# Spectral Action Models of Gravity and Packed Swiss Cheese Cosmologies

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Based on:

• Adam Ball, Matilde Marcolli, *Spectral Action Models of Gravity on Packed Swiss Cheese Cosmology*, arXiv:1506.01401

## Homogeneity versus Isotropy in Cosmology

- Homogeneous and isotropic: Friedmann universe $\mathbb{R} \times S^3$

$$\pm dt^2 + a(t)^2 \left(\sigma_1^2 + \sigma_2^2 + \sigma_3^2\right)$$

with round metric on $S^3$ with $SU(2)$–invariant 1-forms $\{\sigma_i\}$ satisfying relations

$$d\sigma_i = \sigma_j \wedge \sigma_k$$

for all cyclic permutations $(i, j, k)$ of $(1, 2, 3)$

• Homogeneous but not isotropic:
Bianchi IX mixmaster models $\mathbb{R} \times S^3$

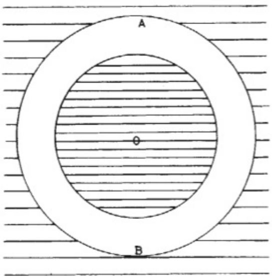
$$F(t)\,(\pm dt^2 + \frac{\sigma_1^2}{W_1^2(t)} + \frac{\sigma_2^2}{W_2^2(t)} + \frac{\sigma_3^2}{W_3^3(t)})$$

with a conformal factor $F(t) \sim W_1(t)W_2(t)W_3(t)$

• Isotropic but not homogeneous?
$\Rightarrow$ Swiss Cheese Models

Main Idea:

• M.J. Rees, D.W. Sciama, *Large-scale density inhomogeneities in the universe*, Nature, Vol.217 (1968) 511–516.




Cut off 4-balls from a FRW spacetime and replace with different density smaller region outside/inside patched across boundary with vanishing Weyl curvature tensor (isotropy preserved)

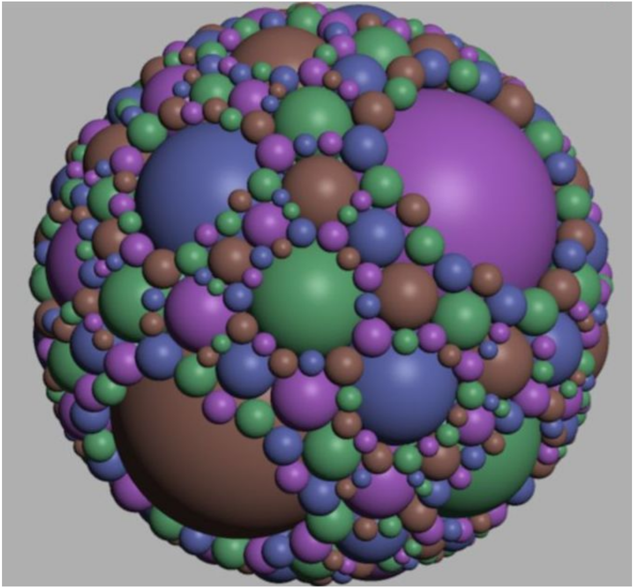
## Packed Swiss Cheese Cosmology

• Iterate construction removing more and more balls $\Rightarrow$ Apollonian sphere packing of 3-dimensional spheres

• Residual set of sphere packing is fractal

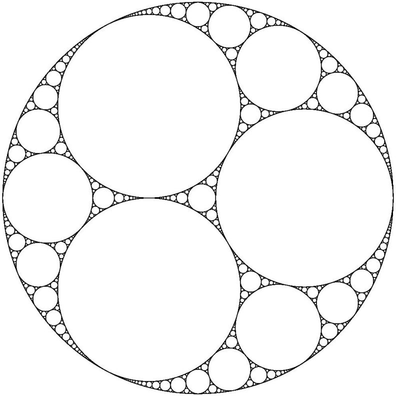
• Proposed as explanation for possible fractal distribution of matter in galaxies, clusters, and superclusters


- F. Sylos Labini, M. Montuori, L. Pietroneo, *Scale-invariance of galaxy clustering*, Phys. Rep. Vol. 293 (1998) N. 2-4, 61–226.
- J.R. Mureika, C.C. Dyer, *Multifractal analysis of Packed Swiss Cheese Cosmologies*, General Relativity and Gravitation, Vol.36 (2004) N.1, 151–184.



## Apollonian sphere packings

- best known and understood case: Apollonian circle packing



Configurations of mutually tanget circles in the plane, iterated on smaller scales filling a full volume region in the unit $2D$ ball: residual set volume zero fractal of Hausdorff dimension 1.30568...

