# The Geometry of Syntax (Part 2)

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## A Mathematical Physicist's adventures in Linguistics

- This talk is in two parts:
  - Today Linguistics: relations between syntactic parameters via Kanerva networks, coding theory, and via Belkin–Niyogi heat kernel dimensional reduction
  - Continuation of Math Colloquium: persistent topology of syntax, algebro-geometric historical linguistics, spin glass models of language evolution

The two parts are largely independent: no need to have seen the previous one (some small amount of repetition if you have seen it)

References for this part:

- Matilde Marcolli, *Syntactic parameters and a coding theory perspective on entropy and complexity of language families*, Entropy 18 (2016), no. 4, Paper No. 110, 17 pp.
- Kevin Shu and Matilde Marcolli, *Syntactic structures and code parameters*, Mathematics in Computer Science 11 (2017) no. 1, 79–90.
- J.J. Park, R. Boettcher, A. Zhao, A. Mun, K. Yuh, V. Kumar, M. Marcolli, *Prevalence and recoverability of syntactic parameters in sparse distributed memories*, in "Geometric Science of Information. Third International Conference GSI 2017", pp. 265–272, Lecture Notes in Computer Science, Vol.10589, Springer 2017.
- Andrew Ortegaray, Robert C. Berwick, Matilde Marcolli, *Heat Kernel Analysis of Syntactic Structures*, arXiv:1803.09832

## Questions and Motivation

- Broad question: can we use computational (mathematical) techniques to better understand how the human brain processes language?
- some of the main questions:
  - Language acquisition (poverty of the stimulus): how does the learning brain converge to *one* grammar?
  - How is language (in particular syntax) stored in the brain?
  - How do languages change and evolve in time? quantitative, predictive modeling?
- focus on the "large scale structure" of language: syntax
- Plan: approach these questions from a mathematical perspective, using tools from geometry and theoretical physics

## Syntax and Syntactic Parameters

• we work here within the framework of the Principles and Parameters model (Chomsky, 1981)

- *principles*: general rules of grammar
- *parameters*: binary variables (on/off switches) that distinguish languages in terms of syntactic structures

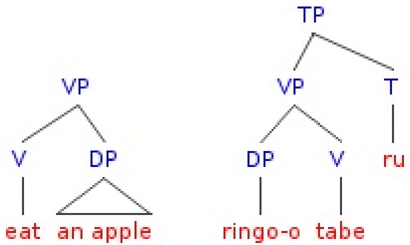
• this idea is very appealing for a mathematician: at the level of syntax a language can be described by a set of coordinates given by binary variables

• however, surprisingly no mathematical model of Principles and Parameters formulation of Linguistics has been developed so far

- Example of parameter: head-directionality

(head-initial versus head-final)

English is head-initial, Japanese is head-final



VP= verb phrase, TP= tense phrase, DP= determiner phrase

- Other examples of parameters:
  - *Subject-side*
  - *Pro-drop*
  - *Null-subject*

## Main Problems

- there is no complete classification of syntactic parameters
- there are hundreds of such binary syntactic variables, but not all of them are “true” syntactic parameters (conflations of deep/surface structure)
- Interdependencies between different syntactic parameters are poorly understood: what is a good independent set of variables, a good set of coordinates?
- syntactic parameters are dynamical: they change historically over the course of language change and evolution
- collecting reliable data is hard! (there are thousands of world languages and analyzing them at the level of syntax is much more difficult for linguists than collecting lexical data; few ancient languages have enough written texts)

## Databases of syntactic structures of world languages

1. Syntactic Structures of World Languages (SSWL) `http://sswl.railsplayground.net/`
2. TerraLing `http://www.terraling.com/`
3. World Atlas of Language Structures (WALS) `http://wals.info/`
4. another set of data from Longobardi–Guardiano, Lingua 119 (2009) 1679-1706
5. more complete set of data by Longobardi, Linguistic Analysis, Vol.41 (2017) N.3-4, 517–556.

• First Step: data analysis of syntax of world languages with various mathematical tools (persistent topology, etc.)

• we used the most extensive database currently available: SSWL with 116 "variables" (syntactic "parameters") and 253 world languages (but... some problems with SSWL)

## Problems of SSWL data

- Very non-uniformly mapped across the languages of the database: some are 100% mapped, while for some only very few of the 116 parameters are mapped
- Linguists criticize the choice of binary variable (not all of them should count as "true" parameters)

• the data of Longobardi–Guardiano are more reliable, with 62 languages (mostly Indo-European) and 83 parameters

• linguistic question: can languages that are far away in terms of historical linguistics end up being close in terms of syntactic parameters?

• Guideline for data use: given what is available at present, use SSWL and Longobardi data (two independent set of syntactic features) keeping limitations in mind and comparing structures of the two datasets

## Parameters from Modularized Global Parameterization Method

- G. Longobardi, *Methods in parametric linguistics and cognitive history*, Linguistic Variation Yearbook, Vol.3 (2003) 101–138
- G. Longobardi, C. Guardiano, *Evidence for syntax as a signal of historical relatedness*, Lingua 119 (2009) 1679–1706.

• Determiner Phrase Module:
- syntactic parameters dealing with person, number, gender (1–6)
- parameters of definiteness (7–16)
- parameters of countability (17–24)
- genitive structure (25–31)
- adjectival and relative modification (32–14)
- position and movement of the head noun (42–50)
- demonstratives and other determiners (51–50 and 6–63)
- possessive pronouns (56–59)

• more parameters added in the more recent publication of a more extensive list of data from Longobardi and collaborators

## What kind of relations exist between syntactic parameters?

- Entailment relations: some explicitly known relations where one state of a parameter (or more) can make another parameter undefined
- Example: $\{p_1, p_2\} = \{\text{Strong Deixis}, \text{Strong Anaphoricity}\}$

|  | $p_1$ | $p_2$ |
|---|---|---|
| $\ell_1$ | $+1$ | $+1$ |
| $\ell_2$ | $-1$ | $0$ |
| $\ell_3$ | $+1$ | $+1$ |
| $\ell_4$ | $+1$ | $-1$ |

$\{\ell_1, \ell_2, \ell_3, \ell_4\} = \{\text{English}, \text{Welsh}, \text{Russian}, \text{Bulgarian}\}$

Strong Deixis $+1$: governs possible positions of demonstratives in the nominal domain
Strong Anaphoricity $+1$: obligatory dependence on an antecedent in a local and asymmetric relation to anaphor

- several entailment relations are recorded in the data of Longobardi–Guardiano
- SSWL database does not record relations between parameters
- relations can be detected through methods of data analysis
- goals: identify a good set of independent variables among syntactic parameters, understand (at least statistically) the "manifold" determined by the relations
- some methods we consider here:

1. Kanerva networks: sparse distributed memories
2. coding theory: code parameters, position in the space of codes
3. heat kernel dimensional reduction: Laplace eigenfunctions

## Syntactic Parameters in Kanerva Networks

• J.J. Park, R. Boettcher, A. Zhao, A. Mun, K. Yuh, V. Kumar, M. Marcolli, *Prevalence and recoverability of syntactic parameters in sparse distributed memories*, in "Geometric Science of Information. Third International Conference GSI 2017", pp. 265–272, Lecture Notes in Computer Science, Vol.10589, Springer 2017.

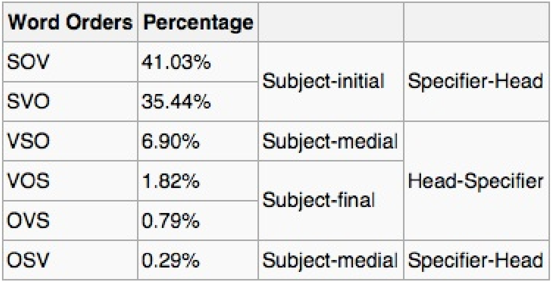
• Select a subset of SSWL parameters with properties:

- Completely mapped for a large number of languages in the database
- Known to have relations, though not of a simple explicit entailment form

• Detect which among these parameters are more or less recoverable from the other ones by testing recoverability in a sparse distributed memory

## Preliminary considerations: Frequency of Expression

- different syntactic parameters have very different frequency of expression among world languages
- Example: Word Order: SOV, SVO, VSO, VOS, OVS, OSV



| Word Orders | Percentage | | |
|---|---|---|---|
| SOV | 41.03% | Subject-initial | Specifier-Head |
| SVO | 35.44% | | |
| VSO | 6.90% | Subject-medial | Head-Specifier |
| VOS | 1.82% | Subject-final | |
| OVS | 0.79% | | |
| OSV | 0.29% | Subject-medial | Specifier-Head |

Very unevenly distributed across world languages

- this creates overall effect (using data that record expression of parameters among world languages): needs to be normalized when searching for abstract syntactic relations among parameters

# Parameters and frequencies (as classified in SSWL)

01 Subject-Verb (0.64957267)

02 Verb-Subject (0.31623933)

03 Verb-Object (0.61538464)

04 Object-Verb (0.32478634)

05 Subject-Verb-Object (0.56837606)

06 Subject-Object-Verb (0.30769232)

07 Verb-Subject-Object (0.1923077)

08 Verb-Object-Subject (0.15811966)

09 Object-Subject-Verb (0.12393162)

10 Object-Verb-Subject (0.10683761)

11 Adposition-Noun-Phrase (0.58974361)

12 Noun-Phrase-Adposition (0.2905983)

13 Adjective-Noun (0.41025642)

14 Noun-Adjective (0.52564102)

15 Numeral-Noun (0.48290598)

16 Noun-Numeral (0.38034189)

17 Demonstrative-Noun (0.47435898)

18 Noun-Demonstrative (0.38461539)

19 Possessor-Noun (0.38034189)

20 Noun-Possessor (0.49145299)

A01 Attributive-Adjective-Agreement (0.46581197)

Kanerva networks (sparse distributed memories)

- P. Kanerva, *Sparse Distributed Memory*, MIT Press, 1988.

