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AN ENUMERATIVE GEOMETRY FRAMEWORK FOR ALGORITHMIC LINE PROBLEMS IN $\mathbb{R}^3$*

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Abstract. We investigate the enumerative geometry aspects of algorithmic line problems when the admissible bodies are balls or polytopes. For this purpose, we study the common tangent lines/transversals to k balls of arbitrary radii and $4-k$ lines in $\mathbb{R}^3$. In particular, we compute tight upper bounds for the maximum number of real common tangents/transversals in these cases. Our results extend the results of Macdonald, Pach, and Theobald who investigated common tangents to four unit balls in $\mathbb{R}^3$ [*Discrete Comput. Geom.*, 26 (2001), pp. 1–17].

Key words. tangents, balls, transversals, lines, enumerative geometry, real solutions, computational geometry

AMS subject classifications. 14N10, 68U05, 51M30, 14P99, 52C45

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1. Introduction. Algorithmic questions involving lines in $\mathbb{R}^3$ belong to the fundamental problems in computational geometry [36, 26], computer graphics [28], and robotics [33]. As an initial reference example from computational geometry, consider the problem of determining which bodies of a given scene cannot be seen from *any* viewpoint outside of the scene. From the geometric point of view, this leads to the problem of determining the *common tangents to four given bodies* in $\mathbb{R}^3$ (cf. section 2). Other algorithmic tasks leading to the same geometric core problem include computing smallest enclosing cylinders [32], computing geometric permutations/stabbing lines [27, 2], controlling a laser beam in manufacturing [26], or solving placement problems in geometric modeling [10, 17].

If the bodies are polytopes, the common tangents are common transversals of edges [27]; so, in fact, the main geometric task is to compute the common transversals to four given lines in $\mathbb{R}^3$. This geometric problem has been well known for many years (see, e.g., [16]). In particular, if a configuration has only finitely many common transversals, then this number is bounded by 2; and it is well known how to characterize the configurations with infinitely many common transversals.

On the other hand, the following theorem in [21] shows that this situation completely changes if the bodies under investigation are unit balls (see also [35, 23]).

PROPOSITION 1. *Four unit balls in $\mathbb{R}^3$ have at most 12 common tangent lines unless their centers are located on the same line. Furthermore, there exists a configuration with 12 tangents; i.e., the upper bound is tight.*

Essentially, this means that algebraically this tangent problem is of degree 12. Note that due to this high degree, proving the characterization of the configurations with infinitely many common tangents is a highly nontrivial task.

However, concerning the class of tangent problems to four given bodies, Proposition 1 solves only one particular case. In the present paper, we develop techniques to analyze a substantially larger class of variants. In particular, we aim at filling the gaps between the two extreme situations mentioned before by considering common

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Table 1

Summary of results and references of known results. For the case of four balls of general radii we are able to provide a formulation with Bézout bound 12 *(which improves the results from* [17] *substantially; see section* 4*).*

	Upper bound # solutions		# Real solutions of our construction		Characterization of degenerate instances	
4 lines	2	(well known)	2	(well known)	yes	(well known)
3 lines, 1 ball	4		4		yes	
2 lines, 2 balls	8		8		–	
1 line, 3 balls	12		12		–	
4 unit balls	12	[21]	12	[21]	yes	[21]
4 balls	12	([17])	12	[21]	–	

tangents/transversals to k balls and $4-k$ lines, $k \in \{0,\ldots,4\}$. For convenience of notation, we consider a transversal of a line as a tangent to the line. Our investigations do not only clarify the exact growth in algebraic degree from 2 to 12 but also provide effective means to tackle these questions when the symmetry (in the sense of identical bodies) is lost. From the algorithmic point of view, these problems of common tangents immediately arise in the mentioned applications when the class of admissible bodies in the scene consists of both balls and polytopes (see section 2).

As the main contribution of this paper, we compute *tight* upper bounds for the number of common tangents to k balls and $4-k$ lines in the finite case, $k \in \{0,\ldots,4\}$. Here, *tightness* refers to the following (quite strong) sense of real algebraic geometry (cf. [34]): On the one hand, for each k we bound the number of solutions by algebraic methods, say, by some number m. Then, on the other hand, we provide a construction which indeed leads to m solutions in *real* space $\mathbb{R}^3$ (which would not be possible if any polynomial formulation contained some complex solutions or solutions at infinity).

The general difficulty of proving tight bounds of this kind may be seen by the following two aspects. For the classical enumerative geometry problem of conics tangent to five given conics (dating back to Steiner in 1847) the existence problem of 3264 *real* solutions had not been solved until a few years ago (see [30] and [14, sect. 7.2]). Furthermore, as pointed out in [34], there are nearly no criteria or general techniques for tackling these type of questions. For these reasons, it is even more remarkable that in *all* (!) of the situations there exists a construction matching the upper bound.

Table 1 summarizes our results and provides references of known results. It shows the upper bounds for the number of solutions and the matching numbers of real solutions in our constructions. The last column shows that only in a few cases are we able to explicitly characterize the configurations with an infinite number of common tangents. Namely, besides the already existing results for four lines and four unit balls, we add the characterization for three lines and one ball. In the entries with a "–" we cannot give such a characterization and will discuss this issue at the end of the paper.

Let us point out that the proofs of these results are of quite different flavors. For $k \in \{1,2\}$, the upper bounds immediately follow from Bézout's theorem. Whereas for $k=1$ it is easy to give a construction matching this bound, the construction for $k=2$ is quite involved. In particular, for $k=2$ we apply tools from algebraic geometry and computer algebra (e.g., standard bases) to prove correctness of the construction. However, proving the tight upper bound for three balls and one line is completely different. Here, the Bézout bound in our formulation will be 16 instead of 12. In order to find a better bound for the number of real solutions, we have to analyze the underlying algebraic geometry of the problem in detail. Finally, in the proof for four balls of general radii we use elementary geometry to find a formulation with Bézout bound 12. Altogether, we think that this variety of techniques can serve to provide many ideas when tackling related problems.

This paper is structured as follows. In section 2, we establish the connection between the algorithmic problems and the geometric tangent problems. Then, after providing some algebraic background on Plücker coordinates in section 3, we prove the necessary results for Table 1 in section 4. We conclude the paper with a short discussion of the remaining open questions.

2. Motivation and algorithmic background. The problem under investigation represents the algebraic core problem within several algorithmic applications mentioned in the introduction. Exemplarily, we describe two of them.

Partial visibility. Consider the following problem from ray-tracing with moving viewpoints. Here, we want to compute information on the viewpoint positions where the visibility topology of the scene changes. As a special case, this includes tackling the following core problem of partial visibility.

