR} \) odd iff
\( s = (1, -1, -1, 1) \) acts by \( \text{Ad}(s) = -1 \)

\[\iff\text{Rep of } \mathcal{B} = (\mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}^{op}_{LR})_p \text{ as } \mathbb{C}\text{-algebra} \]
\[
p = \frac{1}{2}(1 - s \otimes s^0), \text{ with } \mathcal{A}^0 = \mathcal{A}^{op}
\]

\[
\mathcal{B} = \bigoplus^{4\text{-times}} M_2(\mathbb{C}) \oplus M_6(\mathbb{C})
\]

Contragredient bimodule of \( \mathcal{M} \)

\[
\mathcal{M}^0 = \{ \bar{\xi} ; \xi \in \mathcal{M} \}, \quad a \bar{\xi} b = \overline{b^* \xi a^*}
\]
The bimodule $\mathcal{M}_F$
$\mathcal{M}_F = \text{sum of all inequivalent irreducible odd } \mathcal{A}_{LR}\text{-bimodules}$

- $\dim_{\mathbb{C}} \mathcal{M}_F = 32$
- $\mathcal{M}_F = \mathcal{E} \oplus \mathcal{E}^0$

$$\mathcal{E} = 2_L \otimes 1^0 \oplus 2_R \otimes 1^0 \oplus 2_L \otimes 3^0 \oplus 2_R \otimes 3^0$$

- $\mathcal{M}_F \cong \mathcal{M}_F^0$ by antilinear $J_F(\xi, \bar{\eta}) = (\eta, \bar{\xi})$ for $\xi, \eta \in \mathcal{E}$
- $J_F^2 = 1$, $\xi b = J_F b^* J_F \xi$ for $\xi \in \mathcal{M}_F$, $b \in \mathcal{A}_{LR}$

- Sum irreducible representations of $\mathcal{B}$

$$2_L \otimes 1^0 \oplus 2_R \otimes 1^0 \oplus 2_L \otimes 3^0 \oplus 2_R \otimes 3^0$$

$$\oplus 1 \otimes 2^0_L \oplus 1 \otimes 2^0_R \oplus 3 \otimes 2^0_L \oplus 3 \otimes 2^0_R$$

- Grading: $\gamma_F = c - J_F c J_F$ with $c = (0, 1, -1, 0) \in \mathcal{A}_{LR}$

$$J_F^2 = 1, \quad J_F \gamma_F = -\gamma_F J_F$$

Grading and KO-dimension: commutations $\Rightarrow KO\text{-dim 6 mod 8}$
Interpretation as particles (Fermions)

\[
q(\lambda) = \begin{pmatrix}
\lambda & 0 \\
0 & \bar{\lambda}
\end{pmatrix}
\]

\[
q(\lambda)|\uparrow\rangle = \lambda|\uparrow\rangle, \quad q(\lambda)|\downarrow\rangle = \bar{\lambda}|\downarrow\rangle
\]

- **2_L \otimes 1^0**: neutrinos \( \nu_L \in |\uparrow\rangle_L \otimes 1^0 \) and charged leptons \( e_L \in |\downarrow\rangle_L \otimes 1^0 \)
- **2_R \otimes 1^0**: right-handed neutrinos \( \nu_R \in |\uparrow\rangle_R \otimes 1^0 \) and charged leptons \( e_R \in |\downarrow\rangle_R \otimes 1^0 \)
- **2_L \otimes 3^0** (color indices): u/c/t quarks \( u_L \in |\uparrow\rangle_L \otimes 3^0 \) and d/s/b quarks \( d_L \in |\downarrow\rangle_L \otimes 3^0 \)
- **2_R \otimes 3^0** (color indices): u/c/t quarks \( u_R \in |\uparrow\rangle_R \otimes 3^0 \) and d/s/b quarks \( d_R \in |\downarrow\rangle_R \otimes 3^0 \)
- **1 \otimes 2^0_{L,R}**: antineutrinos \( \bar{\nu}_{L,R} \in 1 \otimes |\uparrow\rangle_{L,R}^0 \), and charged antileptons \( \bar{e}_{L,R} \in 1 \otimes |\downarrow\rangle_{L,R}^0 \)
- **3 \otimes 2^0_{L,R}** (color indices): antiquarks \( \bar{u}_{L,R} \in 3 \otimes |\uparrow\rangle_{L,R}^0 \) and \( \bar{d}_{L,R} \in 3 \otimes |\downarrow\rangle_{L,R}^0 \)
Subalgebra and order one condition:

\( N = 3 \) generations (input): \( \mathcal{H}_F = \mathcal{M}_F \oplus \mathcal{M}_F \oplus \mathcal{M}_F \)