- field $\mathbb{F}_2 = \{0,1\}$, vector space $\mathbb{F}_2^N$ large $N$

- uniform random sample of $2^k$ hard locations with $2^k << 2^N$

- median Hamming distance between hard locations

- Hamming spheres of radius slightly larger than median value (access sphere)

- *writing to network*: storing datum $X \in \mathbb{F}_2^N$, each hard location in access sphere of $X$ gets $i$-th coordinate (initialized at zero) incremented depending on $i$-th entry ot $X$

- *reading at a location*: $i$-th entry determined by majority rule of $i$-th entries of all stored data in hard locations within access sphere

Kanerva networks are good at reconstructing corrupted data

## Memory items in SDM: most items unrelated but most pairs linked by few intermediaries

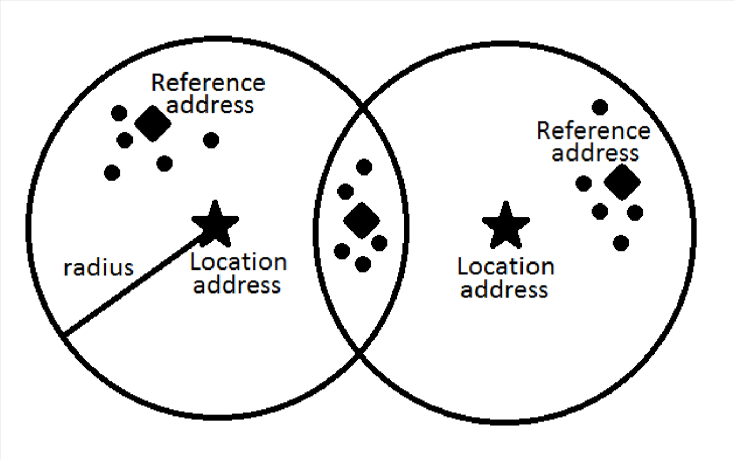


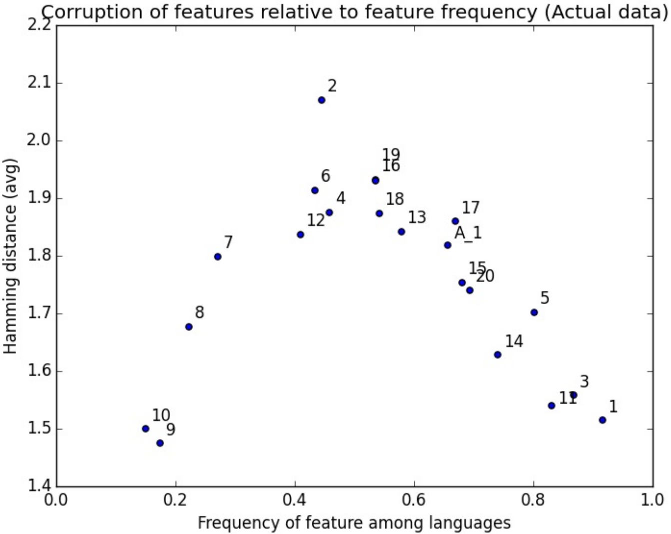
illustration from: Ján Kvak, *Creating and Recognizing Visual Words Using Sparse Distributed Memory*

proposed as a realistic computational model of how information is stored and retrieved in human memory

## Procedure

- Kanerva Network with Boolean space $\mathbb{F}_2^{21}$
- 166 data points (fully mapped SSWL languages)
- Kanerva network with access sphere of $n/4$, with $n$ median Hamming distance between points
- optimal: larger $n$ excessive number of hard locations being in the sphere, computationally intractable
- correct data written to the Kanerva network
- known language bit-string, with a single corrupted bit, used as read location
- result of the read compared to original bit-string to test bit recovery
- average Hamming distance resulting from corruption of a given bit (a particular syntactic parameter) computed across all languages

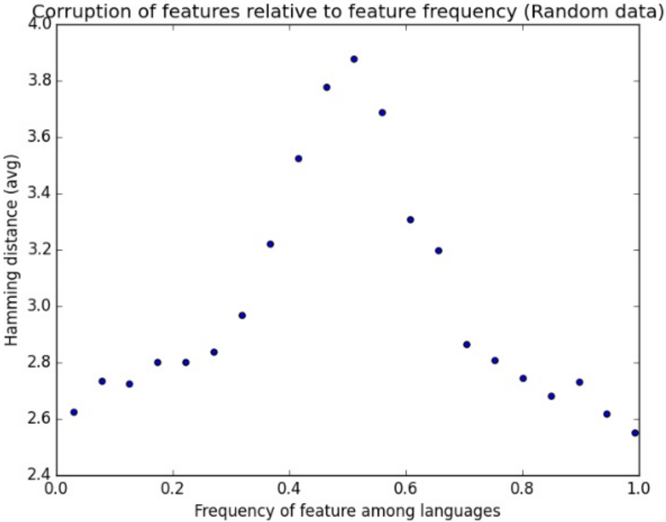
# Recoverability in Kanerva Networks



need to identify effects due to syntax from overall frequency effect

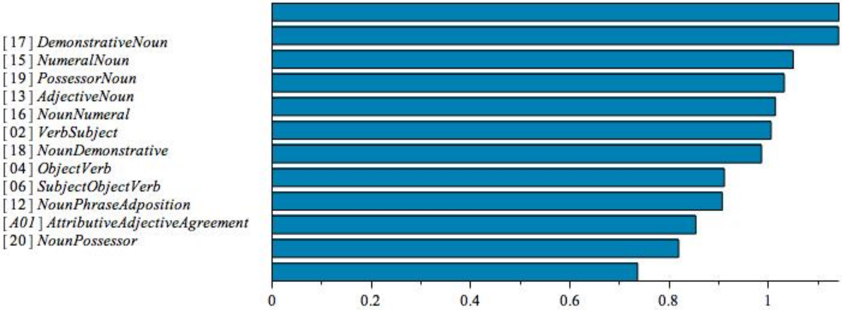
## Normalize for frequency effect

- the recoverability data obtained combine two effects
  - an overall effect depending on the frequency of expression
  - a finer effect due to actual syntactic relations
- Procedure to separate overall frequency effect:
  - for each syntactic parameter subset of languages of fixed size chosen randomly with property that half of the languages have that parameter expressed
  - ignore those parameters with too few languages for which this can be done
  - use a fixed size of 95 languages
  - data of these languages written to Kanerva network and recoverability of corrupted individual parameters tested again
  - test run again with random data generated with an approximately similar distribution of bits



Overall effect related to relative prevalence of a parameter

More refined effect after normalizing for prelavence (extracting effect of syntactic dependencies)



## Additional Remarks

- Overall effect relating recoverability in a Kanerva Network to prevalence of a certain parameter among languages (depends only on frequencies: see in random data with assigned frequencies)

- Additional effects (that deviate from random case) which detect possible dependencies among syntactic parameters: increased recoverability beyond what effect based on frequency

- Possible neuroscience implications? Kanerva Networks as models of human memory (parameter prevalence linked to neuroscience models)

- More refined effects if divided by language families?

