Codes and Complexity

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This lecture is based on:

- Yuri I. Manin, Matilde Marcolli, *Error-correcting codes and phase transitions*, Mathematics in Computer Science (2011) 5:133–170.
- Yuri I. Manin, Matilde Marcolli, *Kolmogorov complexity and the asymptotic bound for error-correcting codes*, Journal of Differential Geometry, Vol.97 (2014) 91–108
- Yuri I. Manin, Matilde Marcolli, *Asymptotic bounds for spherical codes*, arXiv:1801.01552

Error-correcting codes

- *Alphabet*: finite set A with $\#A = q \geq 2$.
- *Code*: subset $C \subset A^n$, *length* $n = n(C) \geq 1$.
- *Code words*: elements $x = (a_1, \ldots, a_n) \in C$.
- *Code language*: $\mathcal{W}_C = \cup_{m \geq 1} \mathcal{W}_{C,m}$, words $w = x_1, \ldots, x_m$; $x_i \in C$.
- *ω-language*: Λ_C, infinite words $w = x_1, \ldots, x_m, \ldots$; $x_i \in C$.
- Special case: $A = \mathbb{F}_q$, *linear codes*: $C \subset \mathbb{F}_q^n$ linear subspace
- in general: *unstructured codes*

Code parameters

- $k = k(C) := \log_q \#C$ and $[k] = [k(C)]$ integer part of $k(C)$

$$q^{[k]} \leq \#C = q^k < q^{[k]+1}$$

- *Hamming distance*: $x = (a_i)$ and $y = (b_i)$ in C

$$d((a_i),(b_i)) := \#\{i \in (1,\dots,n) \,|\, a_i \neq b_i\}$$

- *Minimal distance* $d = d(C)$ of the code

$$d(C) := \min\{d(a,b) \,|\, a, b \in C, a \neq b\}$$

Code parameters

- $R = k/n =$ *transmission rate* of the code
- $\delta = d/n =$ *relative minimum distance* of the code

Small R: fewer code words, easier decoding, but longer encoding signal; small δ: too many code words close to received one, more difficult decoding. Optimization problem: increase R and δ... how good are codes?

- M.A. Tsfasman, S.G. Vladut, *Algebraic-geometric codes*, Mathematics and its Applications (Soviet Series), Vol. 58, Kluwer Academic Publishers, 1991.

The space of code parameters:

- $Codes_q$ = set of all codes C on an alphabet $\#A = q$
- function $cp : Codes_q \to [0,1]^2 \cap \mathbb{Q}^2$ to code parameters $cp : C \mapsto (R(C), \delta(C))$
- the function $C \mapsto (R(C), \delta(C))$ is a *total recursive map* (Turing computable)
- *Multiplicity* of a code point (R, δ) is $\#cp^{-1}(R, \delta)$

Bounds in the space of code parameters

- singleton bound: $R + \delta \leq 1$
- Gilbert–Varshamov line: $R = \frac{1}{2}(1 - H_q(\delta))$

$$H_q(\delta) = \delta \log_q(q-1) - \delta \log_q \delta - (1-\delta) \log_q(1-\delta)$$

q-ary entropy (for linear codes GV line $R = 1 - H_q(\delta)$)

Statistics of codes and the Gilbert–Varshamov bound

Known *statistical* approach to the GV bound: *random codes*

Shannon Random Code Ensemble: ω-language with alphabet A; uniform Bernoulli measure on Λ_A; choose code words of C as independent random variables in this measure

Volume estimate:

$$q^{(H_q(\delta)-o(1))n} \leq Vol_q(n, d = n\delta) = \sum_{j=0}^{d} \binom{n}{j} (q-1)^j \leq q^{H_q(\delta)n}$$

Gives probability of parameter δ for SRCE meets the GV bound with probability exponentially (in n) near 1: expectation

$$\mathbb{E} \sim \binom{q^k}{2} Vol_q(n, d) q^{-n} \sim q^{n(H_q(\delta)-1+2R)+o(n)}$$

Spoiling operations on codes: C an $[n,k,d]_q$ code

- $C_1 := C *_i f \subset A^{n+1}$

$$(a_1,\ldots,a_{n+1}) \in C_1 \text{ iff } (a_1,\ldots,a_{i-1},a_{i+1},\ldots,a_{n+1}) \in C\,,$$

and $a_i = f(a_1,\ldots,a_{i-1},a_{i+1}\ldots,a_{n+1})$
C_1 an $[n+1,k,d]_q$ code (f constant function)

- $C_2 := C*_i \subset A^{n-1}$

$$(a_1,\ldots,a_{n-1}) \in C_2 \text{ iff } \exists b \in A,\ (a_1,\ldots,a_{i-1},b,a_{i+1},\ldots,a_{n-1}) \in C.$$

C_2 an $[n-1,k,d]_q$ code

- $C_3 := C(a,i) \subset C \subset A^n$

$$(a_1,\ldots,a_n) \in C_3 \text{ iff } a_i = a.$$

C_3 an $[n-1, k-1 \le k' < k, d' \ge d]_q$ code

Asymptotic bound

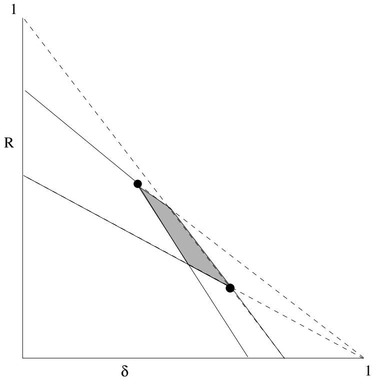
- Yu.I.Manin, *What is the maximum number of points on a curve over* $\mathbb{F}_2$*?* J. Fac. Sci. Tokyo, IA, Vol. 28 (1981), 715–720.