Left action of \( \mathcal{A}_{LR} \) sum of representations \( \pi|_{\mathcal{H}_f} \oplus \pi'|_{\mathcal{H}_f} \) with
\( \mathcal{H}_f = \mathcal{E} \oplus \mathcal{E} \oplus \mathcal{E} \) and \( \mathcal{H}_{\bar{f}} = \mathcal{E}^0 \oplus \mathcal{E}^0 \oplus \mathcal{E}^0 \) and with no equivalent subrepresentations (disjoint)

If \( D \) mixes \( \mathcal{H}_f \) and \( \mathcal{H}_{\bar{f}} \) \( \Rightarrow \) no order one condition for \( \mathcal{A}_{LR} \)

Problem for coupled pair: \( \mathcal{A} \subset \mathcal{A}_{LR} \) and \( D \) with off diagonal terms
maximal \( \mathcal{A} \) where order one condition holds

\( \mathcal{A}_F = \{ (\lambda, q_L, \lambda, m) \mid \lambda \in \mathbb{C}, \ q_L \in \mathbb{H}, \ m \in M_3(\mathbb{C}) \} \)

\( \sim \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \). unique up to \( \text{Aut}(\mathcal{A}_{LR}) \)

\( \Rightarrow \) spontaneous breaking of LR symmetry
Subalgebras with off diagonal Dirac and order one condition

Operator $T : \mathcal{H}_f \rightarrow \mathcal{H}_{\bar{f}}$

$$\mathcal{A}(T) = \{ b \in \mathcal{A}_{LR} \mid \pi'(b)T = T\pi(b) \},$$

$$\pi'(b^*)T = T\pi(b^*) \}$$

involutive unital subalgebra of $\mathcal{A}_{LR}$

$\mathcal{A} \subset \mathcal{A}_{LR}$ involutive unital subalgebra of $\mathcal{A}_{LR}$

- restriction of $\pi$ and $\pi'$ to $\mathcal{A}$ disjoint $\Rightarrow$ no off diag $D$ for $\mathcal{A}$
- $\exists$ off diag $D$ for $\mathcal{A}$ $\Rightarrow$ pair $e, e'$ min proj in commutants of $\pi(\mathcal{A}_{LR})$ and $\pi'(\mathcal{A}_{LR})$ and operator $T$

$$e'Te = T \neq 0 \quad \text{and} \quad \mathcal{A} \subset \mathcal{A}(T)$$

- Then case by case analysis to identify max dimensional
Symmetries
Up to a finite abelian group

$$\text{SU}(A_F) \sim \text{U}(1) \times \text{SU}(2) \times \text{SU}(3)$$

Adjoint action of $\text{U}(1)$ (in powers of $\lambda \in \text{U}(1)$)

$$\uparrow \otimes 1^0 \quad \downarrow \otimes 1^0 \quad \uparrow \otimes 3^0 \quad \downarrow \otimes 3^0$$

$$2_L \quad -1 \quad -1 \quad \frac{1}{3} \quad \frac{1}{3}$$

$$2_R \quad 0 \quad -2 \quad \frac{4}{3} \quad -\frac{2}{3}$$

⇒ correct hypercharges of fermions (confirms identification of $\mathcal{H}_F$ basis with fermions)
Classifying Dirac operators for \((\mathcal{A}_F, \mathcal{H}_F, \gamma_F, J_F)\) all possible \(D_F\) self adjoint on \(\mathcal{H}_F\), commuting with \(J_F\), anticommuting with \(\gamma_F\) and \([[D, a], b^0] = 0, \forall a, b \in \mathcal{A}_F\)

Input conditions (massless photon): commuting with subalgebra

\[
\mathbb{C}_F \subset \mathcal{A}_F, \quad \mathbb{C}_F = \{ (\lambda, \lambda, 0), \lambda \in \mathbb{C} \}
\]

then \(D(Y) = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}\) with \(S = S_1 \oplus (S_3 \otimes 1_3)\)

\[
S_1 = \begin{pmatrix} 0 & 0 & Y_{(\uparrow 1)}^* & 0 \\ 0 & 0 & 0 & Y_{(\downarrow 1)}^* \\ Y_{(\uparrow 1)} & 0 & 0 & 0 \\ 0 & Y_{(\downarrow 1)} & 0 & 0 \end{pmatrix}
\]

same for \(S_3\), with \(Y_{(\downarrow 1)}, Y_{(\uparrow 1)}, Y_{(\downarrow 3)}, Y_{(\uparrow 3)} \in \text{GL}_3(\mathbb{C})\) and \(Y_R\) symmetric:

\[
T : E_R = \uparrow_R \otimes 1^0 \to J_F E_R
\]
Moduli space \( \mathcal{C}_3 \times \mathcal{C}_1 \) \( \mathcal{C}_3 = \text{pairs } (Y_{(\downarrow 3)}, Y_{(\uparrow 3)}) \) modulo

\[ Y'_{(\downarrow 3)} = W_1 Y_{(\downarrow 3)} W_3^*, \quad Y'_{(\uparrow 3)} = W_2 Y_{(\uparrow 3)} W_3^* \]

