Spectral Action Models of Gravity and Packed Swiss Cheese Cosmologies

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Topics in Mathematical Physics
Based on:

Homogeneity versus Isotropy in Cosmology

- Homogeneous and isotropic: Friedmann universe $\mathbb{R} \times S^3$

$$\pm dt^2 + a(t)^2 (\sigma_1^2 + \sigma_2^2 + \sigma_3^2)$$

with round metric on $S^3$ with $SU(2)$-invariant 1-forms $\{\sigma_i\}$ satisfying relations

$$d\sigma_i = \sigma_j \wedge \sigma_k$$

for all cyclic permutations $(i, j, k)$ of $(1, 2, 3)$
• Homogeneous but not isotropic:
  Bianchi IX mixmaster models $\mathbb{R} \times S^3$

  \[ F(t) (\pm dt^2 + \frac{\sigma_1^2}{W_1^2(t)} + \frac{\sigma_2^2}{W_2^2(t)} + \frac{\sigma_3^2}{W_3^2(t)}) \]

  with a conformal factor $F(t) \sim W_1(t)W_2(t)W_3(t)$

• Isotropic but not homogeneous?
  $\Rightarrow$ Swiss Cheese Models
Main Idea:


Cut off 4-balls from a FRW spacetime and replace with different density smaller region outside/inside patched across boundary with vanishing Weyl curvature tensor (isotropy preserved)
Packed Swiss Cheese Cosmology

- Iterate construction removing more and more balls ⇒ Apollonian sphere packing of 3-dimensional spheres
- Residual set of sphere packing is fractal
- Proposed as explanation for possible fractal distribution of matter in galaxies, clusters, and superclusters

Configurations of mutually tangent circles in the plane, iterated on smaller scales filling a full volume region in the unit 2D ball: residual set volume zero fractal of Hausdorff dimension 1.30568...
Many results (geometric, arithmetic, analytic) known about Apollonian circle packings: see for example


Higher dimensional analogs of Apollonian packings: much more delicate and complicated geometry

Some known facts on Apollonian sphere packings

• **Descartes configuration** in $D$ dimensions: $D + 2$ mutually tangent $(D - 1)$-dimensional spheres

• **Example:** start with $D + 1$ equal size mutually tangent $S^{D-1}$ centered at the vertices of $D$-simplex and one more smaller sphere in the center tangent to all

![4-dimensional simplex]
- Quadratic Soddy–Gosset relation between radii $a_k$

\[
\left( \sum_{k=1}^{D+2} \frac{1}{a_k} \right)^2 = D \sum_{k=1}^{D+2} \left( \frac{1}{a_k} \right)^2
\]

- curvature-center coordinates: $(D + 2)$-vector

\[
w = \left( \frac{\|x\|^2 - a^2}{a}, 1, 1, \frac{1}{x_1}, \ldots, \frac{1}{x_D} \right)
\]

(first coordinate curvature after inversion in the unit sphere)

- Configuration space $\mathcal{M}_D$ of all Descartes configuration in $D$ dimensions = all solutions $\mathcal{W}$ to equation

\[
\mathcal{W}^t Q_D \mathcal{W} = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2 I_D \end{pmatrix}
\]

with left and a right action of Lorentz group $O(D + 1, 1)$
• Dual Apollonian group $G_{D}^\perp$ generated by reflections: inversion with respect to the $j$-th sphere

$$S_j^\perp = I_{D+2} + 21_{D+2}e_j^t - 4e_je_j^t$$

ej = j-th unit coordinate vector

• $D \neq 3$: only relations in $G_{D}^\perp$ are $(S_j^\perp)^2 = 1$

• $G_{D}^\perp$ discrete subgroup of $GL(D + 2, \mathbb{R})$

• Apollonian packing $\mathcal{P}_D =$ an orbit of $G_{D}^\perp$ on $M_D$

⇒ iterative construction: at $n$-th step add spheres obtained from initial Descartes configuration via all possible

$$S_{j_1}^\perp S_{j_2}^\perp \cdots S_{j_n}^\perp, \quad j_k \neq j_{k+1}, \forall k$$