- $V_q \subset [0,1]^2$: all code points $(R,\delta) = cp(C)$, $C \in Codes_q$
- U_q: set of limit points of V_q
- Asymptotic bound: U_q all points below graph of a function

$$U_q = \{(R,\delta) \in [0,1]^2 \,|\, R \leq \alpha_q(\delta)\}$$

- Isolated code points: $V_q \smallsetminus (V_q \cap U_q)$

Method: controlling quadrangles



$R = \alpha_q(\delta)$ continuous decreasing function with $\alpha_q(0) = 1$ and $\alpha_q(\delta) = 0$ for $\delta \in [\frac{q-1}{q}, 1]$; has inverse function on $[0, (q-1)/q]$; U_q union of all lower cones of points in $\Gamma_q = \{R = \alpha_q(\delta)\}$

Characterization of the asymptotic bound

• Code points and multiplicities

• Set of code points of infinite multiplicity
$U_q \cap V_q = \{(R,\delta) \in [0,1]^2 \cap \mathbb{Q}^2 \mid R \leq \alpha_q(\delta)\}$ below the asymptotic bound

• Code points of finite multiplicity all above the asymptotic bound $V_q \smallsetminus (U_q \cap V_q)$ and isolated (open neighborhood containing (R,δ) as unique code point)

Questions:

• Is there a characterization of the isolated good codes on or above the asymptotic bound?

Estimates on the asymptotic bound

- Plotkin bound:

$$\alpha_q(\delta) = 0, \quad \delta \geq \frac{q-1}{q}$$

- singleton bound:

$$\alpha_q(\delta) \leq 1 - \delta$$

- Hamming bound:

$$\alpha_q(\delta) \leq 1 - H_q(\frac{\delta}{2})$$

- Gilbert–Varshamov bound:

$$\alpha_q(\delta) \geq 1 - H_q(\delta)$$

Computability question

• Note: only the asymptotic bound marks a significant change of behavior of codes across the curve (isolated and finite multiplicity/accumulation points and infinite multiplicity)

• in this sense it is very different from all the other bounds in the space of code parameters

• .... but no explicit expression for the curve $R = \alpha_q(\delta)$

• ... is the function $R = \alpha_q(\delta)$ computable?

• ... a priori no good statistical description of the asymptotic bound: is there something replacing Shannon entropy characterizing Gilbert–Varshamov curve?

- Yu.I. Manin, *A computability challenge: asymptotic bounds and isolated error-correcting codes*, arXiv:1107.4246

The asymptotic bound and Kolmogorov complexity

• while random codes are related to Shannon entropy (through the GV-bound) good codes and the asymptotic bound are related to Kolmogorov complexity

• the asymptotoc bound $R = \alpha_q(\delta)$ becomes computable given an oracle that can list codes by increasing Kolmogorov complexity

• given such an oracle: iterative (algorithmic) procedure for constructing the asymptotic bound

• ... it is at worst as "non-computable" as Kolmogorov complexity

• asymptotic bound can be realized as phase transition curve of a statistical mechanical system based on Kolmogorov complexity

- Yu.I. Manin, M. Marcolli, *Kolmogorov complexity and the asymptotic bound for error-correcting codes*, Journal of Differential Geometry, Vol.97 (2014) 91–108

Complexity

• How does one measure complexity of a physical system?

• Kolmogorov complexity: measures length of a minimal algorithmic description

... but ... gives very high complexity to completely random things

• Shannon entropy: measures average number of bits, for objects drawn from a statistical ensemble

• There are other proposals for complexity, but more difficult for formulate

• Gell-Mann complexity: complexity is high in an intermediate region between total order and complete randomness

Kolmogorov complexity

- Let $T_{\mathcal{U}}$ be a universal Turing machine (a Turing machine that can simulate any other arbitrary Turing machine: reads on tape both the input and the description of the Turing machine it should simulate)
- Given a string w in an alphabet $\mathfrak{A}$, the Kolmogorov complexity

$$\mathcal{K}_{T_{\mathcal{U}}}(w) = \min_{P:\, T_{\mathcal{U}}(P)=w} \ell(P),$$

minimal length of a program that outputs w

- universality: given any other Turing machine T

$$\mathcal{K}_T(w) = \mathcal{K}_{T_{\mathcal{U}}}(w) + c_T$$

shift by a bounded constant, independent of w; c_T is the Kolmogorov complexity of the program needed to describe T for $T_{\mathcal{U}}$ to simulate it

• any program that produces a description of w is an upper bound on Kolmogorov complexity $\mathcal{K}_{T_{\mathcal{U}}}(w)$