\( W_j \) unitary matrices

\[ \mathcal{C}_3 = (K \times K) \setminus (G \times G)/K \]

\( G = \text{GL}_3(\mathbb{C}) \) and \( K = U(3) \) \( \dim_{\mathbb{R}} \mathcal{C}_3 = 10 = 3 + 3 + 4 \)
(3 + 3 eigenvalues, 3 angles, 1 phase)

\( \mathcal{C}_1 = \text{triplets } (Y_{(\downarrow 1)}, Y_{(\uparrow 1)}, Y_R) \) with \( Y_R \) symmetric modulo

\[ Y'_{(\downarrow 1)} = V_1 Y_{(\downarrow 1)} V_3^*, \quad Y'_{(\uparrow 1)} = V_2 Y_{(\uparrow 1)} V_3^*, \quad Y'_R = V_2 Y_R V_2^* \]

\[ \pi : \mathcal{C}_1 \to \mathcal{C}_3 \) surjection forgets \( Y_R \) fiber symmetric matrices mod

\[ Y_R \mapsto \lambda^2 Y_R \quad \dim_{\mathbb{R}}(\mathcal{C}_3 \times \mathcal{C}_1) = 31 \) (dim fiber 12-1=11)
Physical interpretation: Yukawa parameters and Majorana masses

Representatives in $C_3 \times C_1$:

$$Y_{(\uparrow 3)} = \delta_{(\uparrow 3)} \quad Y_{(\downarrow 3)} = U_{CKM} \delta_{(\downarrow 3)} U_{CKM}^*$$

$$Y_{(\uparrow 1)} = U_{PMNS}^* \delta_{(\uparrow 1)} U_{PMNS} \quad Y_{(\downarrow 1)} = \delta_{(\downarrow 1)}$$

$\delta_{\uparrow}, \delta_{\downarrow}$ diagonal: Dirac masses

$$U = \begin{pmatrix}
  c_1 & -s_1 c_3 & -s_1 s_3 \\
  s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\
  s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta}
\end{pmatrix}$$

angles and phase $c_i = \cos \theta_i$, $s_i = \sin \theta_i$, $e^{i\delta} = \exp(i\delta)$

$U_{CKM} =$ Cabibbo–Kobayashi–Maskawa

$U_{PMNS} =$ Pontecorvo–Maki–Nakagawa–Sakata

$\Rightarrow$ neutrino mixing

$Y_R =$ Majorana mass terms for right-handed neutrinos
CKM matrix very strict experimental constraints (SM compatible)

unitarity triangle (offdiag elements of $V^*V$ adding to 0)

constraints also on neutrino mixing matrix
Geometric point of view:

- CKM and PMNS matrices data: coordinates on moduli space of Dirac operators
- Experimental constraints define subvarieties in the moduli space
- Symmetric spaces \((K \times K) \backslash (G \times G)/K\) interesting geometry
- Get parameter relations from “interesting subvarieties”?
Summary: matter content of the NCG model

\( \nu\text{MSM} \): Minimal Standard Model with additional right handed neutrinos with Majorana mass terms

**Free parameters** in the model:

- 3 coupling constants
- 6 quark masses, 3 mixing angles, 1 complex phase
- 3 charged lepton masses, 3 lepton mixing angles, 1 complex phase
- 3 neutrino masses
- 11 Majorana mass matrix parameters
- 1 QCD vacuum angle

Moduli space of Dirac operators on the finite NC space \( F \): all masses, mixing angles, phases, Majorana mass terms

Other parameters:

- coupling constants: product geometry and action functional
- vacuum angle not there (but quantum corrections...?)
Other particle models
Changing the finite geometry produces other particle models:

- Minimal Standard Model
- Supersymmetric QCD
- QED

Respectively:

- Thijs van den Broek, Walter D. van Suijlekom, *Supersymmetric QCD and noncommutative geometry*, arXiv:1003.3788
Next episode:

- Product geometries
- Almost commutative geometries
- The spectral action
- Asymptotic expansion at large energies
- The Lagrangian
- Renormalization group equations and low energy limit