Coding Theory to study how syntactic structures differ across the landscape of human languages

• Kevin Shu, Matilde Marcolli, *Syntactic Structures and Code Parameters*, arXiv:1610.00311

• Matilde Marcolli, *Syntactic Parameters and a Coding Theory Perspective on Entropy and Complexity of Language Families*, Entropy 2016, 18(4), 110

- select a group of languages $\mathcal{L} = \{\ell_1, \ldots, \ell_N\}$
- with the binary strings of $n$ syntactic parameters form a code $\mathcal{C}(\mathcal{L}) \subset \mathbb{F}_2^n$
- compute code parameters $(R(\mathcal{C}), \delta(\mathcal{C}))$ code rate and relative minimum distance
- analyze position of $(R, \delta)$ in space of code parameters
- get information about "syntactic complexity" of $\mathcal{L}$

• Note: some overlap with my talk "Codes and Complexity" for the Centre for Complex Systems Studies

## Codes and code parameters

error correcting codes $\mathcal{C} \subset \mathbb{F}_2^n$

- transmission rate (encoding)

$$R(\mathcal{C}) = \frac{k}{n}, \quad k = \log_2(\#\mathcal{C}) = \log_2(N)$$

for $q$-ary codes in $\mathbb{F}_q^n$ take $k = \log_q(N)$

- relative minimum distance (decoding)

$$\delta(\mathcal{C}) = \frac{d}{n}, \quad d = \min_{\ell_1 \neq \ell_2} d_H(\ell_1, \ell_2)$$

Hamming distance of binary strings of $\ell_1$ and $\ell_2$

- error correcting codes: optimize for maximal $R$ and $\delta$ but constraints that make them inversely correlated
- bounds in the space of code parameters $(R, \delta)$

## Bounds on code parameters

- singleton bound: $R + \delta \leq 1$

- Gilbert-Varshamov curve (q-ary codes)

$$R = 1 - H_q(\delta), \quad H_q(\delta) = \delta \log_q(q-1) - \delta \log_q \delta - (1-\delta) \log_q(1-\delta)$$

q-ary Shannon entropy: asymptotic behavior of volumes of Hamming balls for large $n$

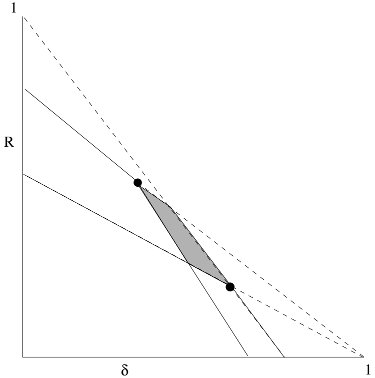
- The Gilbert-Varshamov curve represents the typical behavior of large random codes (Shannon Random Code Ensemble)

- Note: if syntactic parameters really were identically distributed independent random variables, subject to an evolution via a Markov model on a tree (simple assumption of phylogenetic models) then would expect codes from sets of languages to behave like Shannon random codes

- distance from SRCE behavior measures presence of relations that affect distribution of syntactic parameters across languages

## The asymptotic bound

- Yu.I. Manin, *What is the maximum number of points on a curve over $\mathbb{F}_2$?* J. Fac. Sci. Tokyo, IA, Vol. 28 (1981), 715–720.
- existence proved by spoiling operations on codes



$R = \alpha_q(\delta)$ continuous decreasing function with $\alpha_q(0) = 1$ and $\alpha_q(\delta) = 0$ for $\delta \in [\frac{q-1}{q}, 1]$

## Properties of the asymptotic bound

• separates space $[0,1]^2$ of code parameters into region below asymptotic bound $R = \alpha_q(\delta)$ where code points dense and with infinite multiplicity from region above where code points isolated and with finite multiplicity

• the function $R = \alpha_q(\delta)$ may be non-computable, but only as bad as Kolmogorov complexity (becomes computable given an oracle that orders codes by their Kolmogorov complexity)

- Yu.I. Manin, M. Marcolli, *Error-correcting codes and phase transitions*, Mathematics in Computer Science, Vol.5 (2011) 133–170
- Yu.I. Manin, M. Marcolli, *Kolmogorov complexity and the asymptotic bound for error-correcting codes*, Journal of Differential Geometry, Vol.97 (2014) 91–108

## Estimates on the asymptotic bound

- Plotkin bound:

$$\alpha_q(\delta) = 0, \quad \delta \geq \frac{q-1}{q}$$

- singleton bound:

$$\alpha_q(\delta) \leq 1 - \delta$$

- Hamming bound:

$$\alpha_q(\delta) \leq 1 - H_q(\frac{\delta}{2})$$

- Gilbert–Varshamov bound:

$$\alpha_q(\delta) \geq 1 - H_q(\delta)$$

- difficult to construct codes above the asymptotic bound: examples from algebro-geometric codes from curves (but only for $q \geq 49$ otherwise entirely below the GV curve)

## Application to Linguistics: Syntactic Parameters and Coding

- M. Marcolli, *Principles and Parameters: a coding theory perspective*, arXiv:1407.7169

• idea: assign a (binary or ternary) code to a family of languages and use position of code parameters with respect to the asymptotic bound to test relatedness and to test difference in behavior of syntactic parameters from independent random variables

• $N =$ number of syntactic parameters $\Pi = (\Pi_\ell)_{\ell=1}^N$
each $\Pi_\ell$ with values in $\mathbb{F}_2 = \{0, 1\}$
(or $\mathbb{F}_3 = \{-1, 0, +1\}$ if include parameters that are not set in certain languages)

• $\mathcal{F} = \{L_k\}_{k=1}^m$ a set of natural languages (language "family")

• Code $C = C(\mathcal{F})$ in $\mathbb{F}^N$ ($\mathbb{F}_2^N$ or $\mathbb{F}_3^N$) with $m$ code words
$w_k = \Pi(L_k)$ string of syntactic parameters for the language $L_k$

## Interpretation of Code Parameters

• $R = R(C)$ measures ratio between logarithmic size of number of languages in $\mathcal{F}$ and total number of parameters: how $\mathcal{F}$ distributed in the ambient $\mathbb{F}^N$

• $\delta = \delta(C)$ is the minimum, over all pairs of languages $L_i, L_j$ in $\mathcal{F}$ of the relative Hamming distance

$$\delta(C(\mathcal{F})) = \min_{L_i \neq L_j \in \mathcal{F}} \delta_H(L_i, L_j)$$

$$\delta_H(L_i, L_j) = \frac{1}{N} \sum_{\ell=1}^{N} |\Pi_\ell(L_i) - \Pi_\ell(L_j)|$$

• code parameter $\delta$ used in Longobardi's Parameter Comparison Method for reconstruction of phylogenetic trees

## Spoiling operations on binary codes

1. $C_1 = C \star_i a$ associates to a word $c = (a_1, \dots, a_n)$ of $C$ the word $c \star_i a = (a_1, \dots, a_{i-1}, a, a_i, \dots, a_n)$
2. $C_2 = C\star_i$, which is a projection of the code $C$ in the $i$-th direction
3. $C_3 = C(a, i)$ code words with same $i$-th digit equal to $a$

## Interpretation of Spoiling Operations

- first spoiling operation: effect of including one syntactic parameter in the list which is dependent on the other parameters

- second spoiling operation: forgetting one of the syntactic parameters

- third spoiling operation: forming subfamilies by considering languages that have a common value of one of the parameters

Simple Example:

- group of three languages $\mathcal{F} = \{\ell_1, \ell_2, \ell_3\}$: Italian, Spanish, French using first group of 6 parameters
- code $C = C(\mathcal{F})$

| | | | | | | |
|---|---|---|---|---|---|---|
| $\ell_1$ | 1 | 1 | 1 | 0 | 1 | 1 |
| $\ell_2$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $\ell_3$ | 1 | 1 | 1 | 0 | 1 | 0 |

- code parameters: $(R = \log_2(3)/6 = 0.2642, \delta = 1/6)$
- code parameters satisfy $R < 1 - H_2(\delta)$: below the Gilbert–Varshamov curve

Spoiling operations in this example:

- first spoiling operation:

first two parameters same value 1, so
$C = C' \star_1 f_1 = (C'' \star_2 f_2) \star_1 f_1$ with $f_1$ and $f_2$ constant equal to 1 and $C'' \subset \mathbb{F}_2^4$ without first two letters

- second spoiling operation:

conversely, $C'' = C' \star_2$ and $C' = C \star_1$

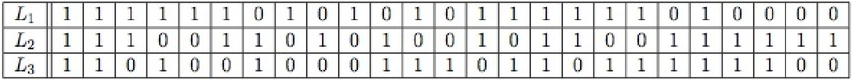
- third spoiling operation:

$C(0,4) = \{\ell_1, \ell_3\}$ and $C(1,6) = \{\ell_2, \ell_3\}$

What if languages are not in the same historical family?

Example: $\mathcal{F} = \{L_1, L_2, L_3\}$: Arabic, Wolof, Basque

- excluding parameters that are not set, or are entailed by other parameters, for these languages: left with 25 parameters from original list (number 1–5, 7, 10, 20–21, 25, 27–29, 31–32, 34, 37, 42, 50–53, 55–57)
- code $C = C(\mathcal{F})$



| | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $L_1$ | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| $L_2$ | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 |
| $L_3$ | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 0 |

- code parameters: $\delta = 0.52$ and $R > 0$ violates Plotkin bound

$\Rightarrow$ isolated code above the asymptotic bound

## Asymptotic bound and language relatedness

- For binary syntactic parameters: a code $C = C(\mathcal{F})$ violates the Plotkin bound if any pair $L_i \neq L_j$ of languages in $\mathcal{F}$ has $\delta_H(L_i, L_j) \geq 1/2$

- $L_i$ and $L_j$ differ in at least half of the parameters: it would not happen in a group of historically related languages

- but what about codes above the asymptotic bound that do not violate the Plotkin bound?