A set $B \subset \mathbb{R}^n$ (say, $n \in \{2,3\}$) is called a *(convex) body* if it is bounded, closed, convex, and contains an inner point. Now we consider a scene consisting of a set $\mathcal{B}$ of (not necessarily disjoint) bodies from a specific class $\mathcal{X}$ in $\mathbb{R}^n$. ($\mathcal{X}$ might be the set of all balls or the set of all polytopes.) A body $B \in \mathcal{B}$ is called *partially visible from a viewpoint* v if there exists a line segment connecting v and B not intersecting with the interior of any other body in $\mathcal{B}$. A body $B \in \mathcal{B}$ is called *partially visible* if B can be seen from some viewpoint "outside" of the scene, i.e., if there exists a ray starting at B not intersecting with the interior of any other body in $\mathcal{B}$. We call such a ray a *visibility ray* for B. Bodies which are not partially visible can be immediately removed from the scene, which reduces the complexity of the visualization process. In case of dense crystals whose atoms are visualized as sufficiently large balls, the reduction in complexity may be quite substantial.

In the two-dimensional case, checking partial visibility of a body B can be reduced to a finite number of geometric problems as follows (cf. the treatment of stabbing lines in [12]). Without loss of generality let $|\mathcal{B}| \geq 2$. If there exists a visibility ray for B, then we can continuously transform (i.e., translate and rotate) the visibility ray until we reach a situation where the underlying line is tangent to at least two of the bodies. (One of them might be B itself.) Hence, it suffices to compute the set of all common tangent lines to a pair of bodies in $\mathcal{B}$ and check whether one of these lines contains a visibility ray. For any pair of bodies, the number of common tangent lines is at most four (which is a very special case of the results in [6, 19] on the number of common supporting hyperplanes in general dimension).

In the three-dimensional case we can essentially proceed analogously. Since a line in $\mathbb{R}^3$ has four degrees of freedom, the core problem is to compute the common tangents to four bodies in $\mathbb{R}^3$ (cf. [27, 2]). However, in the three-dimensional case, there are also some special cases where we can transform a visibility ray only to a situation with two or three bodies, or where a configuration with four bodies has an infinite number of common tangents.

For a polytope P, any tangent to P intersects an edge of P. Hence, if $\mathcal{X}$ contains balls and polytopes, we have to compute common tangents/transversals to k lines and $4-k$ balls, $0 \leq k \leq 4$. An algorithmic treatment of the situations with infinitely many common tangents (depending on the class $\mathcal{X}$ of bodies) requires an *a priori* characterization of the configurations with infinitely many common tangents.

In contrast to some other problems in computational geometry, characterizing these situations cannot be neglected (say, by applying perturbation techniques [11]), since the large algebraic degree involved makes it highly nontrivial to guarantee a correct perturbation.

Envelopes. Let $\mathcal{B}$ be a collection of n convex bodies in $\mathbb{R}^3$. A line l is called a *line transversal* of $\mathcal{B}$ if it intersects every member of $\mathcal{B}$. The set of line transversals of $\mathcal{B}$ can be represented as the region enclosed between an upper and a lower envelope as follows (see [7, 1, 2]). These representations are important in the design of data structures supporting ray shooting queries (i.e., seeking the first body, if any, met by a query ray) [1].

If we exclude lines parallel to the yz-plane, a line l in $\mathbb{R}^3$ can be uniquely represented by its projections on the xy- and xz-planes: $y = \sigma_1 x + \sigma_2$, $z = \sigma_3 x + \sigma_4$. Hence, a line can be represented by the quadruple $(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \mathbb{R}^4$.

Let B be a convex body in $\mathbb{R}^3$. For fixed $\sigma_1, \sigma_2, \sigma_3$, the set of lines $(\sigma_1, \sigma_2, \sigma_3, \sigma_4)$ that intersect B is obtained by translating a line in the z-direction between two extreme values, $(\sigma_1, \sigma_2, \sigma_3, \phi_B^-(\sigma_1, \sigma_2, \sigma_3))$ and $(\sigma_1, \sigma_2, \sigma_3, \phi_B^+(\sigma_1, \sigma_2, \sigma_3))$, which represent lines tangent to B from below and from above, respectively. Hence, the set of line transversals to $\mathcal{B}$ can be represented as

$$\{(\sigma_1, \sigma_2, \sigma_3, \sigma_4) \;:\; \max_{B \in \mathcal{B}} \phi_B^-(\sigma_1, \sigma_2, \sigma_3) \le \sigma_4 \le \min_{B \in \mathcal{B}} \phi_B^+(\sigma_1, \sigma_2, \sigma_3),$$

which is a region enclosed between a lower envelope and an upper envelope in $\mathbb{R}^4$. If the elements of $\mathcal{B}$ are balls or polytopes, then the set of line transversals defines a semialgebraic set in $\mathbb{R}^4$ (see [2]). Assuming general position, the vertices (= zero-dimensional faces) of the boundary of this region correspond to lines which are tangent to four of the bodies in $\mathcal{B}$. Similar to the first scenario, any implementation of this basic step has to cope with the enumerative questions treated in the present paper.

The role of an algebraic oracle. In both of these algorithmic scenarios, the problem is reduced to the core problem of finding the common tangents/transversals to k lines and $4-k$ balls. In literature, core problems of this type are considered to be problems of *constant description complexity* (see, e.g., [2]). Often, it is assumed that one has access to an *algebraic oracle* computing the necessary tangents, and the algorithm is formulated in terms of that oracle. From this point of view, our analysis can be seen as the necessary mathematical investigations on how to build this algebraic oracle.

In particular, any implementation of this algebraic oracle or any interface to a black box subroutine establishing that oracle has to cope with the enumerative questions. From the viewpoint of data structures it is always useful and sometimes even necessary to know a good (i.e., tight) upper bound on the number of these tangent lines. From the viewpoint of program verification, knowing a tight upper bound on the number of tangent lines offers the possibility of strong and valuable consistency checks within a program (in particular with regard to the necessary numerical subroutines; cf. section 5). Finally, from the viewpoint of efficiency, understanding the geometry of the basic problem helps to find the right polynomial formulations for the underlying numerical algorithms.

3. Plücker coordinates.

In several of the proofs, we use the well-known Plücker coordinates of lines in projective space $\mathbb{P}^3$ (see, e.g., [16, 8]). Let $x = (x_0, x_1, x_2, x_3)^T$, $y = (y_0, y_1, y_2, y_3)^T \in \mathbb{P}^3$ be two different points on a line l. Then l can be represented (of course not uniquely) by the 4×2-matrix L whose two columns are x and y. The

Plücker vector $p = (p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})^T \in \mathbb{P}^5$ of the line is defined by the determinants of the 2×2-submatrices of L, i.e., $p_{ij} = x_i y_j - x_j y_i$. It is well known that the set of vectors in $\mathbb{P}^5$ satisfying the Plücker relation

$$p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0 \tag{1}$$

is in one-to-one correspondence with the set of lines in $\mathbb{P}^3$. A line l intersects with a line l' in $\mathbb{P}^3$ if and only if their Plücker vectors p and p' satisfy

$$p_{01}p'_{23} - p_{02}p'_{13} + p_{03}p'_{12} + p_{12}p'_{03} - p_{13}p'_{02} + p_{23}p'_{01} = 0. \tag{2}$$