• Many results (geometric, arithmetic, analytic) known about Apollonian circle packings: see for example

- R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks, C.H.Yan, *Apollonian circle packings: number theory*, J. Number Theory 100 (2003) 1–45
- A. Kontorovich, H. Oh, *Apollonian circle packings and closed horospheres on hyperbolic 3-manifolds*, Journal of AMS, Vol 24 (2011) 603–648.

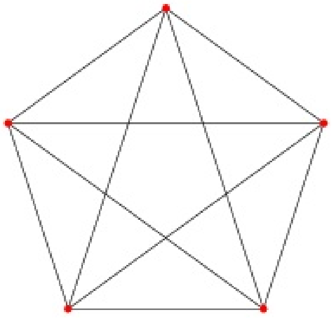
• Higher dimensional analogs of Apollonian packings: much more delicate and complicated geometry

- R.L. Graham, J.C. Lagarias, C.L. Mallows, A.R. Wilks, C.H.Yan, *Apollonian Circle Packings: Geometry and Group Theory III. Higher Dimensions*, Discrete Comput. Geom. 35 (2006) 37–72.

## Some known facts on Apollonian sphere packings

• Descartes configuration in $D$ dimensions: $D+2$ mutually tangent $(D-1)$-dimensional spheres

• Example: start with $D+1$ equal size mutually tangent $S^{D-1}$ centered at the vertices of $D$-simplex and one more smaller sphere in the center tangent to all



4-dimensional simplex

• Quadratic Soddy–Gosset relation between radii $a_k$

$$\left(\sum_{k=1}^{D+2} \frac{1}{a_k}\right)^2 = D \sum_{k=1}^{D+2} \left(\frac{1}{a_k}\right)^2$$

• curvature-center coordinates: $(D+2)$-vector

$$w = (\frac{\|x\|^2 - a^2}{a}, \frac{1}{a}, \frac{1}{a}x_1, \ldots, \frac{1}{a}x_D)$$

(first coordinate curvature after inversion in the unit sphere)

• Configuration space $\mathcal{M}_D$ of all Descartes configuration in $D$ dimensions = all solutions $\mathcal{W}$ to equation

$$\mathcal{W}^t \, Q_D \, \mathcal{W} = \begin{pmatrix} 0 & -4 & 0 \\ -4 & 0 & 0 \\ 0 & 0 & 2\, I_D \end{pmatrix}$$

with left and a right action of Lorentz group $O(D+1,1)$

• Dual Apollonian group $\mathcal{G}_D^\perp$ generated by reflections: inversion with respect to the $j$-th sphere

$$S_j^\perp = I_{D+2} + 2\,1_{D+2} e_j^t - 4\, e_j e_j^t$$

$e_j = j$-th unit coordinate vector

• $D \neq 3$: only relations in $\mathcal{G}_D^\perp$ are $(S_j^\perp)^2 = 1$

• $\mathcal{G}_D^\perp$ discrete subgroup of $\mathrm{GL}(D+2, \mathbb{R})$

• Apollonian packing $\mathcal{P}_D =$ an orbit of $\mathcal{G}_D^\perp$ on $\mathcal{M}_D$

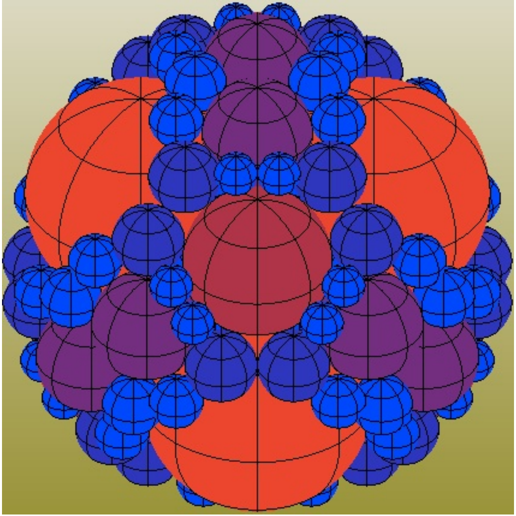
$\Rightarrow$ iterative construction: at $n$-th step add spheres obtained from initial Descartes configuration via all possible

$$S_{j_1}^\perp S_{j_2}^\perp \cdots S_{j_n}^\perp, \quad j_k \neq j_{k+1}, \forall k$$

there are $N_n$ spheres in the $n$-th level

$$N_n = (D+2)(D+1)^{n-1}$$

## iterative construction of sphere packings



- Length spectrum: radii of spheres in packing $\mathcal{P}_D$

$$\mathcal{L} = \mathcal{L}(\mathcal{P}_D) = \{a_{n,k} \,:\, n \in \mathbb{N}, 1 \leq k \leq (D+2)(D+1)^{n-1}\}$$

radii of spheres $S^{D-1}_{a_{n,k}}$

- Melzak's packing constant $\sigma_D(\mathcal{P}_D)$ exponent of convergence of series

$$\zeta_{\mathcal{L}}(s) = \sum_{n=1}^{\infty} \sum_{k=1}^{(D+2)(D+1)^{n-1}} a_{n,k}^s$$