• think of Kolmogorov complexity in terms of data compression

• shortest description of w is also its most compressed form

• can obtain upper bounds on Kolmogorov complexity using data compression algorithms

• finding upper bounds is easy... but NOT lower bounds

Main problem

Kolmogorov complexity is NOT a computable function

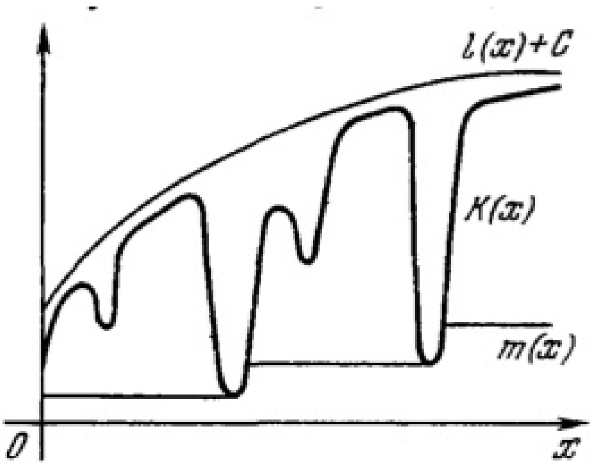
• suppose list programs P_k (increasing lengths) and run through $T_{\mathcal{U}}$: if machine halts on P_k with output w then $\ell(P_k)$ is an upper bound on $\mathcal{K}_{T_{\mathcal{U}}}(w)$

• but... there can be an earlier P_j in the list such that $T_{\mathcal{U}}$ has not yet halted on P_j

• if eventually halts and outputs w then $\ell(P_j)$ is a better approximation to $\mathcal{K}_{T_{\mathcal{U}}}(w)$

• would be able to compute $\mathcal{K}_{T_{\mathcal{U}}}(w)$ if can tell exactly on which programs P_k the machine $T_{\mathcal{U}}$ halts

• but... halting problem is unsolvable



with $m(x) = \min_{y \geq x} \mathcal{K}(y)$

Kolmogorov complexity

X = *infinite constructive world*: have structural numbering computable bijections $\nu : \mathbb{Z}^+ \to X$ principal homogeneous space over group of total recursive permutations $\mathbb{Z}^+ \to \mathbb{Z}^+$

- *Ordering*: $x \in X$ is generated at the $\nu^{-1}(x)$-th step

Optimal partial recursive enumeration $u : \mathbb{Z}^+ \to X$ (Kolmogorov and Schnorr)

$$K_u(x) := \min\{k \in \mathbb{Z}^+ \mid u(k) = x\}$$

Kolmogorov complexity

- changing $u : \mathbb{Z}^+ \to X$ changes $K_u(x)$ up to bounded (multiplicative) constants $c_1 K_v(x) \leq K_u(x) \leq c_2 K_v(x)$
- min length of program generating x (by Turing machine)

Main Idea:

• use characterization of asymptotic bound as separating code points with finite multiplicity from code points with infinite multiplicity

• given the function from codes to code parameter, want an algorithmic procedure that inductively constructs preimage sets with finite/infinite multiplicity

• choose an ordering of code points: at step m list code points in order up to some growing size N_m

• initialize A_1: a set of *a preimage* for each code point up to N_1; initialize $B_1 = \emptyset$

• want to increase at each step A_m and B_m so that the first set only contains code points with multiplicity m

• going from step m to step $m+1$: new code points listed between N_m and N_{m+1} are added to A_m, and then points (previously in A_m or added) that do not have an $m+1$-st preimage are moved to B_{m+1}

• as $m \to \infty$ the sets A_m converge to set of code points of infinite multiplicity and the B_m converge to set of code points of finite multiplicity

• key problem: need to search for the $m+1$-st preimage to detect if a code point stays in A_{m+1} or is moved to B_{m+1}

• ordinarily this would involve an *infinite search*...

• ordering and complexity: use a relation between ordering and complexity that shows that only need to search among bounded complexity codes, so a *complexity oracle* will render the search finite

X, Y infinite constructive worlds, ν_X, ν_Y structural bijections, u, v optimal enumerations, K_u and K_v Kolmogorov complexities

- total recursive function $f : X \to Y \Rightarrow \forall y \in f(X)$, $\exists x \in X$, $y = f(x)$: $\exists$ computable $c = c(f, u, v, \nu_X, \nu_Y) > 0$

$$K_u(x) \leq c \cdot \nu_Y^{-1}(y)$$

Kolmogorov ordering
$\mathbf{K}_u(x) =$ order X by growing Kolmogorov complexity $K_u(x)$

$$c_1 K_u(x) \leq \mathbf{K}_u(x) \leq c_2 K_u(x)$$

So... if know how to generate elements of X in Kolmogorov ordering then can generate all elements of $f(X) \subset Y$ in their structural ordering

In fact... take $F(x) = (f(x), n(x))$ with

$$n(x) = \#\{x' \mid \nu_X^{-1}(x') \leq \nu_X^{-1}(x),\ f(x') = f(x)\}$$

total recursive function $\Rightarrow E = F(X) \subset Y \times \mathbb{Z}^+$ enumerable