In order to characterize lines tangent to balls we consider tangent lines to arbitrary quadrics in $\mathbb{P}^3$. Throughout the presentation, we will identify a quadric surface in $\mathbb{P}^3$ with its symmetric 4×4-representation matrix. For example, the sphere with radius r and center $(c_1, c_2, c_3)^T \in \mathbb{R}^3$, in $\mathbb{P}^3$ described by $(x_1 - c_1x_0)^2 + (x_2 - c_2x_0)^2 + (x_3 - c_3x_0)^2 = r^2x_0^2$, is identified with the matrix

$$\begin{pmatrix} c_1^2 + c_2^2 + c_3^2 - r^2 & -c_1 & -c_2 & -c_3 \\ -c_1 & 1 & 0 & 0 \\ -c_2 & 0 & 1 & 0 \\ -c_3 & 0 & 0 & 1 \end{pmatrix}.$$

LEMMA 2. *Let L be a 4×2-matrix representing the line $l \subset \mathbb{P}^3$. l is tangent to a quadric Q in $\mathbb{P}^3$ if and only if the 2×2-matrix L^TQL is singular.*

Proof. If we denote the two columns of L by x and y, then the line l consists of all points

$$\left\{ z = (z_0, z_1, z_2, z_3)^T \,:\, z = \lambda x + \mu y,\, (\lambda, \mu)^T \in \mathbb{R}^2 \setminus \{(0,0)^T\} \right\}.$$

By definition, l is tangent to Q if and only if this line intersects the quadric exactly once (namely, with multiplicity 2) or if it is contained in the quadric. The homogeneous quadratic equation

$$(\lambda x + \mu y)^T Q (\lambda x + \mu y) = 0$$

can be made affine by setting $\mu = 1$. Since the discriminant of this affine quadratic equation in λ is

$$(2x^TQy)^2 - 4(x^TQx)(y^TQy) = -4\det(L^TQL),$$

the statement follows immediately. $\square$

In order to transfer this condition to Plücker coordinates, we introduce the operator

$$\wedge^2 : \mathbb{R}^{m,n} \to \mathbb{R}^{\binom{m}{2},\binom{n}{2}}$$

(cf. [35]). The row and column indices of the resulting matrix are subsets of cardinality 2 of $\{1, \ldots, m\}$ and $\{1, \ldots, n\}$, respectively. For $I \subset \{1, \ldots, m\}$ and $J \subset \{1, \ldots, n\}$, with $|I| = |J| = 2$,

$$(\wedge^2 A)_{I,J} := \det A_{[I,J]},$$

where $A_{[I,J]}$ denotes the 2×2-submatrix of the given matrix A with row indices I and column indices J. Let l be a line in $\mathbb{P}^3$ and L be a 4×2-matrix representing l.

By interpreting the 6×1-matrix $\wedge^2 L$ as a vector in $\mathbb{P}^5$, we observe $\wedge^2 L = p_l$, where p_l is the Plücker vector of l.

LEMMA 3. *A line $l \subset \mathbb{P}^3$ is tangent to a quadric Q if and only if the Plücker vector p_l of l satisfies*

$$p_l^T (\wedge^2 Q) p_l = 0. \tag{3}$$

Proof. Let L be a 4×2-matrix whose two columns contain different points of l. The Cauchy–Binet formula from multilinear algebra (see, e.g., [22]) implies

$$\begin{aligned}\det(L^T Q L) &= (\wedge^2 L^T)(\wedge^2 Q)(\wedge^2 L)\\ &= (\wedge^2 L)^T (\wedge^2 Q)(\wedge^2 L).\end{aligned}$$

Now the claim follows from Lemma 2. ◻

For a sphere with radius r and center $(c_1, c_2, c_3)^T \in \mathbb{R}^3$ the quadratic form $p_l^T (\wedge^2 Q) p_l$ results in

$$\begin{pmatrix} p_{01} \\ p_{02} \\ p_{03} \\ p_{12} \\ p_{13} \\ p_{23} \end{pmatrix}^T \begin{pmatrix} c_2^2 + c_3^2 - r^2 & -c_1 c_2 & -c_1 c_3 & c_2 & c_3 & 0 \\ -c_1 c_2 & c_1^2 + c_3^2 - r^2 & -c_2 c_3 & -c_1 & 0 & c_3 \\ -c_1 c_3 & -c_2 c_3 & c_1^2 + c_2^2 - r^2 & 0 & -c_1 & -c_2 \\ c_2 & -c_1 & 0 & 1 & 0 & 0 \\ c_3 & 0 & -c_1 & 0 & 1 & 0 \\ 0 & c_3 & -c_2 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} p_{01} \\ p_{02} \\ p_{03} \\ p_{12} \\ p_{13} \\ p_{23} \end{pmatrix}. \tag{4}$$

4. Proofs and constructions.

We show the following theorem.

THEOREM 4. *Given $4 - k$ lines and k balls in $\mathbb{R}^3$, $0 \le k \le 4$. If there exist only finitely many common tangent lines to these four bodies, then the number of these tangents is bounded by*

$$\begin{cases} 2 & \textit{if } k = 0, \\ 4 & \textit{if } k = 1, \\ 8 & \textit{if } k = 2, \\ 12 & \textit{if } k \in \{3, 4\}. \end{cases}$$

These bounds are tight; i.e., for each k there exists a configuration where the number of different real tangent lines matches the stated number. The bounds are tight even if the balls are unit balls.

For brevity, we denote the maximum numbers of tangent lines in the five situations by N_k, $k \in \{0, \dots, 4\}$. Before proving the statements in the following lemmas, let us recall the following version of Bézout's theorem (see, e.g., [8, p. 91]).

THEOREM 5 (Bézout). *Let $f_1, \dots, f_n$ be homogeneous polynomials in $x_0, \dots, x_n$ of degrees $d_1, \dots, d_n > 0$. If $f_1, \dots, f_n$ have a finite number of common zeros in projective n-space $\mathbb{P}^n$, then the number of zeros (counted with multiplicity) is bounded by $d_1 \cdot d_2 \cdots d_n$.*

Note that the upper bounds $N_0 \le 2$, $N_1 \le 4$, $N_2 \le 8$ immediately follow from Bézout's theorem. Namely, since the common tangent lines to three lines and one ball can be formulated by three linear equations of the form (2), one equation of the form (3) as well as the Plücker relation (1) in the six homogeneous variables $p_{01}, \dots, p_{23}$, we obtain $N_1 \le 4$. Analogously, we obtain $N_0 \le 2$, $N_2 \le 8$.