- Residual set: $\mathcal{R}(\mathcal{P}_D) = B^D \smallsetminus \cup_{n,k} B^D_{a_{n,k}}$ with $\partial B^D_{a_{n,k}} = S^{D-1}_{a_{n,k}} \in \mathcal{P}_D$
- Packing $\Rightarrow \mathrm{Vol}_D(\mathcal{R}(\mathcal{P}_D)) = 0 \Rightarrow \sum_{\mathcal{L}} a_{n,k}^D < \infty \Rightarrow \sigma_D(\mathcal{P}_D) \leq D$
- packing constant and Hausdorff dimension:

$$\dim_H(\mathcal{R}(\mathcal{P}_D)) \leq \sigma_D(\mathcal{P}_D)$$

for Apollonian circles known to be same

- Sphere counting function: spheres with given curvature bound

$$\mathcal{N}_\alpha(\mathcal{P}_D) = \#\{S^{D-1}_{a_{n,k}} \in \mathcal{P}_D \,:\, a_{n,k} \geq \alpha\}$$

curvatures $c_{n,k} = a_{n,k}^{-1} \leq \alpha^{-1}$

- for Apollonian circles power law (Kontorovich–Oh)

$$\mathcal{N}_\alpha(\mathcal{P}_2) \sim_{\alpha\to 0} \alpha^{-\dim_H(\mathcal{R}(\mathcal{P}_2))}$$

- for higher dimensions (Boyd): packing constant

$$\limsup_{\alpha\to 0} -\frac{\log \mathcal{N}_\alpha(\mathcal{P}_D)}{\log \alpha} = \sigma_D(\mathcal{P}_D)$$

if limit exists $\mathcal{N}_\alpha(\mathcal{P}_D) \sim_{\alpha\to 0} \alpha^{-(\sigma_D(\mathcal{P}_D)+o(1))}$

## Screens and Windows

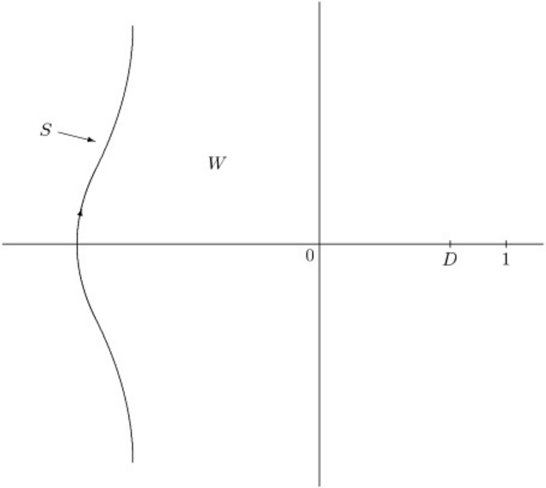
• in general $\zeta_{\mathcal{L}_D}(s)$ need have analytic continuation to meromorphic on whole $\mathbb{C}$

• $\exists$ *screen* $\mathcal{S}$: curve $S(t) + it$ with $S : \mathbb{R} \to (-\infty, \sigma_D(\mathcal{P}_D)]$

• *window* $\mathcal{W}$ = region to the right of screen $\mathcal{S}$ where analytic continuation


- M.L. Lapidus, M. van Frankenhuijsen, *Fractal geometry, complex dimensions and zeta functions. Geometry and spectra of fractal strings*, Second edition. Springer Monographs in Mathematics. Springer, 2013.