- Expect: $C = C(\mathcal{F})$ above the asymptotic bound
$\Rightarrow \mathcal{F}$ not a historical language family
(quantitative test of historical relatedness)

## Why the asymptotic bound?

- Why look at position with respect to asymptotic bound as a test of historical relatedness? because it is the only true "bound" in the space of code parameters across which behavior truly changes

- codes below the asymptotic bound are *easily deformable* (as long as number of syntactic parameters is large)

- if think of language evolution as a process of parameter change, expect languages that have evolved in the same family to determine codes in this zone of the space of code parameters

- codes $C = C(\mathcal{F})$ above the asymptotic bound should be a clear sign that list of languages in $\mathcal{F}$ do *not* belong to same historical family

- though there can be codes $C = C(\mathcal{F})$ below the asymptotic bound that also don't come from historically related languages: converse implication does not hold

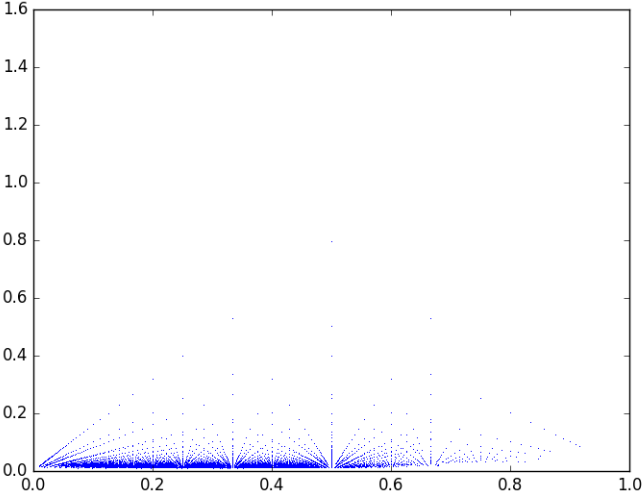
## Code parameters of language sets

- Kevin Shu and Matilde Marcolli, *Syntactic structures and code parameters*, Mathematics in Computer Science 11 (2017) no. 1, 79–90.

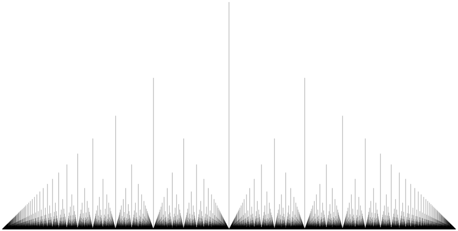
• take all sets of two and three languages in the SSWL database and set of parameters completely mapped for languages in the set

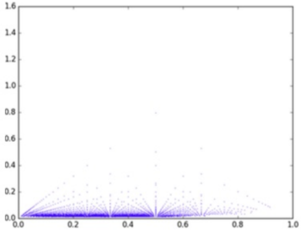
• for each pair/triple compute the code parameters of the resulting code and plot where they lie in the space of code parameters

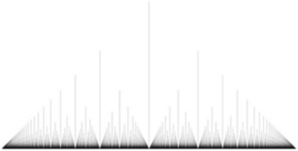
- distribution of code parameters for small sets of languages (pairs or triples) and SSWL data



- in lower region of code parameter space a superposition of two Thomae functions ($f(x) = 1/q$ for $x = p/q$ coprime, zero on irrationals)



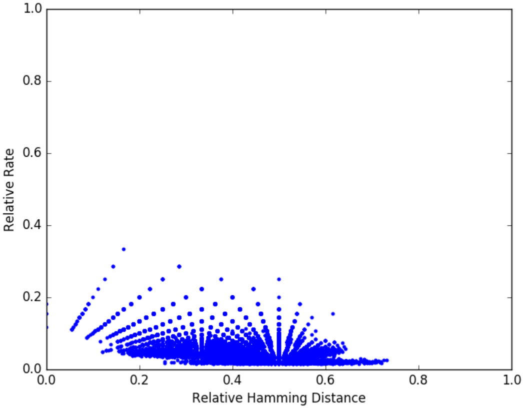




and behaves like the case of random codes with fixed $k = \log_2(N)$

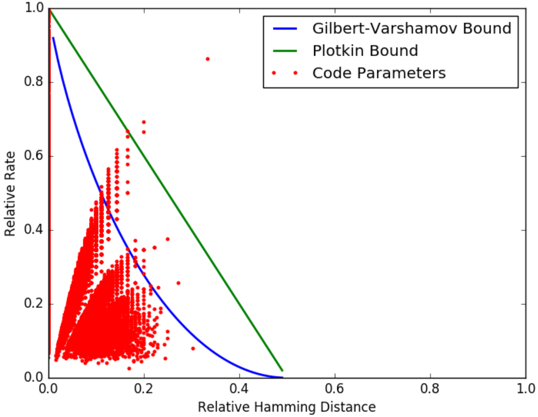
$$(\delta = \frac{d}{n}, R = k \cdot \frac{1}{n})$$

- randomly chosen sets of two or three languages tend to populate the lower region of the Thomae function graph



uniformly random sets of three languages

- more interesting what happens in the upper regions of the code parameter space
- take larger sets of randomly selected languages and syntactic parameters in the SSWL database



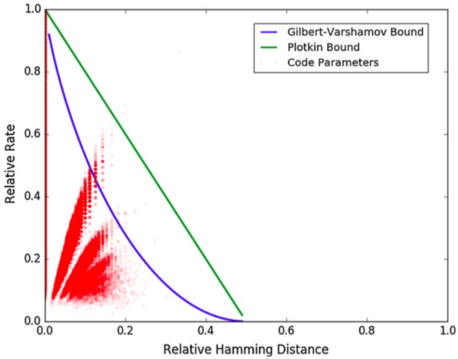
codes better than algebro-geometric above GV, asymptotic, and Plotkin

## Remarks

- construction of binary codes above asymptotic bound through linguistics

- what are the best codes obtained this way? explicit examples with languages that are phylogenetically very distant

- these points are rare compared to typical: find explicitly which languages are involved



What kind of dynamics of language change can lead to this type of distribution of syntactic features (and to codes so high above the GV line)?

• build a simple model of language change based on interaction between languages (bilingualism, code switching)

• tendency of parameters to align if high interaction (spin glass model)

• if *no relations*: independent set of uncoupled Ising models on a graph, for each syntactic parameter... convergence to most prevalent configuration in initial condition

• crucial role of relations between parameters in giving interesting dynamics and interesting equilibrium configurations (not all parameters aligned)

# Spin Glass Models of Syntax

• Karthik Siva, Jim Tao, Matilde Marcolli, *Syntactic Parameters and Spin Glass Models of Language Change*, Linguistic Analysis, Vol. 41 (2017) N. 3-4, 559–608.

• historical examples: Sanskrit flipped some syntactic parameters by influence of Dravidian languages...

• physicist viewpoint: binary variables (up/down spins) that flip by effect of interactions: Spin Glass Model

– focus on linguistic change caused by language interactions

– think of syntactic parameters as spin variables

– spin interaction tends to align (ferromagnet)

– strength of interaction proportional to bilingualism (MediaLab)

– role of temperature parameter: probabilistic interpretation of parameters & amount of code-switching in bilingual populations

– not all parameters are independent: entailment relations

– Metropolis–Hastings algorithm: simulate evolution

The Ising Model of spin systems on a graph $G$

- graph: vertices = languages, edges = language interaction (strength proportional to bilingual population); over each vertex a set of spin variables (syntactic parameters)

- configurations of spins $s : V(G) \to \{\pm 1\}$

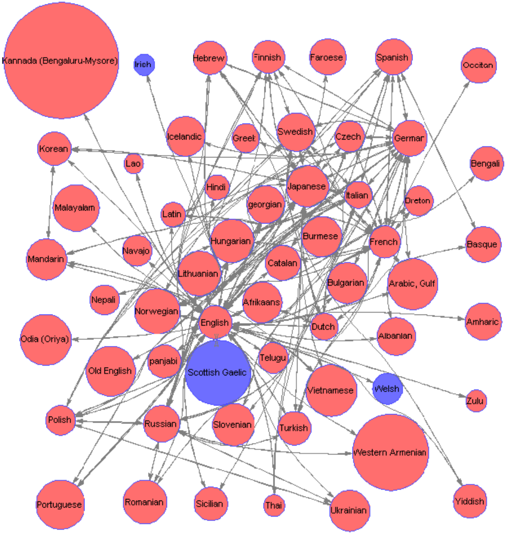
- if only one syntactic parameter, would have an Ising model on the graph $G$: configurations $s : V(G) \to \{\pm 1\}$ set the parameter at all the locations on the graph

- variable interaction energies along edges (some pairs of languages interact more than others)

$$H(s) = -\sum_{e \in E(G) : \partial(e) = \{v, v'\}} \sum_{i=1}^{N} J_e \, s_{v,i} \, s_{v',i}$$