Further, note that the common transversals to four given lines in three-dimensional space are a well-studied problem in enumerative geometry, and it is well known that

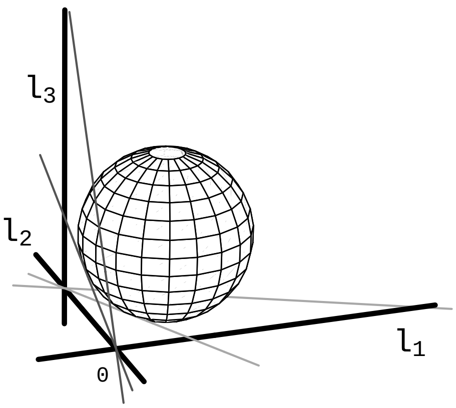

Fig. 1. *The figure shows a configuration with three lines l_1, l_2, l_3, and one ball of radius* 11/10, *leading to four common tangent lines. The two tangent lines in the x_1x_2-plane are drawn in light grey, whereas the two tangent lines in the x_2x_3-plane are drawn in dark grey.*

the upper bound of 2 can actually be achieved in real space $\mathbb{R}^3$ (see, e.g., [16]); hence $N_0 = 2$. The number of common transversals is finite if and only if the Plücker vectors of the four given lines are linearly independent.

In the following, let $B(c, r)$ denote the (closed) ball with center c and radius r.

Lemma 6. *$N_1 = 4$.*

Proof. Since $N_1 \leq 4$, it suffices to give a construction with three lines and one ball, leading to four common tangents. Let l_1 be the x_1-axis, l_2 be the x_2-axis, and l_3 be parallel to the x_3-axis and passing through $(0, 2, 0)^T$ (see Figure 1); hence $l_1 \cap l_2 = \{(0, 0, 0)^T\}$ and $l_2 \cap l_3 = \{(0, 2, 0)^T\}$.

Each line intersecting the three lines l_1, l_2, and l_3 is located in the x_1x_2-plane (in which case it passes through $(0, 2, 0)^T$) or is located in the x_2x_3-plane (in which case it passes through the origin). For $1 < r < \sqrt{2}$ the ball $B((1, 1, 1)^T, r)$ intersects both the x_1x_2-plane and the x_2x_3-plane but does not intersect with any of the lines l_1, l_2, l_3. Hence, since there are two tangents to the ball passing through the origin and lying in the x_1x_2-plane, and since there are two tangents to the ball passing through $(0, 2, 0)^T$ and lying in the x_1x_3-plane, there are four common tangents altogether. Figure 1 shows a configuration with $1 < r = 11/10 < \sqrt{2}$. We remark that by appropriate scaling, the ball can be transformed into a unit ball. Furthermore, by slightly perturbing the configuration, the lines can be made pairwise skew. □

To complete the entries for three lines and one ball in Table 1, it remains to characterize the configurations with infinitely many common tangent lines. If the three lines are not pairwise skew, then all common tangent lines lie in the same plane or pass through a point of intersection. Since the resulting characterization can be easily established, we can assume that the three lines are pairwise skew.

It is well known that the common transversals of three pairwise skew lines define a hyperboloid (see, e.g., [31, 3]). By applying a translation and a rotation, the hyperboloid can be transformed into

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} - \frac{x_3^2}{c^2} = 1 \qquad \text{with } a, b, c > 0. \tag{5}$$

This transformation changes the center of the ball into some new center $(p_1, p_2, p_3)^T \in \mathbb{R}^3$. Now the characterization of infinitely many common tangent lines is given by the following lemma.

LEMMA 7. *Let l_1, l_2, l_3 be three pairwise skew lines whose common transversals generate a hyperboloid of the form* (5), *and let B_4 be a ball with center $(p_1, p_2, p_3)^T$ and radius $r > 0$. Then there exist infinitely many common tangents to l_1, l_2, l_3, B_4 if and only if $p_1 = p_2 = 0$, $a = b$, and in the x_1x_3-plane the circle $x_1^2 + (x_3 - p_3)^2 = r^2$ is a tangent circle to both branches of the hyperbola $x_1^2/a^2 - x_3^2/c^2 = 1$.*

Proof. The hyperboloid (5) can be parametrized by one of the two sets of generating lines. In particular, this hyperboloid is generated by the set of lines

$$\left\{ (x_1, x_2, 0)^T + \lambda \left(-\frac{a}{bc} x_2, \frac{b}{ac} x_1, 1 \right)^T \,:\, \lambda \in \mathbb{R} \right\}, \tag{6}$$

$$\text{where } \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1 \tag{7}$$

(see, e.g., [18]). In order to characterize those lines which are tangent to the ball, we can apply Lemma 3 to the lines (6) and obtain a polynomial equation in x_1, x_2 of degree at most 4. After bringing the terms of even degree in x_1 to the left side and the terms of odd degree in x_1 to the right side, squaring the equation yields a new equation in which every term is of even degree in x_1. Now we can use (7) to eliminate x_2 and obtain a polynomial equation of degree at most 8 in x_1. Since a univariate polynomial with infinitely many common zeros is the zero polynomial, this polynomial formulation in a single variable implies that if the hyperboloid contains infinitely many tangent lines to the ball, then *all* lines in the parametrization are tangent lines to the ball.

Since the intersection of the hyperboloid with any plane parallel to the x_1x_2-axis is symmetric with respect to the origin, a necessary condition for infinitely many common tangents is $p_1 = p_2 = 0$. In this situation, a configuration with infinitely many common tangents further implies $a = b$. Hence, since $p_1 = p_2 = 0$ and $a = b$, both the hyperboloid and the ball are rotational symmetric with respect to the x_3-axis, and it suffices to consider the section through the x_1x_3-plane. In this section, the circle $x_1^2 + (x_3 - p_3)^2 = r^2$ must be a tangent circle to both branches of the hyperbola $x_1^2/a^2 - x_3^2/c^2 = 1$.

If, conversely, $p_1 = p_2 = 0$, $a = b$, and in the x_1x_3-plane the circle $x_1^2 + (x_3 - p_3)^2 = r^2$ is a tangent circle to the hyperbola $x_1^2/a^2 - x_3^2/c^2 = 1$, then the rotational symmetry implies that every line in the hyperboloid $x_1^2/a^2 + x_2^2/b^2 - x_3^2/c^2 = 1$ is tangent to the ball B_4. Hence, there are infinitely many common tangents. ∎

LEMMA 8. $N_2 = 8$.