- $X_m := \{x \in X \mid n(x) = m\}$ and $Y_m := f(X_m) \subset Y$ enumerable
- for $x \in X_1$ and $y = f(x)$: complexity $K_u(x) \leq c \cdot \nu_Y^{-1}(y)$ (using inequalities for complexity under composition)

Multiplicity: $mult(y) := \#f^{-1}(y)$

$$Y_\infty \subset \cdots f(X_{m+1}) \subset f(X_m) \subset \cdots \subset f(X_1) = f(X)$$

$Y_\infty = \cap_m f(X_m)$ and $Y_{fin} = f(X) \smallsetminus Y_\infty$

Key Step: $y \in Y_\infty$ and $m \geq 1$: $\exists$ unique $x_m \in X$, $y = f(x_m)$, $n(x_m) = m$ and $c = c(f, u, v, \nu_X, \nu_Y) > 0$

$$K_u(x_m) \leq c \cdot \nu_Y^{-1}(y)\, m \log(\nu_Y^{-1}(y) m)$$

Oracle mediated recursive construction of Y_∞ and Y_{fin}

- Choose sequence (N_m, m), $m \geq 1$, $N_{m+1} > N_m$
- Step 1: $A_1 =$ list $y \in f(X)$ with $\nu_Y^{-1}(y) \leq N_1$; $B_1 = \emptyset$
- Step $m+1$: Given A_m and B_m, list $y \in f(X)$ with $\nu_Y^{-1}(y) \leq N_{m+1}$; $A_{m+1} =$ elements in this list for which $\exists\, x \in X$, $y = f(x)$, $n(x) = m+1$; $B_{m+1} =$ remaining elements in the list
- oracle: search for $x \in X$, $y = f(x)$, $n(x) = m+1$ only among those x with complexity bounded by function of $\nu_Y^{-1}(y)$ as above
- $A_m \cup B_m \subset A_{m+1} \cup B_{m+1}$, union is all $f(X)$; $B_m \subset B_{m+1}$ and $Y_{fin} = \cup_m B_m$, while $Y_\infty = \cup_{m\geq 1}(\cap_{n\geq 0} A_{m+n})$
- from A_m to A_{m+1} first add all new y with $N_m < \nu_Y^{-1}(y) \leq N_{m+1}$ then subtract those that have no more elements in the fiber $f^{-1}(y)$: these will be in B_{m+1}

Structural numbering for codes

- $X = Codes_q$, $Y = [0,1]^2 \cap \mathbb{Q}^2$ and $f : X \to Y$ is $cp : C \mapsto (R(C), \delta(C))$ code parameters map
- $A = \{0, \dots, q-1\}$ ordered, A^n lexicographically; computable total order ν_X:
(i) if $n_1 < n_2$ all $C \subset A^{n_1}$ before all $C' \subset A^{n_2}$;
(ii) $k_1 < k_2$ all $[n, k_1, d]_q$-codes before $[n, k_2, d']_q$-codes;
(iii) fixed n and q^k: lexicographic order of code words, concatenated into single word $w(C)$ (determines code): order all the $w(C)$ lexicographically
- total recursive map $cp : Codes_q \to [0,1]^2 \cap \mathbb{Q}^2$
- fixed enumeration ν_Y of rational points in $[0,1]^2$

... inductively building the asymptotic bound using the described oracle mediated procedure

- Question: is there a statistical view of this procedure?

Partition function for code complexity

$$Z(X,\beta) = \sum_{x\in X} K_u(x)^{-\beta}$$

weights elements in constructive world X by inverse complexity; $\beta =$ inverse temperature, thermodynamic parameter

Convergence properties

- Kolmogorov complexity and Kolmogorov ordering

$$c_1\, \mathbf{K}_u(x) \leq K_u(x) \leq c_2\, \mathbf{K}_u(x)$$

- convergence of $Z(X,\beta)$ controlled by series

$$\sum_{x\in X} \mathbf{K}_u(x)^{-\beta} = \sum_{n\geq 1} n^{-\beta} = \zeta(\beta)$$

- Partition function $Z(X,\beta)$ convergence for $\beta > 1$; phase transition at pole $\beta = 1$

Asymptotic bound as a phase transition

- $X' \subset X$ infinite decidable subset of a constructive world
- $i : X' \hookrightarrow X$ total recursive function; also $j : X \to X'$ identity on X' constant on complement

$$K_u(i(x')) \leq c_1 K_v(x') \quad \text{and} \quad K_v(j(x)) \leq c_2 K_u(x)$$

- $\delta = \beta_q(R)$ inverse of $\alpha_q(\delta)$ on $R \in [0, 1 - 1/q]$
- Fix $R \in \mathbb{Q} \cap (0,1)$ and $\Delta \in \mathbb{Q} \cap (0,1)$

$$Z(R, \Delta; \beta) = \sum_{C : R(C) = R; 1 - \Delta \leq \delta(C) \leq 1} K_u(C)^{-\beta + \delta(C) - 1}$$

Phase transition at the asymptotic bound

- $1 - \Delta > \beta_q(R)$: partition function $Z(R, \Delta; \beta)$ real analytic in β
- $1 - \Delta < \beta_q(R)$: partition function $Z(R, \Delta; \beta)$ real analytic for $\beta > \beta_q(R)$ and divergence for $\beta \to \beta_q(R)_+$

Another view of the asymptotic bound as a phase transition

- Yuri I. Manin, Matilde Marcolli, *Error-correcting codes and phase transitions*, Mathematics in Computer Science (2011) 5:133–170.