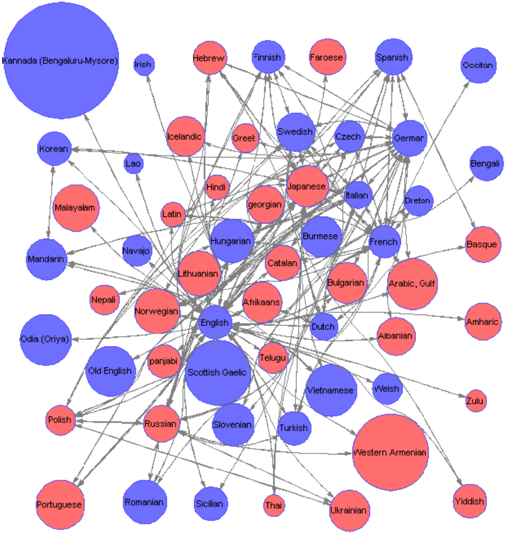
- if all $N$ parameters are independent, then it would be like having $N$ non-interacting copies of same Ising model on the same graph $G$

# Example: Single parameter dynamics *Subject-Verb* parameter



Initial configuration: most languages in SSWL have +1 for *Subject-Verb*; use interaction energies from MediaLab data

Equilibrium: low temperature all aligned to $+1$; high temperature:



Temperature: fluctuations in bilingual users between different structures (“code-switching” in Linguistics)

## Entailment relations among parameters: toy model example

- Example: $\{p_1, p_2\} = \{\text{Strong Deixis}, \text{Strong Anaphoricity}\}$

| | $p_1$ | $p_2$ |
|---|---|---|
| $\ell_1$ | $+1$ | $+1$ |
| $\ell_2$ | $-1$ | $0$ |
| $\ell_3$ | $+1$ | $+1$ |
| $\ell_4$ | $+1$ | $-1$ |

$\{\ell_1, \ell_2, \ell_3, \ell_4\} = \{\text{English}, \text{Welsh}, \text{Russian}, \text{Bulgarian}\}$

## Modeling Entailment

• key idea: change the Hamiltonian (which determines the dynamics) by adding terms that make the configurations $(p_1 = 1, p_2 = \pm 1)$ and $(p_1 = -1, p_2 = 0)$ energetically favored over others

• variables: $S_{\ell,p_1} = \exp(\pi i X_{\ell,p_1}) \in \{\pm 1\}$, $S_{\ell,p_2} \in \{\pm 1, 0\}$ and $Y_{\ell,p_2} = |S_{\ell,p_2}| \in \{0,1\}$ and Hamiltonian $H = H_E + H_V$

$$H_E = H_{p_1} + H_{p_2} = - \sum_{\ell,\ell' \in \text{languages}} J_{\ell\ell'} \left( \delta_{S_{\ell,p_1}, S_{\ell',p_1}} + \delta_{S_{\ell,p_2}, S_{\ell',p_2}} \right)$$

$$H_V = \sum_\ell H_{V,\ell} = \sum_\ell J_\ell \, \delta_{X_{\ell,p_1}, Y_{\ell,p_2}}$$

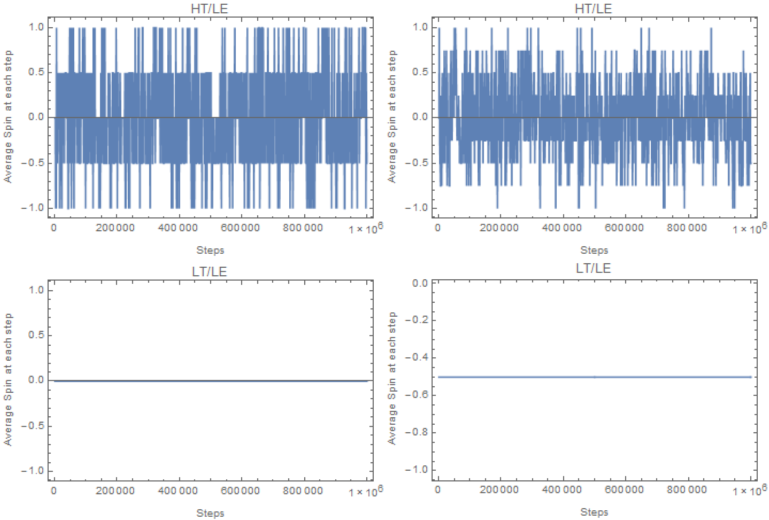
if freeze $p_1$ and evolution for $p_2$: Potts model with external magnetic field

- two parameters: *temperature* (amount of code switching) and coupling *energy of entailment* (how strongly enforced the entailment relations are)

Equilibrium configuration

| $(p_1, p_2)$ | HT/HE | HT/LE | LT/HE | LT/LE |
|---|---|---|---|---|
| $\ell_1$ | $(+1, 0)$ | $(+1, -1)$ | $(+1, +1)$ | $(+1, -1)$ |
| $\ell_2$ | $(+1, -1)$ | $(-1, -1)$ | $(+1, +1)$ | $(+1, -1)$ |
| $\ell_3$ | $(-1, 0)$ | $(-1, +1)$ | $(+1, +1)$ | $(-1, 0)$ |
| $\ell_4$ | $(+1, +1)$ | $(-1, -1)$ | $(+1, +1)$ | $(-1, 0)$ |

## Average value of spin



$p_1$ left and $p_2$ right in low entailment energy case

- when consider more realistic models (at least the 28 languages and 63 parameters of Longobardi–Guardiano with all their entailment relations) very slow convergence of the Metropolis–Hastings dynamics even for low temperature

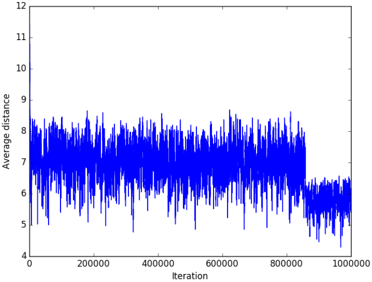
- how to get better information on the dynamics? consider set of languages as codes and an induced dynamics in the space of code parameters

## Space of Code Parameters and dynamics of syntactic parameters

• Spin Glass Model dynamics for a set of languages $\mathcal{L}$ induces dynamics on codes $\mathcal{C}(\mathcal{L})$ and on code parameters $(R, \delta)$

- no entailment (independent parameters): fixed $R$ and $\delta$ flows towards zero (spoiling code)
- entailment: dynamics can improve code making $\delta$ larger ($R$ fixed)

• for large number of parameters see dynamics more easily on code parameter than with average magnetization of spin glass model



## Heat Kernel dimensional reduction

- Andrew Ortegaray, Robert C. Berwick, Matilde Marcolli, *Heat Kernel Analysis of Syntactic Structures*, arXiv:1803.09832

- Geometric methods of dimensional reduction: *Belkin–Niyogi heat kernel method*

- M. Belkin, P. Niyogi, *Laplacian eigenmaps for dimensionality reduction and data representation*, Neural Comput. 15 (6) (2003) 1373–1396.

- Question: low dimensional representations of data sampled from a probability distribution on a manifold

- Want more efficient methods than Principal Component Analysis

- Main Idea: build a graph with neighborhood information, use Laplacian of graph, obtain low dimensional representation that maintains the local neighborhood information using eigenfunctions of the Laplacian

- setting: data points $x_1, \dots, x_k \in \mathcal{M} \subset \mathbb{R}^\ell$ on a manifold; find points $y_1, \dots, y_k$ in a low dimensional $\mathbb{R}^m$ ($m << \ell$) that *represent* the data points $x_i$

- Step 1 (a): adjacency graph ($\epsilon$-neighborhood): an edge $e_{ij}$ between $x_i$ and $x_j$ if $\|x_i - x_j\|_{\mathbb{R}^\ell} < \epsilon$

- Step 1 (b): adjacency graph ($n$ nearest neighborhood): egde $e_{ij}$ between $x_i$ and $x_j$ if $x_i$ is among the $n$ nearest neighbors of $x_j$ or viceversa

- Step 2: weights on edges: heat kernel

$$W_{ij} = \exp\left( -\frac{\|x_i - x_j\|^2}{t} \right)$$

if edge $e_{ij}$ and $W_{ij} = 0$ otherwise; heat kernel parameter $t > 0$

- Step 3: Eigenfunctions for connected graph (or on each component)

$$L\psi = \lambda D \psi$$

diagonal matrix of weights $D_{ii} = \sum_j W_{ji}$; Laplacian $L = D - W$ with $W = (W_{ij})$; eigenvalues $0 = \lambda_0 \leq \lambda_1 \leq \cdots \leq \lambda_{k-1}$ and $\psi_j$ eigenfuctions

$$\psi_i : \{1, \ldots, k\} \to \mathbb{R}$$

defined on set of vertices of graph

- Step 4: Mapping by Laplace eigenfunctions

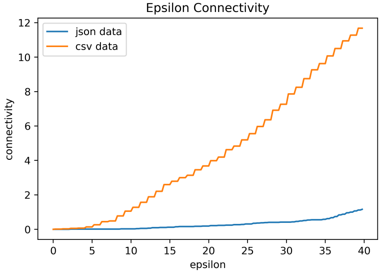
$$\mathbb{R}^\ell \supset \mathcal{M} \ni x_i \mapsto (\psi_1(i), \ldots, \psi_m(i)) \in \mathbb{R}^m$$