Proof. Since $N_2 \leq 8$, it suffices to give a construction with two lines and two balls of the same radius, leading to eight common tangent lines. We start from the following configuration with six different common tangent lines. The two balls are symmetrically located on the x_1-axis: $c_3 = (\gamma, 0, 0)^T$, $c_4 = (-\gamma, 0, 0)^T$; the radius r will be specified below. The lines l_1 and l_2 are chosen in a plane $x_2 = \beta$ for some $\beta > 0$ such that the lines intersect in $(0, \beta, 0)^T$. Hence, every common transversal of the two lines either lies in the plane $x_2 = \beta$ or passes through the point $(0, \beta, 0)^T$. If the two balls intersect with each other, and $\beta < r$, and $(0, \beta, 0)^T$ is not contained in the union of the balls $B(c_3, r)$, $B(c_4, r)$, then there are exactly six different common tangents (see Figure 2): two tangents pass through $(0, \beta, 0)^T$ and lie in the plane $x_1 = 0$; two tangents lie in the plane $x_2 = \beta$ and are parallel to the x_1-axis; and two tangents lie in the plane $x_2 = \beta$ and pass through $(0, \beta, 0)^T$. For the following considerations it is quite useful to have a succinct description of the last two tangents and also to work with integer coefficients for β, γ, and r. In particular, we will force

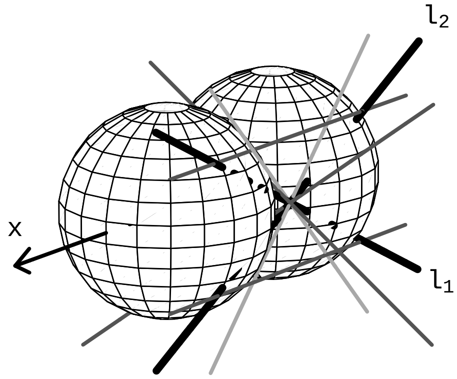

FIG. 2. *The figure shows a construction with two lines and two balls, leading to six different tangent lines. The two tangents lying in the plane $x_2 = \beta$ and passing through $(0, \beta, 0)^T$ are drawn in light grey. The other four tangents are drawn in dark grey.*

the two tangents in the plane $x_2 = \beta$ and passing through $(0, \beta, 0)^T$ to be of the form $(0, \beta, 0)^T + \lambda(1, 0, \pm 1)^T$. In order to obtain these tangents, β, γ, and r have to satisfy $\beta^2 + \gamma^2/2 = r^2$ and $r > \gamma$. An appropriate choice is $\beta = 7$, $\gamma = 8$, and $r = 9$, so that the tangents of the last type are

$$t_1 := \left\{(0,7,0)^T + \lambda(1,0,1)^T \,:\, \lambda \in \mathbb{R}\right\} \text{ and } t_2 := \left\{(0,7,0)^T + \lambda(1,0,-1)^T \,:\, \lambda \in \mathbb{R}\right\}.$$

Now the key observation is that these two tangents have multiplicity 2. In order to prove this we consider the system of Plücker equations stemming from (2) and (4). Independent of the specific choice of lines l_1, l_2 with the above properties, the common transversals of l_1 and l_2 are given by the common zeros of the two linear, homogeneous polynomials

$$\begin{aligned} f_1 &= -7p_{03} + p_{23}, \\ f_2 &= 7p_{01} + p_{12}. \end{aligned}$$

The quadratic equations resulting from the balls $B(c_3, r)$ and $B(c_4, r)$ are

$$\begin{aligned} f_3 &= -81p_{01}^2 - 17p_{02}^2 - 17p_{03}^2 - 16p_{02}p_{12} + p_{12}^2 - 16p_{03}p_{13} + p_{13}^2 + p_{23}^2, \\ f_4 &= -81p_{01}^2 - 17p_{02}^2 - 17p_{03}^2 + 16p_{02}p_{12} + p_{12}^2 + 16p_{03}p_{13} + p_{13}^2 + p_{23}^2. \end{aligned}$$

Furthermore, let $f_5 = p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12}$ be the polynomial of the Plücker relation (1).

The tangent t_1 has Plücker coordinate $(1, 0, 1, -7, 0, 7)^T$. In order to compute the multiplicity of this tangent, we follow the method and the notation in [9, sect. 4.4]. First we pass to an affine version of the polynomials by adding the polynomial $f_6 = p_{01} - 1$; this forces $p_{01} = 1$ in any common zero of the system. Then we move the point t_1 to the origin by applying the linear variable transformation

$$(p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})^T = (q_{01}, q_{02}, q_{03}, q_{12}, q_{13}, q_{23})^T + (1, 0, 1, -7, 0, 7)^T.$$

The local intersection multiplicity μ can be computed as the vector space dimension of the quotient ring

$$\mu = \dim R_l / I_l,$$

where $R_l := \mathbb{C}[q_{01}, \ldots, q_{23}]_{\langle q_{01},\ldots,q_{23}\rangle}$ is the local ring whose elements are the rational functions in $q_{01}, \ldots, q_{23}$ with nonvanishing denominator at 0. I_l is the ideal defined by $f_1, \ldots, f_6$ in the local ring R_l.

In order to compute μ, we use the fact that in case of finite dimension

$$\dim R_l/I_l = \dim R_l/\langle \mathrm{LT}(I_l)\rangle,$$

where $\langle \mathrm{LT}(I_l)\rangle$ denotes the ideal generated by the leading terms of I_l (see, e.g., [9, Chap. 4, Cor. 4.5]). This dimension can be easily extracted from a standard basis of I_l. (For the convenience of the reader, a short review of standard bases can be found in the appendix.) Since by our choice of β, γ, and r all coefficients are integers, we can apply a computer algebra package (e.g., `Singular` [15]) to compute a standard basis $\{h_1, \ldots, h_6\}$ of the ideal I_l with respect to antigraded reverse lexicographical order:

$$\begin{aligned}
h_1 &= q_{01},\\
h_2 &= 112q_{02} + 34q_{03} + 14q_{12} - 16q_{13},\\
h_3 &= 14q_{03} + q_{12},\\
h_4 &= q_{12},\\
h_5 &= 64q_{23},\\
h_6 &= 112q_{13}^2.
\end{aligned}$$

Hence, the leading monomials of $h_1, \ldots, h_6$ with respect to antigraded reverse lexicographical order are q_{01}, q_{02}, q_{03}, q_{12}, q_{23}, q_{13}^2. The desired multiplicity μ is the cardinality of the set of cosets $\{1 + I_l, q_{13} + I_l\}$, which implies $\mu = 2$. By symmetry, the tangent t_2 has multiplicity 2 as well.

Now we choose one particular configuration of the presented class, namely the one with $l_1 := t_1$ and $l_2 := t_2$. By perturbing this configuration, the two double tangent lines will split into four different tangent lines: first, we slightly increase the x_2-coordinate of the line l_2 so that the resulting line l_2' becomes $(0, \beta', 0)^T + \lambda(1, 0, -1)^T$ for some $\beta' > \beta$. In this process, the double tangent t_1 splits into two tangents t_1^a and t_1^b intersecting l_1 and l_2' in different orders; i.e., one of the tangents t_1^a, t_1^b touches l_1, l_2, B_3, and B_4 in the order (B_3, l_1, l_2, B_4) and one of them in the order (B_3, l_2, l_1, B_4). However, the tangent t_2 is still a double zero of the system of polynomials, since the parallel lines t_2 and l_2' intersect in the plane at infinity of $\mathbb{P}^3$.