• when constructing random codes (Shannon Random Code Ensemble): choose code words as equally distributed independent random variables

• imagine passing from classical to quantum systems, where the code words remain the fundamental degrees of freedom

• the basic quantum system of this kind is a system of independent harmonic oscillators: creation/annihilation operators associated to the basic independent degrees of freedom

Single Code: algebra of creation/annihilation operators

- for a single code C: code words are degrees of freedom
- Algebra of observable of a single code: Toeplitz algebra on code words

$$\mathcal{T}_C : \quad T_x, \quad x \in C, \quad T_x^* T_x = 1$$

$T_x T_x^*$ mutually orthogonal projectors

- Fock space representation $\mathcal{H}_C$ spanned by ϵ_w, words $w = x_1, \ldots, x_N$ in code language $\mathcal{W}_C$

$$T_x \, \epsilon_w = \epsilon_{xw}$$

Quantum Statistical Mechanics of a single code

- algebra of observables $\mathcal{T}_C$; time evolution $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}_C)$

$$\sigma_t(T_x) = K_u(C)^{it}\, T_x$$

- Hamiltonian $\pi(\sigma_t(T)) = q^{itH}\pi(T)q^{-itH}$

$$H\,\epsilon_w = \ell(w)\, log_q K_u(C)\,\epsilon_w$$

in Fock representation, $\ell(w)$ length of word (# of code words)

- Partition function

$$Z(C,\sigma,\beta) = \mathrm{Tr}(e^{-\beta H}) = \sum_m (\# W_{C,m}) K_u(C)^{-\beta m}$$

$$= \sum_m q^{m(nR-\beta \log_q K_u(C))} = \frac{1}{1-q^{nR}K_u(C)^{-\beta}}$$

- Convergence: $\beta > nr/\log_q K_u(C)$

QSM system at a code point (R, δ)

- Different codes $C \in cp^{-1}(R, \delta)$ as independent subsystems
- Tensor product of Toeplitz algebras $\mathcal{T}_{(R,\delta)} = \otimes_{C \in cp^{-1}(R,\delta)} \mathcal{T}_C$
- Shift on single code temperature so that

$$Z(C, \sigma, n(\beta - \delta + 1)) \leq (1 - K_u(C)^{-\beta})^{-1}$$

by *singleton bound* on codes $R + \delta - 1 \leq 0$

- Fock space $\mathcal{H}_{(R,\delta)} = \otimes \mathcal{H}_C$; time evolution $\sigma = \otimes \sigma^C$
- Partition function (variable temperature)

$$Z(cp^{-1}(R, \delta), \sigma; \beta) = \prod_{C \in cp^{-1}(R,\delta)} Z(C, \sigma, n(\beta - \delta + 1))$$

- Convergence controlled by $\prod_C (1 - K_u(C)^{-\beta})^{-1}$; in turned controlled by the classical zeta function $Z(cp^{-1}(R, \delta), \beta) = \sum_{C \in cp^{-1}(R,\delta)} K_u(C)^{-\beta}$

first versus second quantization

- Bosonic second quantization: example of primes p and integers $n \in \mathbb{N}$; independent degrees of freedom (primes) quantized by isometries $\tau_p^* \tau_p = 1$; tensor product of Toeplitz algebras $\otimes_p \mathcal{T}_p = C^*(\mathbb{N})$ semigroup algebra; $\sigma_t(\tau_p) = p^{it}\tau_p$, partition function $\zeta(\beta) = \prod_p (1 - p^{-\beta})^{-1}$ prod of partition functions individual systems
- Infinite tensor product: second quantization; finite tensor product: quantum mechanical (finitely many degrees of freedom) first quantization
- $(\mathcal{T}_{(R,\delta)}, \sigma)$ is quantum mechanical above the asymptotic bound; bosonic QFT below asymptotic bound

Asymptotic bound boundary between first and second quantization

Asymptotic bound as a phase transition (QSM point of view)

• Variable temperature partition function: $\mathcal{A} = \otimes_{s\in S}\mathcal{A}_s$, $\sigma = \otimes_s \sigma_s$; $\beta : S \to \mathbb{R}_+$; $Z(\mathcal{A},\sigma,\beta) = \prod_s Z(\mathcal{A}_s,\sigma_s,\beta(s))$

• fix a code point (R,δ); partition function (variable β)

$$Z((R,\delta),\sigma;\beta) = \prod_{C\in cp^{-1}(R,\delta)} (1 - q^{(R-\beta)n_C})^{-1}$$

• if (R,δ) above bound finite product; if below bound convergence governed by $\sum_C q^{(R-\beta)n_C}$, for $\beta > R$.