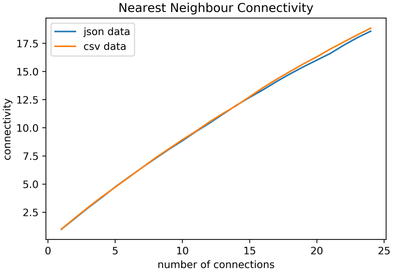
map by first $m$ eigenfunctions

- Belkin–Niyogi: *optimality* of embedding by Laplace eigenfunctions

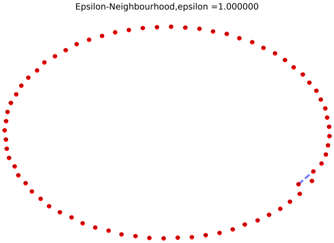
# Heat Kernel analysis of Syntactic Parameters

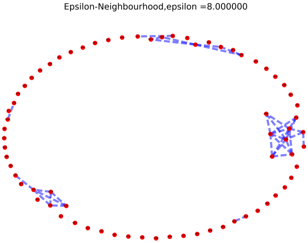
- Connectivity in $\epsilon$-neighborhood and nearest-neighbor (difference between SSWL data (json) and Longobardi data (csv)



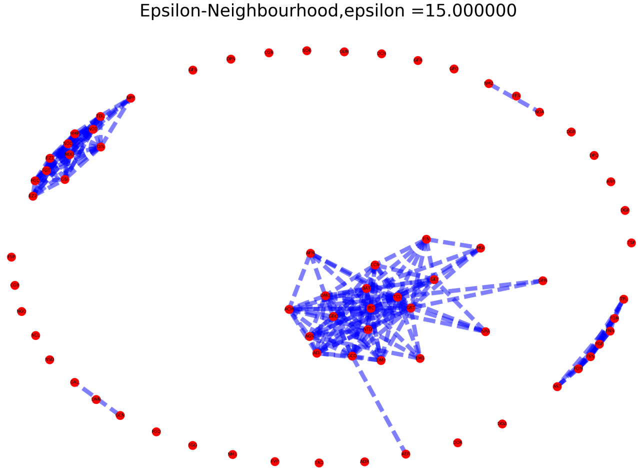


# Graphs with $\epsilon$-neighborhood Longobardi data


Epsilon-Neighbourhood,epsilon =1.000000

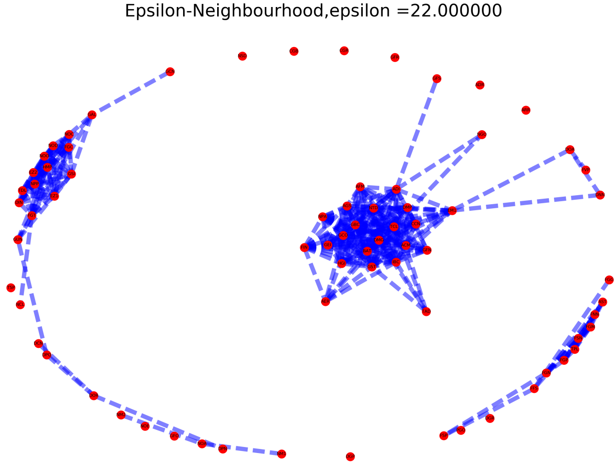

Epsilon-Neighbourhood,epsilon =8.000000

# Graphs with $\epsilon$-neighborhood Longobardi data

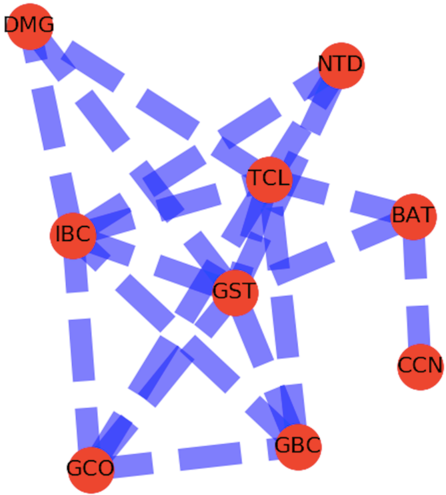


Epsilon-Neighbourhood,epsilon =15.000000

# Graphs with $\epsilon$-neighborhood Longobardi data

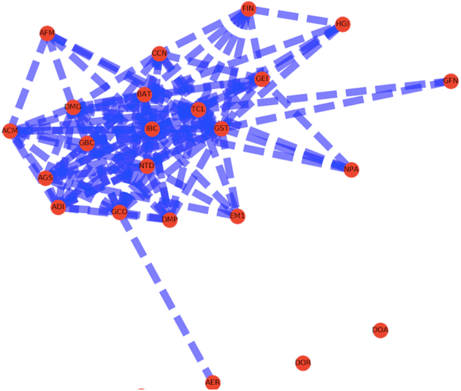

Epsilon-Neighbourhood,epsilon =22.000000

# Closer look at some of these structures



Examples of parameters in this structure: DMG (def. matching genitives), GCO (gramm. collective number), GST (grammaticalised Genitive) ...

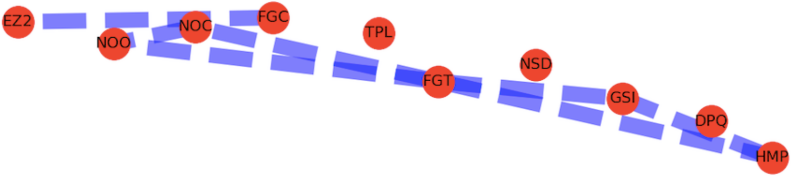
# Closer look at some of these structures



The same structure looked at a different $\epsilon$-scale has acquired further connections to other parameters

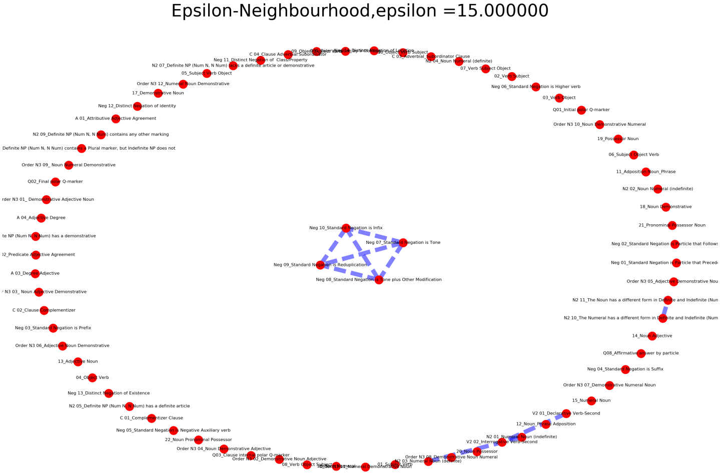
Most vertices in these structures have high *centrality* in the graph

## Closer look at some of these structures



Another structure in the Longobardi data involving parameters like EZ2 (non-clausal linker), FGC (gramm. classi- fier), FGT (gramm. temporality), GSI (grammaticalised inalienability), HMP (NP-heading modifier) ...