Similarly, we can make the double tangent t_2 split into two tangents by slightly decreasing the x_2-coordinate of the line l_1; denote the resulting line by l_1'. Figure 3 shows the configuration for l_1' passing through the points $(0, 6.5, 0)^T$, $(2, 6.5, 2)^T$ and l_2' passing through the points $(0, 7.5, 0)^T$, $(2, 7.5, -2)^T$. □

For N_3 the situation is more involved. The Bézout bound gives 16, but, in fact, the number of real common tangents is bounded by 12. Our proof is based on some algebraic-geometric investigations of the common tangents to four unit balls by Macdonald [20]. By appropriately applying these considerations to the situation with three balls and one line, it will turn out that there are always two solutions at infinity with multiplicity at least 2. For the general background on the algebraic and geometric concepts used in the subsequent proofs, easily accessible introductions can be found in [25, 29].

We start with the following observation in [35]. The sphere with center $(c_1, c_2, c_3)^T \in \mathbb{R}^3$ and radius r has the homogeneous equation in $\mathbb{P}^3$

$$(x_1 - c_1x_0)^2 + (x_2 - c_2x_0)^2 + (x_3 - c_3x_0)^2 = r^2x_0^2.$$

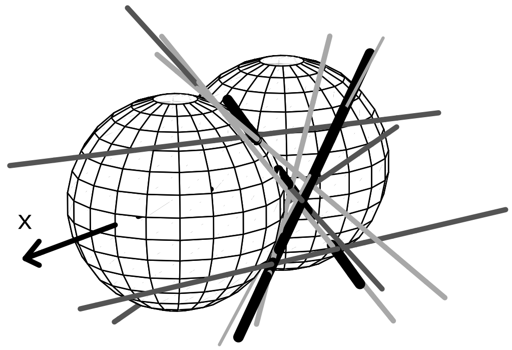

FIG. 3. *Construction with two lines and two balls, leading to eight common tangent lines.*

In the plane at infinity $x_0 = 0$, this gives the equation

$$x_1^2 + x_2^2 + x_3^2 = 0,$$

which is independent of the center and the radius. Let ω denote this conic section in the plane at infinity. Later in the proof, we will work in the space of lines in $\mathbb{P}^3$. In that situation, we will have to consider those tangents through any point $z \in \omega$ in the plane at infinity rather than z itself. For this reason, we provide a characterization of these tangents.

LEMMA 9. *Let $z = (0, \zeta_1, \zeta_2, \zeta_3)^T \in \omega$. The tangent to the conic ω at z which lies in the plane at infinity has Plücker coordinate*

$$(p_{01}, p_{02}, p_{03}, p_{12}, p_{13}, p_{23})^T = (0, 0, 0, \zeta_3, -\zeta_2, \zeta_1)^T.$$

In particular, the tangent contains the points $(0, -\zeta_2, \zeta_1, 0)^T$, $(0, \zeta_3, 0, -\zeta_1)^T$, and $(0, 0, -\zeta_3, \zeta_2)^T$.

Proof. Since $\zeta_0 = 0$ we can compute in projective plane $\mathbb{P}^2$; so let $\overline{z} = (\zeta_1, \zeta_2, \zeta_3)^T$. The conic section

$$x^T A x = 0 \qquad \text{with } A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is regular in $\overline{z}$ with tangent $\{y = (y_1, y_2, y_3)^T \in \mathbb{P}^2 : \overline{z}^T A y = 0\}$. In particular, $(-\zeta_2, \zeta_1, 0)^T$, $(\zeta_3, 0, -\zeta_1)^T$, $(0, -\zeta_3, \zeta_2)^T$, and $\overline{z}$ itself lie on this tangent. Now any two of these points can be used to compute the Plücker coordinate of the tangent line. ∎

Consider a configuration with a line l_1 and three spheres Q_2, Q_3, and Q_4 in $\mathbb{R}^3$. The idea for proving the solutions at infinity is to transfer the geometry of ω to the space of lines in $\mathbb{P}^3$. More precisely, let t be a tangent to ω at z in the plane at infinity. Since the quadrics $\wedge^2 Q_2, \wedge^2 Q_3, \wedge^2 Q_4 \in \mathbb{P}^5$ characterize the tangents to Q_2, Q_3, Q_4, the Plücker vector p_t of t is contained in $\wedge^2 Q_2$, $\wedge^2 Q_3$, and $\wedge^2 Q_4$. Let Ω denote the quadric in $\mathbb{P}^5$ defined by the Plücker equation (1). Since t is a line in $\mathbb{P}^3$, t is also contained in Ω. We will show that the tangent hyperplanes to the quadrics $\wedge^2 Q_2$, $\wedge^2 Q_3$, $\wedge^2 Q_4$, Ω at p_t contain a common subspace of dimension 2. In connection with the linear form defined by the transversals of the line l_1, this will prove the multiplicity of at least 2.

Let us investigate the spheres Q_2, Q_3, Q_4 first. For $i \in \{2,3,4\}$, we are looking for lines whose Plücker vectors lie in the tangent hyperplane of $\wedge^2 Q_i$ at p_t. The geometric concept behind this relation is *polarity*. Recall that the polar plane of a point $a \in \mathbb{P}^n$ with respect to an arbitrary quadric Q is defined by

$$\{y \in \mathbb{P}^n \,:\, a^T Q y = 0\}.$$

If $a \in Q$, then the polar hyperplane is a tangent hyperplane. The polar line of a line $l \in \mathbb{P}^3$ is defined by

$$\{y \in \mathbb{P}^3 \,:\, a^T Q y = 0 \text{ for all } a \in l\}.$$

The following lemma establishes a connection between the tangent hyperplanes to $\wedge^2 Q$ and the concept of polarity for a quadric Q.

LEMMA 10. *Let t be a tangent line to a quadric $Q \subset \mathbb{P}^3$, and let the point $a \in \mathbb{P}^3$ be contained in the polar line of t. Then, for any line l containing a, the Plücker vector p_l of l is contained in the tangent hyperplane to $\wedge^2 Q$ at p_t, i.e., $p_t^T(\wedge^2 Q)p_l = 0$.*

Proof. Let T be a representation of t by a 4×2-matrix as described in section 3. Further, let b be a point on l with $b \neq a$, and let $L = (a, b)$ be a representation of l by a 4×2-matrix. Since a is contained in the polar line of t, we have $T^T Q a = (0,0)^T$. Hence, by reasoning as in Lemma 3, we can conclude

$$p_t^T(\wedge^2 Q)p_l = \det(T^T Q L) = 0. \qquad \square$$

In particular, the following version of a well-known relationship (see, e.g., [25]) shows that the precondition of Lemma 10 is satisfied if $a = t \cap Q$.

LEMMA 11. *If t is tangent to a quadric Q at some point a, then a is contained in the polar line of t.*

Proof. Let $y \neq a$ be a point on t. Since t lies on the polar plane (namely, the tangent plane) of a with respect to Q, we have $a^T Q y = 0$. Since also $a^T Q a = 0$, a lies on the polar line of t with respect to Q. $\square$

Finally, we are ready to prove the upper bound for N_3.

LEMMA 12. $N_3 \leq 12$.