• change of behavior of the system at $R = \alpha_q(\delta)$ asymptotic bound

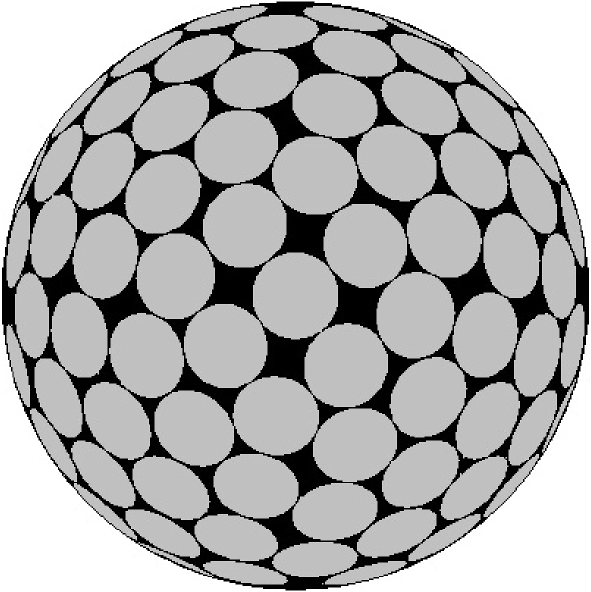
Spherical Codes

- Yuri I. Manin, Matilde Marcolli, *Asymptotic bounds for spherical codes*, arXiv:1801.01552

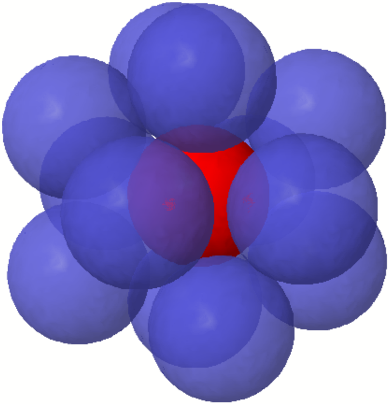
- spherical code: finite set X of points on unit sphere $S^{n-1} \subset \mathbb{R}^n$
- spherical code X has minimal angle ϕ if $\forall x \neq y \in X$

$$\langle x, y \rangle \leq \cos \phi$$

- $A(n, \phi) =$ max number of points on S^{n-1} with minimal angle ϕ



Relation to sphere packings and kissing number



example of sphere configuration with kissing nunber 12

Spherical codes from binary codes

• C binary $[n,k,d]_2$-code

• identifying $\mathbb{Z}/2\mathbb{Z} = \{\pm 1\}$: code words as subset of the vertices of n-cube centered at origin in $\mathbb{R}^n$ inscribed in sphere S^{n-1} (normalization factor)

• binary code C gives spherical code X_C with parameters

$$\cos\phi = 1 - \frac{2d}{n} \Leftrightarrow \delta(C) = \frac{d}{n} = \sin^2(\phi/2) = \frac{1-\cos\phi}{2}$$

$$R(C) = \frac{\log_2 \# X_C}{n}$$

with maximum (for fixed n and d)

$$R(C)_{max}(n,d) = \frac{\log_2 A(n,\phi(n,d))}{n}$$

• Question: is there an asymptotic bound for spherical codes?

Space of code parameters

- binary codes: $[0,1]^2 \cap \mathbb{Q}$ coordinates (δ, R)
- spherical codes:
  - code rate $R = n^{-1} \log_2 \# X_C$
  - minimum angle $\phi = \phi_{X_C}$ (or $\cos \phi$)
- unbounded: ϕ smaller maximal number of points $A(n, \phi)$ grows, so R unbounded near $\phi \to 0$
- space $\mathbb{R}_+ \times [0, \pi]$

Regions in the space of code parameters

- code points of some spherical code X

$$\mathcal{P} = \{P = (R,\phi) \,|\, \exists X \subset S^{n-1} \,:\, (R,\phi) = (R(X) = \frac{1}{n}\log_2 \#X, \phi_X)\}$$

- accumulation points of set of code parameters

$$\mathcal{A} = \{P = (R,\phi) \,|\, \exists (R_i,\phi_i) \in \mathcal{P} \,:\, (R,\phi) = \lim_i (R_i,\phi_i),\, (R_i,\phi_i) \neq (R,\phi)\}$$

- points surrounded by a 2-ball densely filled by code parameters

$$\mathcal{U} = \{P = (R,\phi) \,|\, \exists \epsilon > 0 \,:\, B(P,\epsilon) \subset \mathcal{A}\}$$

- asymptotic bound:

$$\Gamma = \{(R = \alpha(\phi), \phi) \,|\, \alpha(\phi) = \sup\{R \in \mathbb{R}_+ \,:\, (R,\phi) \in \mathcal{U}\}\,\}$$

with $\alpha(\phi) = 0$ if $\{R \in \mathbb{R}_+ \,:\, (R,\phi) \in \mathcal{U}\} = \emptyset$

New phenomena with respect to binary codes

• the two regions $\mathcal{A}$ and $\mathcal{U}$ do not coincide

• asymptotic bound is the boundary of the region $\mathcal{U}$ (not of $\mathcal{A}$)

• the part of the region $\mathcal{A}$ that is not in $\mathcal{U}$ consists of sequences of horizontal segments not contained in $\mathcal{U} \cup \Gamma$