there are $N_n$ spheres in the $n$-th level

$$N_n = (D + 2)(D + 1)^{n-1}$$
iterative construction of sphere packings
Length spectrum: radii of spheres in packing $\mathcal{P}_D$

$$\mathcal{L} = \mathcal{L}(\mathcal{P}_D) = \{a_{n,k} : n \in \mathbb{N}, 1 \leq k \leq (D + 2)(D + 1)^{n-1}\}$$

radii of spheres $S^{D-1}_{a_n,k}$

Melzak's packing constant $\sigma_D(\mathcal{P}_D)$ exponent of convergence of series

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \sum_{k=1}^{(D+2)(D+1)^{n-1}} a_{n,k}^s$$

Residual set: $\mathcal{R}(\mathcal{P}_D) = B^D \setminus \bigcup_{n,k} B^D_{a_n,k}$ with

$$\partial B^D_{a_n,k} = S^{D-1}_{a_n,k} \in \mathcal{P}_D$$

Packing $\Rightarrow$ $\text{Vol}_D(\mathcal{R}(\mathcal{P}_D)) = 0 \Rightarrow \sum_{\mathcal{L}} a_{n,k}^D < \infty \Rightarrow \sigma_D(\mathcal{P}_D) \leq D$

packing constant and Hausdorff dimension:

$$\dim_H(\mathcal{R}(\mathcal{P}_D)) \leq \sigma_D(\mathcal{P}_D)$$

for Apollonian circles known to be same
• Sphere counting function: spheres with given curvature bound

\[ N_\alpha(P_D) = \# \{ S_{a_n,k}^{D-1} \in P_D : a_{n,k} \geq \alpha \} \]

curvatures \( c_{n,k} = a_{n,k}^{-1} \leq \alpha^{-1} \)

• for Apollonian circles power law (Kontorovich–Oh)

\[ N_\alpha(P_2) \sim_{\alpha \to 0} \alpha^{-\dim_H(\mathcal{R}(P_2))} \]

• for higher dimensions (Boyd): packing constant

\[ \limsup_{\alpha \to 0} \frac{- \log N_\alpha(P_D)}{\log \alpha} = \sigma_D(P_D) \]

if limit exists \( N_\alpha(P_D) \sim_{\alpha \to 0} \alpha^{- (\sigma_D(P_D) + o(1))} \)
Screens and Windows

- in general $\zeta_{\mathcal{L}_D}(s)$ need have analytic continuation to meromorphic on whole $\mathbb{C}$
- $\exists$ screen $S$: curve $S(t) + it$ with $S : \mathbb{R} \to (-\infty, \sigma_D(\mathcal{P}_D)]$
- window $\mathcal{W} =$ region to the right of screen $S$ where analytic continuation

Screens and windows

$S \rightarrow W$
Some additional assumptions

- **Definition:**
  Apollonian packing $\mathcal{P}_D$ of $(D - 1)$-spheres is *analytic* if
  1. $\zeta_L(s)$ has analytic to meromorphic function on a region $\mathcal{W}$ containing $\mathbb{R}_+$
  2. $\zeta_L(s)$ has only one pole on $\mathbb{R}_+$ at $s = \sigma_D(\mathcal{P}_D)$.
  3. Pole at $s = \sigma_D(\mathcal{P}_D)$ is simple

- Also assume: $\exists \lim_{\alpha \to 0} -\frac{\log N_\alpha(\mathcal{P}_D)}{\log \alpha} = \sigma_D(\mathcal{P}_D)$

- **Question:** in general when are these satisfied for packings $\mathcal{P}_D$?