# Graphs with $\epsilon$-neighborhood SSWL data



Epsilon-Neighbourhood,epsilon =15.000000

C 04_Clause Adverbial Subordinator
09_Object Subject Verb
Neg 14_Distinct Negation of Location
10_Object Verb Subject
C 03_Adverbial_Subordinator Clause
N2 04_Noun Numeral (definite)
Neg 11_Distinct Negation of  Class/Property
N2 07_Definite NP (Num N, N Num) lacks a definite article or demonstrative
05_Subject Verb Object
Order N3 12_Numeral Noun Demonstrative
17_Demonstrative Noun
Neg 12_Distinct Negation of identity
A 01_Attributive Adjective Agreement
N2 09_Definite NP (Num N, N Num) contains any other marking
Definite NP (Num N, N Num) contains a Plural marker, but Indefinite NP does not
Order N3 09_ Noun Numeral Demonstrative
Q02_Final polar Q-marker
rder N3 01_ Demonstrative Adjective Noun
A 04_Adjective Degree
te NP (Num N, N Num) has a demonstrative
)2_Predicate Adjective Agreement
A 03_Degree Adjective
r N3 03_ Noun Adjective Demonstrative
C 02_Clause Complementizer
Neg 03_Standard Negation is Prefix
Order N3 06_Adjective Noun Demonstrative
13_Adjective Noun
04_Object Verb
Neg 13_Distinct Negation of Existence
N2 05_Definite NP (Num N, N Num) has a definite article
C 01_Complementizer Clause
Neg 05_Standard Negation is Negative Auxiliary verb
22_Noun Pronominal Possessor
Order N3 04_Noun Demonstrative Adjective
Q03_Clause internal polar Q-marker
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08_Verb Object Subject
Order N3 11_Numeral Demonstrative Noun
01_Subject Verb
N2 03_Numeral Noun (definite)
Order N3 08_Demonstrative Noun Numeral
20_Noun Possessor
V2 02_Interrogative Verb-Second
N2 01_Numeral Noun (indefinite)
12_Noun_Phrase Adposition
V2 01_Declarative Verb-Second
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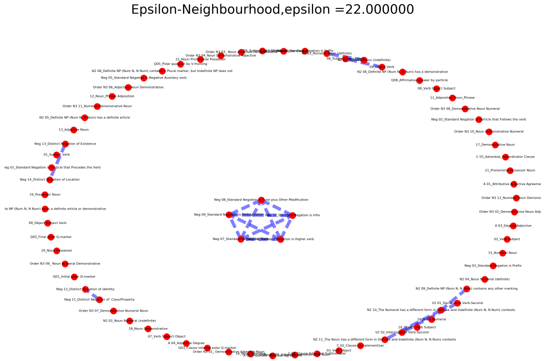
N2 10_The Numeral has a different form in Definite and Indefinite (Num
N2 11_The Noun has a different form in Definite and Indefinite (Num
Order N3 05_Adjective Demonstrative Nou
Neg 01_Standard Negation is Particle that Precede
Neg 02_Standard Negation is Particle that Follows
21_Pronominal Possessor Noun
18_Noun Demonstrative
N2 02_Noun Numeral (indefinite)
11_Adposition Noun_Phrase
06_Subject Object Verb
19_Possessor Noun
Order N3 10_Noun Demonstrative Numeral
Q01_Initial polar Q-marker
03_Verb Object
Neg 06_Standard Negation is Higher verb
02_Verb Subject
07_Verb Subject Object
Neg 10_Standard Negation is Infix
Neg 07_Standard Negation is Tone
Neg 09_Standard Negation is Reduplication
Neg 08_Standard Negation is Tone plus Other Modification

# Graphs with $\epsilon$-neighborhood SSWL data


Epsilon-Neighbourhood,epsilon =22.000000
Order N3 03_ Noun Adjective Demonstrative
Order N3 04_Noun Demonstrative Adjective
22_Noun Pronominal Possessor
Q05_Polar question by V-fronting
N2 08_Definite NP (Num N, N Num) contains a Plural marker, but Indefinite NP does not
Neg 05_Standard Negation is Negative Auxiliary verb
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Neg 13_Distinct Negation of Existence
01_Subject Verb
leg 01_Standard Negation is Particle that Precedes the Verb
Neg 14_Distinct Negation of Location
19_Possessor Noun
te NP (Num N, N Num) lacks a definite article or demonstrative
09_Object Subject Verb
Q02_Final polar Q-marker
20_Noun Possessor
Order N3 09_ Noun Numeral Demonstrative
Q01_Initial polar Q-marker
Neg 12_Distinct Negation of identity
Neg 11_Distinct Negation of Class/Property
Order N3 07_Demonstrative Numeral Noun
N2 02_Noun Numeral (indefinite)
18_Noun Demonstrative
07_Verb Subject Object
A 04_Adjective Degree
Q03_Clause internal polar Q-marker
Order N3 01_ Demonstrative Adjective Noun
03_Verb Object
C 02_Clause Complementizer
N2 11_The Noun has a different form in Definite and Indefinite (Num N, N Num) contexts
V2 02_Interrogative Verb-Second
10_Object Verb Subject
16_Noun Numeral
N2 10_The Numeral has a different form in Definite and Indefinite (Num N, N Num) contexts
V2 01_Declarative Verb-Second
N2 09_Definite NP (Num N, N Num) contains any other marking
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Neg 03_Standard Negation is Prefix
15_Numeral Noun
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A 03_Degree Adjective
Order N3 02_Demonstrative Noun Adj
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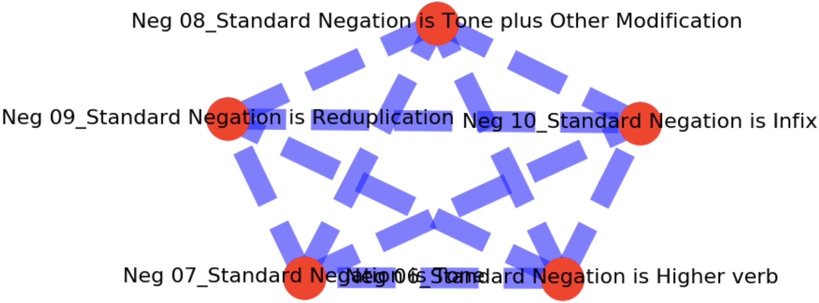
N2 06_Definite NP (Num N, N Num) has a demonstrative
04_Object Verb
N2 01_Numeral Noun (indefinite)
06_Subject Object Verb
N2 03_Numeral Noun (definite)
Negation is Suffix
Neg 08_Standard Negation is Tone plus Other Modification
Neg 09_Standard Negation is Reduplication
Neg 10_Standard Negation is Infix
Neg 07_Standard
Negation is Higher verb

The $\epsilon$-neighborhood construction is better suited to gain connectivity information in the Longobardi data: the SSWL data remain highly disconnected (only small local structures)

# Closeup of SSWL $\epsilon$-structure

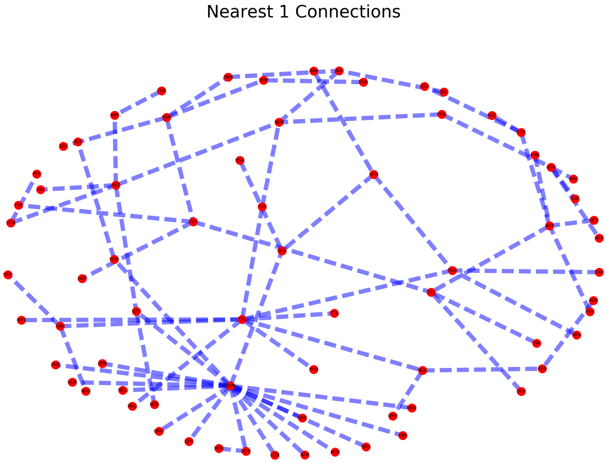

Neg 08_Standard Negation is Tone plus Other Modification
Neg 09_Standard Negation is Reduplication
Neg 10_Standard Negation is Infix
Neg 07_Standard Negation is Tone
Neg 06_Standard Negation is Higher verb

In SSWL $\epsilon$-structures only involve closely related parameters (Neg7,8,9 expressed in some Niger-Congo languages)

More interesting results using the $n$-neighborhood method

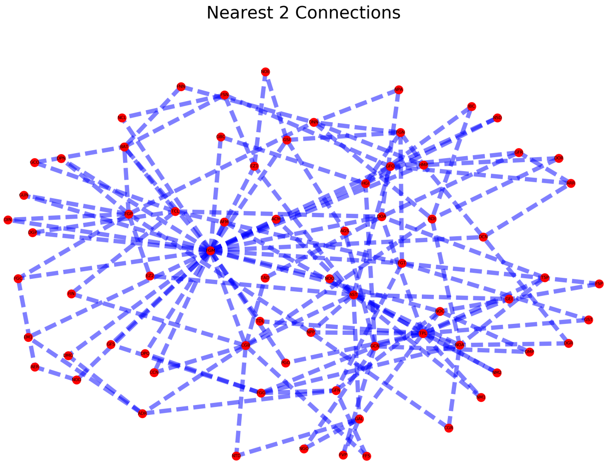
## Graphs with *n*-neighborhood Longobardi data

Fewer vertices with high centrality: FGM node (gramm. Case)



# Graphs with *n*-neighborhood Longobardi data

More high centrality vertices: AST (structured APs), CGB (unbounded sg N), FGP (gramm. person) ...