• also the asymptotic bound is only non-trivial in a "small angle region"

- small angles region: $0 \leq \phi \leq \pi/2$
- large angle region: $\pi/2 < \phi \leq \pi$

Large angle region $\pi/2 < \phi \leq \pi$

- Rankin bound: for $\pi/2 < \phi \leq \pi$

$$A(n,\phi) \leq (\cos\phi - 1)/\cos\phi$$

- bound realized for $-1 \leq \cos\phi \leq -1/n$ while for $-1/n \leq \cos\phi < 0$ one has $A(n,\phi) = n+1$
- code points lie below the curve

$$R = \frac{1}{n}\log_2(\min\{n+1, \frac{\cos\phi - 1}{\cos\phi}\})$$

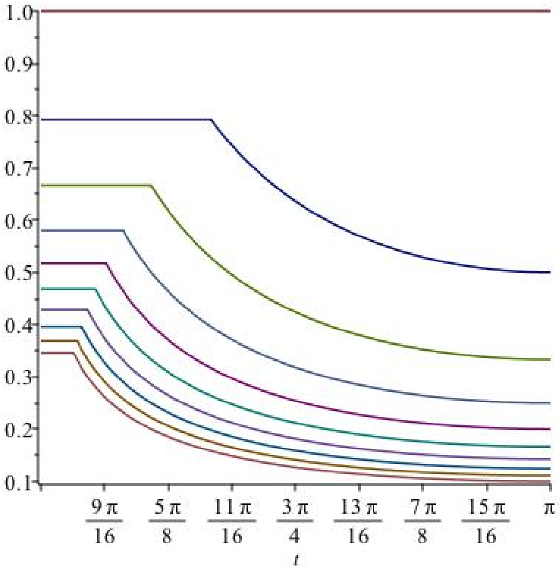
- large $n \to \infty$ behavior

$$R = \frac{\log_2 \#X}{n} \leq \frac{\log_2 A(n,\phi)}{n} \to 0, \qquad \pi/2 \leq \phi \leq \pi$$

$\Rightarrow$ no interesting asymptotic bound in this region

- still contains code points in $\mathcal{A} \smallsetminus \mathcal{U}$ and $\mathcal{P} \smallsetminus \mathcal{A}$

Plots for $n = 1, \dots, 10$



Estimates in the small angle region

- Kabatiansky–Levenshtein bound: large $n \to \infty$

$$R \leq \frac{\log_2 A(n,\phi)}{n} \leq \frac{1+\sin\phi}{2\sin\phi} \log_2(\frac{1+\sin\phi}{2\sin\phi}) - \frac{1-\sin\phi}{2\sin\phi} \log_2(\frac{1-\sin\phi}{2\sin\phi})$$

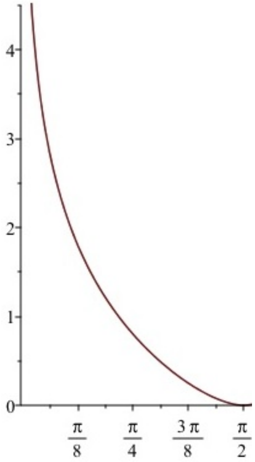
for minimum angle $0 \leq \phi \leq \pi/2$

- for large $n \to \infty$ code parameter in the undergraph

$$\mathcal{S} := \{(R,\phi) \in \mathbb{R}_+ \times [0,\pi] \,:\, R \leq H(\phi)\}$$

$$H(\phi) = \frac{1+\sin\phi}{2\sin\phi} \log_2(\frac{1+\sin\phi}{2\sin\phi}) - \frac{1-\sin\phi}{2\sin\phi} \log_2(\frac{1-\sin\phi}{2\sin\phi})$$

Graph of $H(\phi)$:



• either cutoff on minimum angle $\phi \geq \phi_0$ (e.g. case of sphere packings) or cutoff on $R = \frac{1}{n} \log_2 \#X \leq T$ (more natural for spoiling operations)

Spoiling operations for spherical codes

1. first spoiling operations:
   - binary codes: $C_1 = C \star_i a$ associates to a word $c = (a_1, \dots, a_n)$ of C the word $c \star_i a = (a_1, \dots, a_{i-1}, a, a_i, \dots, a_n)$
   - spherical codes: take code $X_C \subset S^{n-1}$ and inserts S^{n-1} as hyperplane section of S^n
2. second spoiling operation:
   - binary codes: $C_2 = C\star_i$, which is a projection of the code C in the i-th direction
   - spherical codes: $\cos\theta = \langle v_k, v_r \rangle$ angle between two points of code X_C: orthogonal projection along x_i-axis

$$\cos\tilde{\theta} = \frac{n}{n-1}\langle v_k^{\perp_i}, v_r^{\perp_i}\rangle = \frac{n}{n-1}(\cos\theta - \langle v_{k,i}, v_{r,i}\rangle)$$

3. third spoiling operation:
   - binary codes: $C_3 = C(a,i)$ code words with i-th digit a
   - spherical codes: line ℓ and orthogonal hyperplane L through origin of $\mathbb{R}^n$, with $X_3 := X_\ell^\pm = X \cap S^{n-1}_{\ell,\pm}$ intersection with one of the two hemispheres