- Focus on $D = 4$ cases with these conditions
Rough estimate of the packing constant

- $\mathcal{P} = \mathcal{P}_4$ Apollonian packing of 3-spheres $S^3_{a_{n,k}}$
- at level $n$: average curvature

$$\frac{\gamma_n}{N_n} = \frac{1}{6 \cdot 5^{n-1}} \sum_{k=1}^{6 \cdot 5^{n-1}} \frac{1}{a_{n,k}}$$

- estimate $\sigma_4(\mathcal{P}_4)$ with averaged version: $\sum_n N_n \left( \frac{\gamma_n}{N_n} \right)^{-s}$

$$\sigma_{4,av}(\mathcal{P}) = \lim_{n \to \infty} \frac{\log(6 \cdot 5^{n-1})}{\log \left( \frac{\gamma_n}{6 \cdot 5^{n-1}} \right)}$$

- generating function of the $\gamma_n$ known (Mallows)

$$G_{D=4} = \sum_{n=1}^{\infty} \gamma_n x^n = \frac{(1 - x)(1 - 4x)}{1 - \frac{22}{3} x - 5x^2}$$

$u =$ sum of the curvatures of initial Descartes configuration
• obtain explicitly ($u = 1$ case)

\[ \gamma_n = \frac{(11 + \sqrt{166})^n(-64 + 9\sqrt{166}) + (11 - \sqrt{166})^n(64 + 9\sqrt{166})}{3^n \cdot 10 \cdot \sqrt{166}} \]

• this gives a value

\[ \sigma_{4,av}(\mathcal{P}) = 3.85193 \ldots \]

• in Apollonian circle case where $\sigma(\mathcal{P})$ known this method gives larger value, so expect $\sigma_4(\mathcal{P}) < \sigma_{4,av}(\mathcal{P})$

• constraints on the packing constant:

\[ 3 < \dim_H(\mathcal{R}(\mathcal{P})) \leq \sigma_4(\mathcal{P}) < \sigma_{4,av}(\mathcal{P}) = 3.85193 \ldots \]
Models of (Euclidean, compactified) spacetimes

1. Homogeneous Isotropic cases: $S^1_\beta \times S^3_a$

2. Cosmic Topology cases: $S^1_\beta \times Y$ with $Y$ a spherical space form $S^3/\Gamma$ or a flat Bieberbach manifold $T^3/\Gamma$ (modulo finite groups of isometries)

3. Packed Swiss Cheese: $S^1_\beta \times \mathcal{P}$ with Apollonian packing of 3-spheres $S^3_{a_{n,k}}$

4. Fractal arrangements with cosmic topology
Fractal arrangements with cosmic topology

- Example: Poincaré homology sphere, dodecahedral space $S^3/\mathcal{I}_{120}$, fundamental domain dodecahedron
• considered a likely candidate for cosmic topology

• build a fractal model based on dodecahedral space
Fractal configurations of dodecahedra (Sierpinski dodecahedra)
• spherical dodecahedron has $\text{Vol}(Y) = \text{Vol}(S^3_a/\mathcal I_{120}) = \frac{\pi^2}{60} a^3$

• simpler than sphere packings because uniform scaling at each step: $20^n$ new dodecahedra, each scaled by a factor of $(2 + \phi)^{-n}$

$$\dim_H(\mathcal P_{\mathcal I_{120}}) = \frac{\log(20)}{\log(2 + \phi)} = 2.32958...$$

• close up all dodecahedra in the fractal identifying edges with $\mathcal I_{120}$: get fractal arrangement of Poincaré spheres $Y_a(2+\phi)^{-n}$

• zeta function has analytic continuation to all $\mathbb C$

$$\zeta_L(s) = \sum_n 20^n (2 + \phi)^{-ns} = \frac{1}{1 - 20(2 + \phi)^{-s}}$$

exponent of convergence $\sigma = \dim_H(\mathcal P_{\mathcal I_{120}}) = \frac{\log(20)}{\log(2+\phi)}$ and poles

$$\sigma + \frac{2\pi im}{\log(2 + \phi)}, \quad m \in \mathbb Z$$
Spectral action models of gravity (modified gravity)

- **Spectral triple**: $(\mathcal{A}, \mathcal{H}, D)$
  1. Unital associative algebra $\mathcal{A}$
  2. Represented as bounded operators on a Hilbert space $\mathcal{H}$
  3. Dirac operator: self-adjoint $D^* = D$ with compact resolvent, with bounded commutators $[D, a]$