Proof. Let L_1 be the hyperplane (2) in $\mathbb{P}^5$ characterizing the transversals of the line l_1; that is, any point on L_1 which satisfies the Plücker relation is the Plücker coordinate of a transversal to l_1. Let $\wedge^2 Q_2, \wedge^2 Q_3, \wedge^2 Q_4$ be the quadrics (4) characterizing the tangents to the three balls. Further, let $z = (0, \zeta_1, \zeta_2, \zeta_3)^T \in \omega$, and let $\pi \subset \Omega \subset \mathbb{P}^5$ be the set of Plücker vectors whose corresponding lines in $\mathbb{P}^3$ pass through z. π can be written as the image of the projective mapping $h : \mathbb{P}^3 \to \Omega \subset \mathbb{P}^5$,

$$h(y_0, y_1, y_2, y_3) = \wedge^2 \begin{pmatrix} 0 & y_0 \\ \zeta_1 & y_1 \\ \zeta_2 & y_2 \\ \zeta_3 & y_3 \end{pmatrix}.$$

Since h is linear, it follows that π is a two-dimensional plane in $\mathbb{P}^5$ with $\pi \subset \Omega$.

Let t be the tangent to ω at z in the plane at infinity. By Lemmas 11 and 10, π is contained in the tangent hyperplane to $\wedge^2 Q_i$ at p_t, $2 \leq i \leq 4$.

In order to show that π is also contained in the tangent hyperplane to Ω at p_t, let y be a point different from z, and let l be a line through z and y. Then, by Lemma 9, the Plücker vectors p_t and p_l satisfy

$$\begin{aligned} p_t^T \Omega p_l &= (0,0,0,\zeta_3,-\zeta_2,\zeta_1) \cdot \frac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} -\zeta_1 y_0 \\ -\zeta_2 y_0 \\ -\zeta_3 y_0 \\ \zeta_1 y_2 - \zeta_2 y_1 \\ \zeta_1 y_3 - \zeta_3 y_1 \\ \zeta_2 y_3 - \zeta_3 y_2 \end{pmatrix} \\ &= -\frac{1}{2} y_0 (\zeta_1^2 + \zeta_2^2 + \zeta_3^2) \\ &= 0\,. \end{aligned}$$

Hence, the four tangent hyperplanes of $\wedge^2 Q_2$, $\wedge^2 Q_3$, $\wedge^2 Q_4$, Ω at p_t contain a common subspace of dimension at least 2. By Lemma 9, the tangents to the conic ω lie on a conic $\overline{\omega}$, namely on

$$p_{12}^2 + p_{13}^2 + p_{23}^2 = 0,$$

in the two-dimensional subspace of $\mathbb{P}^5$ given by $p_{01} = p_{02} = p_{03} = 0$. The restriction of the hyperplane L_1 to the subspace $p_{01} = p_{02} = p_{03} = 0$ defines a one-dimensional subspace $\overline{L_1}$. Since $\overline{L_1}$ is one-dimensional, it intersects with $\overline{\omega}$ at two points $b_1, b_2 \in \mathbb{P}^5$ in the plane $p_{01} = p_{02} = p_{03} = 0$. Further, since b_1 and b_2 satisfy the Plücker relation, they are Plücker vectors of some tangents t_1 and t_2 to ω. Altogether, the five tangent hyperplanes of $\wedge^2 Q_2$, $\wedge^2 Q_3$, $\wedge^2 Q_4$, Ω, L_1 at b_1 and b_2 contain a common subspace of dimension at least 1. Hence, the tangent hyperplanes are not independent, which implies that the multiplicity of intersection in b_1 and b_2 is at least 2 (see, e.g., [24, p. 115]). $\square$

In order to show that $N_3 = 12$ it remains to give a construction with one line l_1 and three balls B_2, B_3, B_4 of the same radius r, leading to 12 common tangents. Let l_1 be the x_3-axis, and let the centers c_2, c_3, c_4 of the balls constitute an equilateral triangle with edge length 1 in the x_1x_2-plane, say $c_2 = (\sqrt{3}/3, 0, 0)^T$, $c_3 = (-\sqrt{3}/6, 1/2, 0)^T$, $c_4 = (-\sqrt{3}/6, -1/2, 0)^T$ (see Figure 4). For $1/2 < r < \sqrt{3}/3$, the balls are non-disjoint, and none of them contains the origin.

Let t be a line which intersects l_1, and let H be the plane containing t and l_1. The three cuts $H \cap B_1$, $H \cap B_2$, and $H \cap B_3$ are discs (maybe degenerated to single points or empty sets). Unless H is equidistant to two of the centers, one of these discs is strictly contained in one of the other two. Hence, any common tangent to the line and the three balls lies in one of the three planes which contain the x_3-axis and which are equidistant to two of the centers.

For example, one of these planes is the x_1x_3-plane, which is equidistant to c_2 and c_3. The section through this plane contains two disjoint discs: one representing the (identical) intersections of the plane with B_2 and B_3, and the second one because of B_1. These two discs are separated by the line l_1. Hence, in this plane there are four common tangents. Altogether, since there are three planes of this kind, we have 12 common tangents.

Finally, it remains to analyze the common tangents to four balls (with arbitrary radii) in $\mathbb{R}^3$. Of course, this problem can also be formulated in Plücker coordinates. However, since the solutions of these equations have a common component at infinity

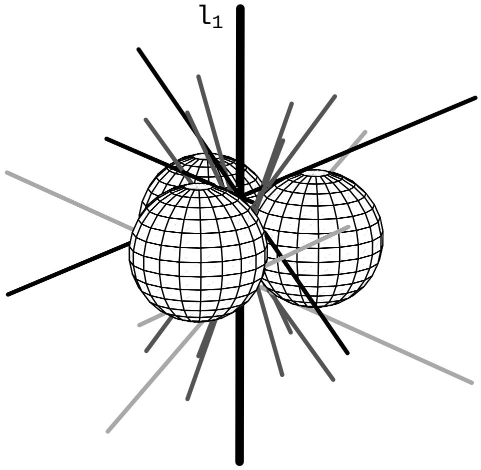

FIG. 4. *Construction with one line and three balls, leading to* 12 *tangents.*

[35], we prefer to compute the number of tangents by an elementary approach. Recently, in [17] the common tangents to four balls have been formulated by polynomial equations with Bézout number 24. We improve this result by giving a polynomial formulation with Bézout number 12; this is optimal by Proposition 1.

The idea for obtaining the system with Bézout bound 12 is to generalize the approach for unit balls in [21]. Note that in the proof we will always refer to the generic case. For this reason—in contrast to Proposition 1—the proof does not provide a precise characterization of the cases with infinitely many common tangent lines.

LEMMA 13. $N_4 \leq 12$.

Proof. Let $c_1, \ldots, c_4$ be affinely independent, and, without loss of generality, let r_4 be the smallest of the radii. We consider functions $\rho_i : [0, r_4] \to \mathbb{R}$ with $\rho_i(0) = 0$, $\rho_i(r_4) = r_i$. Let $\rho_4(t) = t$, while $\rho_i(t)$ for $1 \leq i \leq 3$ will be specified below. First we describe the set of lines which are tangent to the balls $B(c_i, \rho_i(t))$ for $t > 0$.