Main differences: continuous parameters in spoiling operations

• first spoiling operation extends with *continuous parameters* (choice of a hyperplane H): scaling the sphere S^{n-1} and identifying it with the section $H \cap S^n$ to embed new code $X_1 = X \star H$ in S^n

• parameters: $k(X_1) = k(X)$, $n(X_1) = n(X) + 1$ and

$$\cos\phi_{X_1} = \rho_H^2 \cos\phi_X + (1 - \rho_H^2)$$

ρ_H radius of scaled sphere $S_\rho^{n-1} = H \cap S^n$

• second spoiling operation: L hyperplane through origin in $\mathbb{R}^n$ with orthogonal ℓ not containing code points; orthogonal projection $P_L : \mathbb{R}^n \to L \simeq \mathbb{R}^{n-1}$ and normalize vectors: $X_2 = X\star_L \subset S^{n-2}$

• code parameters: $k(X_2) = k(X)$ and $n(X_2) = n(X) - 1$

$$\cos\phi_{X_2} = (1+u)\cos\phi_X + u, \qquad u = (1 - \xi_{X,L}^2)/\xi_{X,L}^2$$

with $\xi_{X,\ell} = \mathrm{dist}(X, \ell)$

- third spoiling operation also continuous choice of ℓ, L with $X_3 := X_\ell^\pm = X \cap S^{n-1}_{\ell,\pm}$ one hemisphere

- code parameters: $\exists \ell$ with $k(X) - 1 \leq k(X_3) < k(X)$ and minimum angle $\phi(X_3) \geq \phi(X)$

controlling cones: starting with X with code parameters $[n, k, \cos\phi]$

- use spoling operations to obtain code parameters to obtain
  1. $[n+1, k, \lambda \cos\phi + 1 - \lambda]$, for all $\lambda \in [0,1]$;
  2. $[n-1, k, (1+u)\cos\phi \pm u]$ for $u = (1 - \xi_{X,L})^2/\xi^2_{X,L}$;
  3. $[n-1, k-a, \cos\phi]$, for $0 < a < k$.

for $0 \leq \phi \leq \pi/2$

- consequence: if (R, ϕ) code point all line segment

$$\ell_{n,k,\cos\phi} = \{(\frac{n}{n+1} R, \lambda \cos\phi + 1 - \lambda) \,:\, \lambda \in [0,1]\}$$

also made of code points: in $\mathcal{A}$ not always in $\mathcal{U}$

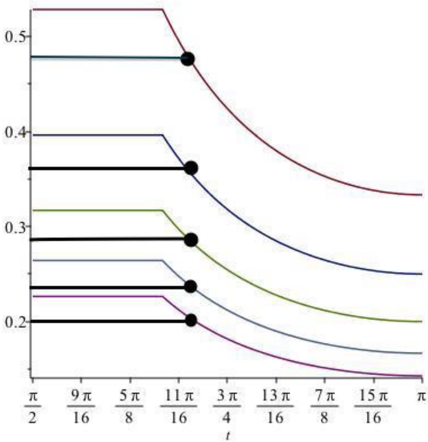
Example of segments in $\mathcal{A}$ not in $\mathcal{U}$

- Rankin examples of spherical codes realizing bound (large angles)

$R(X) = \frac{1}{n} \log_2(\frac{\cos\phi - 1}{\cos\phi})$ for $-1 \leq \cos\phi \leq -1/n$ and

$R(X) = \frac{1}{n} \log_2(n+1)$ for $-1/n \leq \cos\phi < 0$

- apply first spoiling:



Existence of the asymptotic bound

• construct controlling regions $\mathcal{R}_{L,c}(P)$, $\mathcal{R}_{R,c}(P)$, $\mathcal{R}_{U,c}(P)$, $\mathcal{R}_{D,c}(P)$ in a cutoff of undergraph of $H(\phi)$

• use these to constrain position of the asymptotic bound: Γ graph of continuous decreasing $R = \alpha(\phi)$ with $\alpha(\phi) \to \infty$ for $\phi \to 0$ and $\alpha(\pi/2) = 0$.

• set $\mathcal{U}$ is undergraph of this function

$$\mathcal{U} = \{(R, \phi) \,:\, R \leq \alpha(\phi)\}$$

union of all the lower controlling regions $\mathcal{R}_L(P)$ of all points $P \in \Gamma$

• code point $P = (R, \phi) \notin \Gamma$ in region $\mathcal{U}$ iff infinite multiplicity and $\exists$ sequence X_i of spherical codes with $(R(X_i), \phi_{X_i}) = (R, \phi)$ and $n_i \to \infty$ and $\#X_i \to \infty$.

Questions

• applications to sphere packings? (maximal density sphere packings)

• interplay between classical binary (q-ary?) codes and spherical codes

• asymptotic bound and complexity: spherical codes and complexity

• classical to quantum codes (for binary and q-ary: CSSR algorithm): what about spherical codes?

• for binary codes: strange examples of codea above the asymptotic bound coming from linguistics (see my talk in the Linguistics and AI seminar)