- Prototype: $(\mathcal{C}^\infty(M), L^2(M, S), D_M)$

- Extends to non smooth objects (fractals) and noncommutative (NC tori, quantum groups, NC deformations, etc.)
Action functional

- Suppose finitely summable \( ST = (\mathcal{A}, \mathcal{H}, D) \)

\[
\zeta_D(s) = \text{Tr}(|D|^{-s}) < \infty, \quad \Re(s) >> 0
\]

- Spectral action (Chamseddine–Connes)

\[
S_{ST}(\Lambda) = \text{Tr}(f(D/\Lambda)) = \sum_{\lambda \in \text{Spec}(D)} \text{Mult}(\lambda)f(\lambda/\Lambda)
\]

\( f = \) smooth approximation to (even) cutoff
Asymptotic expansion (Chamseddine–Connes) for (almost) commutative geometries:

\[ \text{Tr}(f(D/\Lambda)) \sim \sum_{\beta \in \Sigma_{ST}^+} f_{\beta} \Lambda^{\beta} \int |D|^{-\beta} + f(0) \zeta_D(0) \]

- Residues

\[ \int |D|^{-\beta} = \frac{1}{2} \text{Res}_{s=\beta} \zeta_D(s) \]

- Momenta \( f_{\beta} = \int_0^\infty f(v) v^{\beta-1} dv \)

- Dimension Spectrum \( \Sigma_{ST} \) poles of zeta functions

\[ \zeta_{a,D}(s) = \text{Tr}(a |D|^{-s}) \]

- positive dimension spectrum \( \Sigma_{ST}^+ = \Sigma_{ST} \cap \mathbb{R}^*_+ \)

Warning: for fractal spaces also oscillatory terms coming from part of \( \Sigma_{ST} \) off the real line
Zeta function and heat kernel (manifolds)

- **Mellin transform**

\[
|D|^{-s} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-tD^2} t^{s-1} dt
\]

- **heat kernel expansion**

\[
\text{Tr}(e^{-tD^2}) = \sum_\alpha t^\alpha c_\alpha \quad \text{for } t \to 0
\]

- **zeta function expansion**

\[
\zeta_D(s) = \text{Tr}(|D|^{-s}) = \sum_\alpha \frac{c_\alpha}{\Gamma(s/2)(\alpha + s/2)} + \text{holomorphic}
\]

- **taking residues**

\[
\text{Res}_{s=-2\alpha} \zeta_D(s) = \frac{2c_\alpha}{\Gamma(-\alpha)}
\]
Example spectral action of the round 3-sphere $S^3$

$$S_{S^3}(\Lambda) = \operatorname{Tr}(f(D_{S^3}/\Lambda)) = \sum_{n \in \mathbb{Z}} n(n+1)f((n + \frac{1}{2})/\Lambda)$$

- zeta function

$$\zeta_{D_{S^3}}(s) = 2\zeta(s - 2, \frac{3}{2}) - \frac{1}{2}\zeta(s, \frac{3}{2})$$

$\zeta(s, q) = \text{Hurwitz zeta function}$

- by asymptotic expansion

$$S_{S^3}(\Lambda) \sim \Lambda^3 f_3 - \frac{1}{4} \Lambda f_1$$

- can also compute using Poisson summation formula (Chamseddine–Connes): estimate error term $O(\Lambda^{-\infty})$
Example: round 3-sphere $S^3_a$ radius $a$

$$\zeta_{D_{S^3_a}}(s) = a^s(2\zeta(s - 2, \frac{3}{2}) - \frac{1}{2}\zeta(s, \frac{3}{2}))$$

$$S_{S^3_a}(\Lambda) \sim (\Lambda a)^3 f_3 - \frac{1}{4}(\Lambda a)f_1$$

Example: spherical space form $Y = S^3_a/\Gamma$ (Čačić, Marcolli, Teh)

$$S_Y(\Lambda) \sim \frac{1}{\#\Gamma} S_{S^3_a}(\Lambda)$$
Why a model of (Euclidean) Gravity?