A line l will be specified by its homogeneous direction vector $s = (s_1, s_2, s_3)^T$ and its closest point p to the origin.

The line l has distance $\rho_i(t)$ from some point c_i if and only if the line $l - p$ (which passes through the origin) has distance $\rho_i(t)$ from $c_i - p$, i.e., if and only if

$$((c_i - p) \times s)^2 = \rho_i^2(t) s^2.$$

Introducing the moment vector $m := p \times s$ and applying Lagrange's identity gives

$$(c_i \times s)^2 - 2\langle c_i, p\rangle s^2 + m^2 - \rho_i^2(t) s^2 = 0. \tag{8}$$

Choosing c_4 to be at the origin and subtracting (8) for index 4 from this equation for index $i \in \{1, 2, 3\}$ yields linear equations in p:

$$\langle c_i, p\rangle = \frac{1}{2s^2}(c_i \times s)^2 - \frac{1}{2}(\rho_i^2(t) - t^2), \qquad 1 \leq i \leq 3. \tag{9}$$

Setting $M := (c_1, c_2, c_3)^T$, we obtain the vector equation

$$p = \frac{1}{2s^2} M^{-1} \begin{pmatrix} (c_1 \times s)^2 \\ (c_2 \times s)^2 \\ (c_3 \times s)^2 \end{pmatrix} - \frac{1}{2} M^{-1} \begin{pmatrix} \rho_1^2(t) - t^2 \\ \rho_2^2(t) - t^2 \\ \rho_3^2(t) - t^2 \end{pmatrix}. \tag{10}$$

Now the key idea is that if we choose parametrizations $\rho_i(t)$ with $\rho_i^2(t) - t^2 = C_i$ for some constants $C_i \in \mathbb{R}$, $1 \leq i \leq 3$, then the vector p is uniquely determined by the direction vector s. Furthermore, the conditions $\rho_i(r_4) = r_i$ imply $C_i = r_i^2 - r_4^2$; hence, $\rho_i^2(t) = t^2 + (r_i^2 - r_4^2)$. By Cramer's rule,

$$M^{-1} = \frac{1}{6V}(c_2 \times c_3, c_3 \times c_1, c_1 \times c_2),$$

where $V := \det(c_1, c_2, c_3)/6$ denotes the oriented volume of the tetrahedron $c_1c_2c_3c_4$. By introducing the normal vectors

$$n_1 := (c_2 \times c_3)/2, \quad n_2 := (c_3 \times c_1)/2, \quad n_3 := (c_1 \times c_2)/2,$$

and substituting (10) into $\langle p, s \rangle = 0$, we can eliminate p and obtain a homogeneous cubic condition for the direction vector s:

$$\sum_{i=1}^{3} \left((c_i \times s)^2 + s^2(r_i^2 - r_4^2) \right) \langle n_i, s \rangle = 0.$$

Any solution s of this equation is the direction vector of a line with distances $\rho_i(t)$ from the four centers for *some* parameter t. Substituting the radius condition $||p|| = r_4$ into (10) gives an equation of degree 4. Since $\rho_i(r_4) = r_i$, $1 \leq i \leq 4$, any common solution of the cubic and the quartic equation gives a common tangent to the four balls $B(c_i, r_i)$. By Bézout's theorem, the formulation of the tangent problem by a cubic and a quartic equation implies $N_4 \leq 12$. $\square$

5. Conclusion and open questions. We have investigated the enumerative geometry questions for the common tangents to four bodies in $\mathbb{R}^3$ when the bodies are balls or polytopes. These results reflect the algebraic complexity inherent in the mentioned applications. In other words, whenever we want to focus on exact computations for the visibility or envelope problems described in section 2, we have to cope with solving systems of polynomial equations of the stated degrees.

The main open problem is to complete the characterization of the degenerate instances in Table 1. For example, in the case of four balls with arbitrary radii there are some obvious situations with infinitely many common tangent lines: whenever the four centers are collinear and the four balls are inscribed in the same hyperboloid H. We conjecture that there does not exist any configuration with four balls of arbitrary radii, noncollinear centers, and infinitely many common tangent lines. However, we were not able to prove this.

From the practical point of view, actually computing the numerical values of the solutions (which has, e.g., been done in finding the constructions given in this paper) requires either multidimensional numerical methods such as homotopy methods or combinations of symbolic techniques with univariate polynomial solvers. (For an introduction into all these techniques see [9].) Since generally these techniques are still computationally expensive, it is important to apply the most appropriate polynomial formulations of the concrete problems. From this point of view, our results provide optimal formulations. Finally, let us mention that there are many research efforts in improving the efficiency of the two mentioned numerical polynomial solving techniques. In particular, for recent improvements and the state of the art of the first technique see [37], and with regard to the second technique see [4, 5, 13].

Appendix: Standard bases. We review the definitions of a standard basis, starting from Gröbner basis theory (see [9]). The theory of Gröbner bases provides computational methods to find "nice" generators for an ideal I in a polynomial ring $\mathbb{C}[x_1, \ldots, x_n]$. The theory of *standard bases* extends this theory for ideals in local rings. More precisely, let $R_l := \mathbb{C}[x_1, \ldots, x_n]_{\langle x_1, \ldots, x_n \rangle}$ be the set of rational functions f/g in $x_1, \ldots, x_n$ with $g(0, \ldots, 0) \neq 0$. R_l defines a *local* ring; i.e., it contains exactly one maximal ideal. Since the algebraic-geometric definitions of intersection multiplicities are related to the concept of local rings, standard bases provide a powerful tool to effectively compute intersection multiplicities.

From the various possible term orders, we restrict ourselves to considering the *antigraded reverse lexicographical order* (arevlex). For $\alpha, \beta \in \mathbb{N}_0^n$, we have $x^\alpha >_{arevlex} x^\beta$ if and only if

$$\sum_{i=1}^n \alpha_i < \sum_{i=1}^n \beta_i$$

or

$$\sum_{i=1}^n \alpha_i = \sum_{i=1}^n \beta_i \quad \text{and} \quad x^\alpha >_{revlex} x^\beta,$$

where $>_{revlex}$ denotes the reverse lexicographical order of Gröbner basis theory. For any polynomial f, the leading term of f, denoted $\mathrm{LT}(f)$, is the maximal term of f with regard to the arevlex-order.

For an ideal I in R_l, the *set of leading terms of* I, abbreviated $\mathrm{LT}(I)$, is the set of leading terms of elements of I.

A *standard basis* of I is a set $\{g_1, \ldots, g_t\} \subset I$ such that $\langle \mathrm{LT}(I) \rangle = \langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_t) \rangle$. Given a set of polynomial generators of I, a standard basis of I can be effectively computed by variants of the Buchberger algorithm.

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