# Screens and windows



## Some additional assumptions

- Definition:

Apollonian packing $\mathcal{P}_D$ of $(D-1)$-spheres is *analytic* if

1. $\zeta_{\mathcal{L}}(s)$ has analytic to meromorphic function on a region $\mathcal{W}$ containing $\mathbb{R}_+$
2. $\zeta_{\mathcal{L}}(s)$ has only one pole on $\mathbb{R}_+$ at $s = \sigma_D(\mathcal{P}_D)$.
3. pole at $s = \sigma_D(\mathcal{P}_D)$ is simple

- Also assume: $\exists \lim_{\alpha\to 0} -\frac{\log \mathcal{N}_\alpha(\mathcal{P}_D)}{\log \alpha} = \sigma_D(\mathcal{P}_D)$
- Question: in general when are these satisfied for packings $\mathcal{P}_D$?
- focus on $D = 4$ cases with these conditions

# Rough estimate of the packing constant

- $\mathcal{P} = \mathcal{P}_4$ Apollonian packing of 3-spheres $S^3_{a_{n,k}}$
- at level $n$: average curvature

$$\frac{\gamma_n}{N_n} = \frac{1}{6 \cdot 5^{n-1}} \sum_{k=1}^{6\cdot 5^{n-1}} \frac{1}{a_{n,k}}$$

- estimate $\sigma_4(\mathcal{P}_4)$ with averaged version: $\sum_n N_n(\frac{\gamma_n}{N_n})^{-s}$

$$\sigma_{4,av}(\mathcal{P}) = \lim_{n\to\infty} \frac{\log(6 \cdot 5^{n-1})}{\log\left(\frac{\gamma_n}{6\cdot 5^{n-1}}\right)}$$

- generating function of the $\gamma_n$ known (Mallows)

$$G_{D=4} = \sum_{n=1}^{\infty} \gamma_n x^n = \frac{(1-x)(1-4x)u}{1 - \frac{22}{3}x - 5x^2}$$

$u$ = sum of the curvatures of initial Descartes configuration

• obtain explicitly ($u = 1$ case)

$$\gamma_n = \frac{(11+\sqrt{166})^n(-64+9\sqrt{166})+(11-\sqrt{166})^n(64+9\sqrt{166})}{3^n \cdot 10 \cdot \sqrt{166}}$$

• this gives a value

$$\sigma_{4,av}(\mathcal{P}) = 3.85193\ldots$$

• in Apollonian circle case where $\sigma(\mathcal{P})$ known this method gives larger value, so expect $\sigma_4(\mathcal{P}) < \sigma_{4,av}(\mathcal{P})$

• constraints on the packing constant:

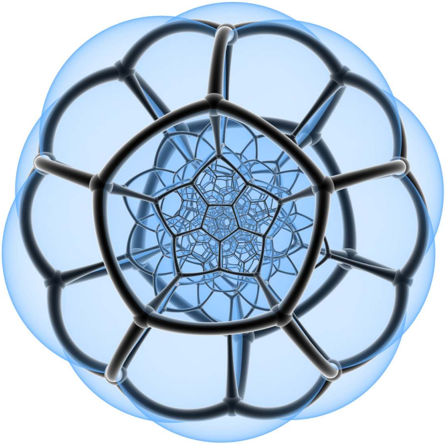
$$3 < \dim_H(\mathcal{R}(\mathcal{P})) \leq \sigma_4(\mathcal{P}) < \sigma_{4,av}(\mathcal{P}) = 3.85193\ldots$$

## Models of (Euclidean, compactified) spacetimes

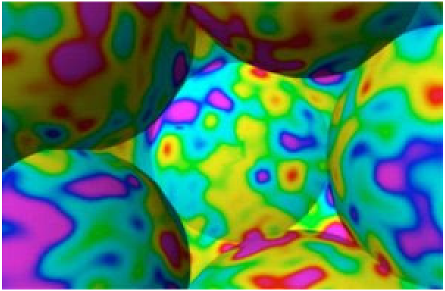
1. Homogeneous Isotropic cases: $S^1_\beta \times S^3_a$
2. Cosmic Topology cases: $S^1_\beta \times Y$ with $Y$ a spherical space form $S^3/\Gamma$ or a flat Bieberbach manifold $T^3/\Gamma$ (modulo finite groups of isometries)
3. Packed Swiss Cheese: $S^1_\beta \times \mathcal{P}$ with Apollonian packing of 3-spheres $S^3_{a_{n,k}}$
4. Fractal arrangements with cosmic topology

# Fractal arrangements with cosmic topology

- Example: Poincaré homology sphere, dodecahedral space $S^3/\mathcal{I}_{120}$, fundamental domain dodecahedron