- $M$ compact Riemannian 4-manifold

$$\text{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4$$

coefficients $a_0$, $a_2$ and $a_4$:
- cosmological term

$$f_4 \Lambda^4 \int |D|^{-4} = \frac{48 f_4 \Lambda^4}{\pi^2} \int \sqrt{g} \, d^4x$$

- Einstein–Hilbert term

$$f_2 \Lambda^2 \int |D|^{-2} = \frac{96 f_2 \Lambda^2}{24\pi^2} \int R \sqrt{g} \, d^4x$$

- modified gravity terms (Weyl curvature and Gauss–Bonnet)

$$f(0) \zeta_D(0) = \frac{f_0}{10\pi^2} \int \left( \frac{11}{6} R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \right) \sqrt{g} \, d^4x$$

$$C^{\mu\nu\rho\sigma} = \text{Weyl curvature and } R^* R^* = \frac{1}{4} \varepsilon_{\mu\nu\rho\sigma} \varepsilon^{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}_{\rho\sigma}$$

momenta: (effective) gravitational and cosmological constant
Spectral action on a fractal spacetime:

- $S^1_\beta \times \mathcal{P}$: Apollonian packing
- $S^1_\beta \times \mathcal{P}_Y$: fractal dodecahedral space

1. Construct a spectral triple for the geometries $\mathcal{P}$ and $\mathcal{P}_Y$
2. Compute the zeta function
3. Compute the asymptotic form of the spectral action
4. Effect of product with $S^1_\beta$

⇒ look for new terms in the spectral action (in addition to usual gravitational terms) that detect presence of fractality
The spectral triple of a fractal geometry

- case of Sierpinski gasket: Christensen, Ivan, Lapidus
- similar case for $\mathcal{P}$ and $\mathcal{P}_Y$
- for $D$-dim packing

$$\mathcal{P}_D = \{S_{a_n,k}^{D-1} : n \in \mathbb{N}, 1 \leq k \leq (D+2)(D+1)^{n-1}\}$$

$$(\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{\mathcal{P}_D}, \mathcal{D}_{\mathcal{P}_D}) = \bigoplus_{n,k}(\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{S_{a_n,k}^{D-1}}, \mathcal{D}_{S_{a_n,k}^{D-1}})$$

- for $\mathcal{P}_Y$ with $Y_a = S^3/\mathbb{I}_{120}$:

$$(\mathcal{A}_{\mathcal{P}_Y}, \mathcal{H}_{\mathcal{P}_Y}, \mathcal{D}_{\mathcal{P}_Y}) = (\mathcal{A}_{\mathcal{P}_Y}, \bigoplus_{n} \mathcal{H}_{Y_{a_n}}, \bigoplus_{n} \mathcal{D}_{Y_{a_n}})$$

with $a_n = a(2 + \phi)^{-n}$
Zeta functions for Apollonian packing of 3-spheres:

- **Lengths zeta function** (fractal string)

\[ \zeta_L(s) := \sum_{n \in \mathbb{N}} \sum_{k=1}^{6 \cdot 5^{n-1}} a_{n,k}^s \]

with \( L = L_4 = \{a_{n,k} | n \in \mathbb{N}, k \in \{1, \ldots, 6 \cdot 5^{n-1}\}\} \)

- **Zeta function of Dirac operator of the spectral triple**

\[ \text{Tr}(|D_P|^{-s}) = \sum_{n=1}^{\infty} \sum_{k=1}^{6 \cdot 5^{n-1}} \text{Tr}(|D_{S^3_{a_{n,k}}}|^{-s}) \]

Each term \( \text{Tr}(|D_{S^3_{a_{n,k}}}|^{-s}) = a_{n,k}^s (2\zeta(s - 2, \frac{3}{2}) - \frac{1}{2} \zeta(s, \frac{3}{2})) \) gives

\[ \text{Tr}(|D_P|^{-s}) = \left(2\zeta(s - 2, \frac{3}{2}) - \frac{1}{2} \zeta(s, \frac{3}{2})\right) \sum_{n,k} a_{n,k}^s \]

\[ = \left(2\zeta(s - 2, \frac{3}{2}) - \frac{1}{2} \zeta(s, \frac{3}{2})\right) \zeta_L(s) \]
Spectral action for Apollonian packing of 3-spheres:
(under good conditions on $\zeta_L(s)$)