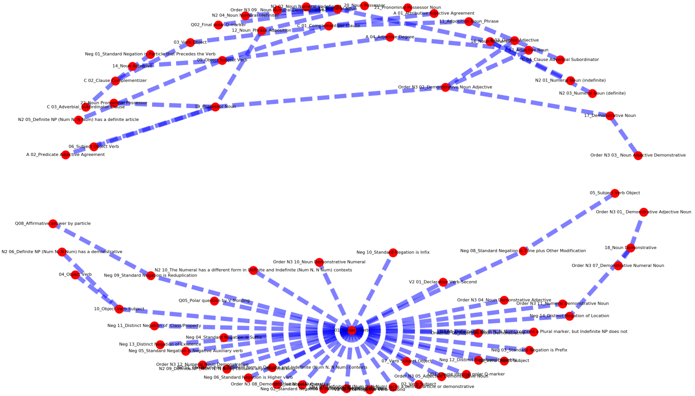


Nearest 2 Connections

# Graphs with *n*-neighborhood SSWL data

## Two main components, lower one highest centrality SubjectVerb

Nearest 1 Connections



N2 02_Noun Numeral (indefinite)
20_Noun Possessor
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N2 04_Noun Numeral (definite)
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Q02_Final polar Q-marker
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06_Subject Object Verb
A 02_Predicate Adjective Agreement
Order N3 03_ Noun Adjective Demonstrative
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Order N3 07_Demonstrative Numeral Noun
04_Object Verb
Neg 09_Standard Negation is Reduplication
V2 01_Declarative Verb-Second
Q05_Polar question by V-fronting
Order N3 04_Noun Demonstrative Adjective
Order N3 11_Numeral Demonstrative Noun
10_Object Verb Subject
Neg 14_Distinct Negation of Location
Neg 11_Distinct Negation of Class/Property
01_Subject Verb
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Neg 13_Distinct Negation of Existence
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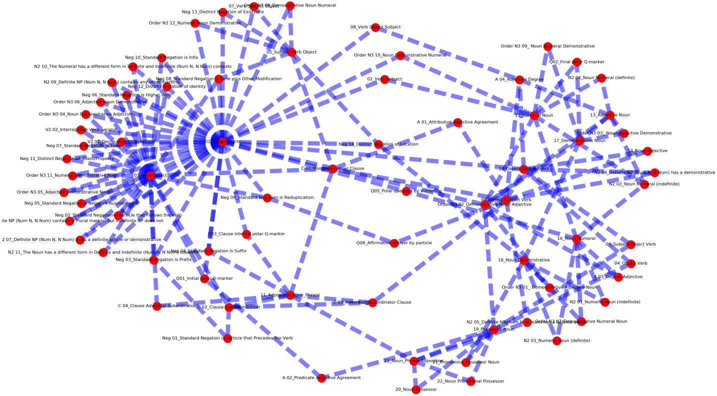
Order N3 08_Demonstrative Noun Numeral
Neg 02_Standard Negation
N2 07_Definite NP (Num N, N Num) lacks a definite article or demonstrative
V2 02_Interrogative Verb-Second

# Graphs with *n*-neighborhood SSWL data

components merge, high centrality SubjectVerb, VerbObject

Nearest 2 Connections


07_Verb Subject Object
Order N3 08_Demonstrative Noun Numeral
Neg 13_Distinct Negation of Existence
Order N3 12_Numeral Noun Demonstrative
08_Verb Object Subject
05_Subject Verb Object
Order N3 09_ Noun Numeral Demonstrative
Order N3 10_Noun Demonstrative Numeral
Q02_Final polar Q-marker
Neg 10_Standard Negation is Infix
N2 10_The Numeral has a different form in Definite and Indefinite (Num N, N Num) contexts
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Neg 11_Distinct Negation of Class/Property
C 01_Complementizer Clause
10_Object Verb Subject
N2 06_Definite NP (Num N, N Num) has a demonstrative
Order N3 11_Numeral Demonstrative Noun
03_Verb Object
N2 02_Noun Numeral (indefinite)
Q05_Polar question by V-fronting
Order N3 05_Adjective Demonstrative Noun
Neg 09_Standard Negation is Reduplication
09_Object Subject Verb
Order N3 02_Demonstrative Noun Adjective
Neg 05_Standard Negation is Negative Auxiliary verb
Neg 02_Standard Negation is Particle that Follows the verb
ite NP (Num N, N Num) contains a Plural marker, but Indefinite NP does not
Q03_Clause internal polar Q-marker
2 07_Definite NP (Num N, N Num) lacks a definite article or demonstrative
16_Noun Numeral
Q08_Affirmative answer by particle
06_Subject Object Verb
N2 11_The Noun has a different form in Definite and Indefinite (Num N, N Num) contexts
Neg 04_Standard Negation is Suffix
Neg 03_Standard Negation is Prefix
18_Noun Demonstrative
04_Object Verb
A 03_Degree Adjective
Q01_Initial polar Q-marker
Order N3 01_ Demonstrative Adjective Noun
11_Adposition Noun_Phrase
C 03_Adverbial_Subordinator Clause
N2 01_Numeral Noun (indefinite)
C 04_Clause Adverbial Subordinator
C 02_Clause Complementizer
N2 05_Definite NP (Num N, N Num) has a definite article
Order N3 07_Demonstrative Numeral Noun
19_Possessor Noun
Neg 01_Standard Negation is Particle that Precedes the Verb
N2 03_Numeral Noun (definite)
12_Noun_Phrase Adposition
21_Pronominal Possessor Noun
A 02_Predicate Adjective Agreement
22_Noun Pronominal Possessor
20_Noun Possessor

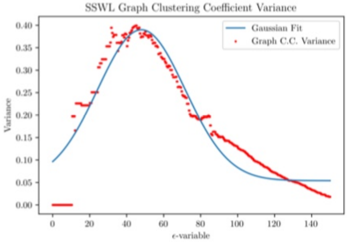
# Clustering behavior of the $\epsilon$-graphs

- neighborhood $V_i$ of $i$-th node of valence $d_i$

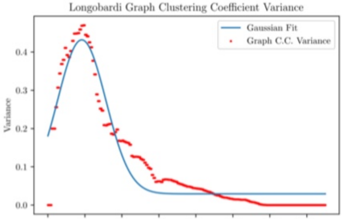
$$K_i = \#\{e \in E(G) \, : \, \partial(e) \cap V_i\}$$

measure of clustering in the region $C_i = K_i / \binom{d_i}{2}$

- variance of clustering coefficients $C_i$ as function of $\epsilon$

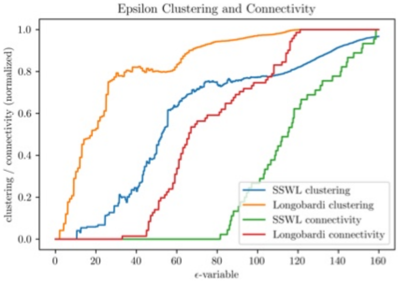


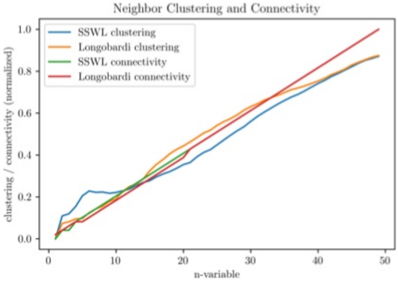
SSWL Graph Clustering Coefficient Variance



Longobardi Graph Clustering Coefficient Variance

# $\epsilon$ and $n$-neighborhood clustering and connectivity



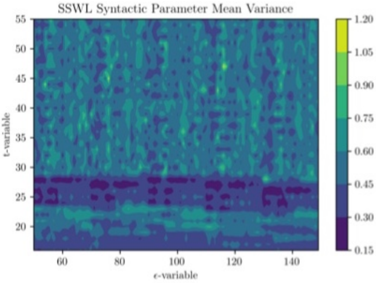


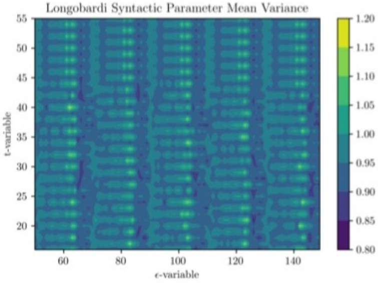
# Regions of $\epsilon$-$t$ space

- Graphs depend on $\epsilon$-neighborhood and on $t$-heat kernel variable

- explore $\epsilon$-$t$ space: determine regions where high variance in distribution of each parameter under the heat kernel mapping

- high variance in a parameter suggests it is highly independent (similar to PCA method)





## Further Questions

- an in depth linguistic analysis of the meaning of these clustering structures is still needed (ongoing work)

- comparison of the heat kernel technique with other dimensional reduction techniques (PCA etc.)

- more detailed discussion of different regions of the $\epsilon$-$t$ space in the heat kernel model (specific parameters with high independence measure)

- manifold $\mathcal{M}$ reconstruction? Belkin-Niyogi results

## Conclusions (for now)

• import a set of different mathematical techniques (phylogenetic algebraic geometry, persistent topology, coding theory, statistical mechanics, geometric models of associative memory) in order to *study natural languages as dynamical objects*

• longer term goals: create mathematical and computational models of

1. how languages are acquired?
2. how languages are stored in the brain?
3. how languages change and evolve dynamically in time?

*for human languages viewed at the level of their syntactic structures*

## Further Related Work

- Algebro-Geometric Models of Computational Semantics
  - Yuri Manin, Matilde Marcolli, *Semantic Spaces*, Mathematics in Computer Science, 10 (2016) N.4, 459–477
- Generative Grammars and Renormalization
  - Matilde Marcolli, Alexander Port, *Graph Grammars, Insertion Lie Algebras, and Quantum Field Theory*, arXiv:1502.07796, Mathematics in Computer Science 9 (2015), no. 4, 391–408
  - Colleen Delaney, Matilde Marcolli, *Dyson-Schwinger equations in the theory of computation*, arXiv:1302.5040, in "Feynman amplitudes, periods and motives", pp.79–107, Contemporary Mathematics, 648, Amer. Math. Soc., 2015
  - Matilde Marcolli, *Linguistic Merge and Dyson–Schwinger equations in Renormalization*, in preparation