- Positive Dimension Spectrum: $\Sigma_{ST_{PSC}}^+ = \{1, 3, \sigma_4(\mathcal{P})\}$
- Asymptotic spectral action

$$\text{Tr}(f(D_{\mathcal{P}}/\Lambda)) \sim \Lambda^3 \zeta_L(3) f_3 - \Lambda^{1/4} \zeta_L(1) f_1$$

$$+ \Lambda^\sigma \left( \zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4} \zeta(\sigma, \frac{3}{2}) \right) R_\sigma f_\sigma + S^{osc}_\Lambda$$

$\sigma = \sigma_4(\mathcal{P})$ packing constant; residue $R_\sigma = \text{Res}_{s=\sigma} \zeta_L(s)$, and momenta $f_\beta = \int_0^\infty \nu^{\beta-1} f(\nu) d\nu$

- Additional term $S^{osc}_\Lambda$ coming from series of contributions of poles of zeta function off the real line: oscillatory terms
Oscillatory terms (fractals)

- zeta function $\zeta_{\mathcal{L}}(s)$ on fractals in general has additional poles off the real line (position depends on Hausdorff and spectral dimension: depending on how homogeneous the fractal)
- best case exact self-similarity: $s = \sigma + \frac{2\pi im}{\log \ell}, m \in \mathbb{Z}$
- heat kernel on fractals has additional log-oscillatory terms in expansion
\[
\frac{C}{t^\sigma} (1 + A \cos \left( \frac{2\pi}{\log \ell} \log t + \phi \right)) + \cdots
\]
for constants $C, A, \phi$: series of terms for each complex pole
Log-oscillatory terms in expansion of the spectral action:


effect of product with $S^1_{\beta}$ (leading term without oscillations)

- case of $S^1_{\beta} \times S^3_a$ (Chamseddine–Connes)

$$DS^1_{\beta} \times S^3_a = \begin{pmatrix} 0 & DS^3_a \otimes 1 + i \otimes DS^1_{\beta} \\ DS^3_a \otimes 1 - i \otimes DS^1_{\beta} & 0 \end{pmatrix}$$

Spectral action

$$\text{Tr}(h(D^2_{S^1_{\beta} \times S^3_a}/\Lambda)) \sim 2\beta \Lambda \text{Tr}(\kappa(D^2_{S^3_a}/\Lambda)),$$

test function $h(x)$, and test function

$$\kappa(x^2) = \int_{\mathbb{R}} h(x^2 + y^2) dy$$
• Case of $S_{\beta}^{1} \times P$:

$$S_{\beta}^{1} \times P(\Lambda) \sim 2\beta \left( \Lambda^{4} \zeta_{L}(3) \mathcal{h}_{3} - \Lambda^{2} \frac{1}{4} \zeta_{L}(1) \mathcal{h}_{1} \right)$$

$$+ 2\beta \Lambda^{\sigma+1} \left( \zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4} \zeta(\sigma, \frac{3}{2}) \right) R_{\sigma} \mathcal{h}_{\sigma}$$

with momenta

$$\mathcal{h}_{3} := \pi \int_{0}^{\infty} h(\rho^{2})\rho^{3} d\rho, \quad \mathcal{h}_{1} := 2\pi \int_{0}^{\infty} h(\rho^{2})\rho d\rho$$

$$\mathcal{h}_{\sigma} = 2 \int_{0}^{\infty} h(\rho^{2})\rho^{\sigma} d\rho$$
Interpretation:

- Term $2\Lambda^4 \beta a^3 h_3 - \frac{1}{2} \Lambda^2 \beta a h_1$, cosmological and Einstein–Hilbert terms, replaced by
  \[ 2\Lambda^4 \beta \zeta_L(3) h_3 - \frac{1}{2} \Lambda^2 \beta \zeta_L(1) h_1 \]
  zeta regularization of divergent series of spectral actions of 3-spheres of packing
- Additional term in gravity action functional: corrections to gravity from fractality
  \[ 2\beta \Lambda^{\sigma+1} \left( \zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4} \zeta(\sigma, \frac{3}{2}) \right) R_\sigma h_\sigma \]
Case of fractal dodecahedral space $\mathcal{P}_Y$

- Zeta functions

$$\zeta_L(\mathcal{P}_Y)(s) = \sum_{n \geq 0} 20^n (2 + \phi)^{-ns}$$

$$\zeta_{\mathcal{D}_Y}(s) = \frac{a^s}{120} \left( 2\zeta(s - 2, \frac{3}{2}) - \frac{1}{2} \zeta(s, \frac{3}{2}) \right) \zeta_L(\mathcal{P}_Y)(s)$$

- Spectral action:

$$\text{Tr}(f(\mathcal{D}_{\mathcal{P}_Y}/\Lambda)) \sim (\Lambda a)^3 \frac{\zeta_L(\mathcal{P}_Y)(3)}{120} f_3 - \Lambda a \frac{\zeta_L(\mathcal{P}_Y)(1)}{120} f_1$$

$$+ (\Lambda a)^\sigma \frac{\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4} \zeta(\sigma, \frac{3}{2})}{120 \log(2 + \phi)} f_\sigma + S_{\mathcal{Y},\Lambda}^{osc}$$

$$\sigma = \dim_H(\mathcal{P}_Y) = \frac{\log(20)}{\log(2 + \phi)} = 2.3296...$$
• on product geometry \( S^1_{\beta} \times \mathcal{P}_Y \)

\[
S_{S^1_{\beta} \times \mathcal{P}_Y}(\Lambda) \sim 2\beta \left( \Lambda^4 \frac{a^3 \zeta_L(\mathcal{P}_Y)(3)}{120} \hbar_3 - \Lambda^2 \frac{a \zeta_L(\mathcal{P}_Y)(1)}{120} \hbar_1 \right) \\
+ 2\beta \Lambda^{\sigma+1} \frac{a^\sigma(\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4} \zeta(\sigma, \frac{3}{2}))}{120 \log(2 + \phi)} \hbar_\sigma + S^{osc}_{S^1_{\beta} \times Y, \Lambda}
\]

• **Note**: correction term now at different \( \sigma \) than Apollonian \( \mathcal{P} \)

• oscillatory terms \( S^{osc}_{Y, \Lambda} \) more explicit than in the Apollonian case
Oscillatory terms: dodecahedral case

• zeros of zeta function $\zeta_L(s)$

$$s_m = \sigma + \frac{2\pi im}{\log(2 + \phi)}, \quad m \in \mathbb{Z}$$

with $\sigma = \log(20)/\log(2 + \phi)$

• contribution to heat kernel expansion of non-real zeros:

$$\mathcal{C} \frac{t^\sigma}{(a_0 + 2\Re(a_1 t^{-2\pi i/\log(2+\phi)}) + \ldots)}$$

with coefficients $a_m$ proportional to $\Gamma(s_m)$: for fixed real part $\sigma$ decays exponentially fast along vertical line

• oscillatory terms are small
Slow-roll inflation potential from the spectral action

- perturb the Dirac operator by a scalar field $D^2 + \phi^2 \Rightarrow$ spectral action gives potential $V(\phi)$

- shape of $V(\phi)$ distinguishes most cosmic topologies: spherical forms and Bieberbach manifolds (Marcolli, Pierpaoli, Teh)
Fractality corrections to potential $V(\phi)$

- additional term in potential

$$U_\sigma(x) = \int_0^\infty u^{(\sigma-1)/2}(h(u + x) - h(u))du$$

depends on $\sigma$ fractal dimension

- size of correction depends on (leading term)

$$\left(\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2})\right)R_\sigma$$

- further corrections to $U_\sigma$ come from the oscillatory terms

$\Rightarrow$ presence of fractality (in this spectral action model of gravity) can be read off the slow-roll potential (hence the slow-roll coefficients, which depend on $V, V', V''$)