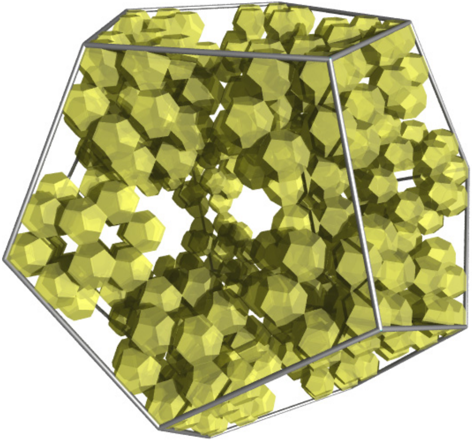


- considered a likely candidate for cosmic topology
  - S. Caillerie, M. Lachièze-Rey, J.P. Luminet, R. Lehoucq, A. Riazuelo, J. Weeks, *A new analysis of the Poincaré dodecahedral space model*, Astron. and Astrophys. 476 (2007) N.2, 691–696



- build a fractal model based on dodecahedral space

# Fractal configurations of dodecahedra (Sierpinski dodecahedra)



• spherical dodecahedron has $Vol(Y) = Vol(S_a^3/\mathcal{I}_{120}) = \frac{\pi^2}{60} a^3$

• simpler than sphere packings because uniform scaling at each step: $20^n$ new dodecahedra, each scaled by a factor of $(2+\phi)^{-n}$

$$\dim_H(\mathcal{P}_{\mathcal{I}_{120}}) = \frac{\log(20)}{\log(2+\phi)} = 2.32958\ldots$$

• close up all dodecahedra in the fractal identifying edges with $\mathcal{I}_{120}$: get fractal arrangement of Poincaré spheres $Y_{a(2+\phi)^{-n}}$

• zeta function has analytic continuation to all $\mathbb{C}$

$$\zeta_{\mathcal{L}}(s) = \sum_n 20^n \, (2+\phi)^{-ns} = \frac{1}{1 - 20(2+\phi)^{-s}}$$

exponent of convergence $\sigma = \dim_H(\mathcal{P}_{\mathcal{I}_{120}}) = \frac{\log(20)}{\log(2+\phi)}$ and poles

$$\sigma + \frac{2\pi i m}{\log(2+\phi)}, \quad m \in \mathbb{Z}$$

Spectral action models of gravity (modified gravity)

- Spectral triple: $(\mathcal{A}, \mathcal{H}, D)$

1. unital associative algebra $\mathcal{A}$
2. represented as bounded operators on a Hilbert space $\mathcal{H}$
3. Dirac operator: self-adjoint $D^* = D$ with compact resolvent, with bounded commutators $[D, a]$

- prototype: $(C^\infty(M), L^2(M, S), \not{D}_M)$

- extends to non smooth objects (fractals) and noncommutative (NC tori, quantum groups, NC deformations, etc.)

## Action functional

- Suppose *finitely summable* $ST = (\mathcal{A}, \mathcal{H}, D)$

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) < \infty, \quad \Re(s) >> 0$$

- Spectral action (Chamseddine–Connes)

$$\mathcal{S}_{ST}(\Lambda) = \mathrm{Tr}(f(D/\Lambda)) = \sum_{\lambda \in \mathrm{Spec}(D)} \mathrm{Mult}(\lambda) f(\lambda/\Lambda)$$

$f$ = smooth approximation to (even) cutoff

Asymptotic expansion (Chamseddine–Connes) for (almost) commutative geometries:

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\beta \in \Sigma_{ST}^+} f_\beta \, \Lambda^\beta \, ⨍ |D|^{-\beta} \; + \; f(0) \, \zeta_D(0)$$

- Residues

$$⨍ |D|^{-\beta} = \frac{1}{2} \mathrm{Res}_{s=\beta} \; \zeta_D(s)$$

- Momenta $f_\beta = \int_0^\infty f(v) \, v^{\beta-1} \, dv$

- Dimension Spectrum $\Sigma_{ST}$ poles of zeta functions $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$

- positive dimension spectrum $\Sigma_{ST}^+ = \Sigma_{ST} \cap \mathbb{R}_+^*$

Warning: for fractal spaces also oscillatory terms coming from part of $\Sigma_{ST}$ off the real line

## Zeta function and heat kernel (manifolds)

- Mellin transform

$$|D|^{-s} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-tD^2} t^{\frac{s}{2}-1} dt$$

- heat kernel expansion

$$\mathrm{Tr}(e^{-tD^2}) = \sum_\alpha t^\alpha c_\alpha \quad \text{for} \quad t \to 0$$

- zeta function expansion

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) = \sum_\alpha \frac{c_\alpha}{\Gamma(s/2)(\alpha + s/2)} + \text{holomorphic}$$

- taking residues

$$\mathrm{Res}_{s=-2\alpha} \zeta_D(s) = \frac{2c_\alpha}{\Gamma(-\alpha)}$$

Example spectral action of the round 3-sphere $S^3$

$$\mathcal{S}_{S^3}(\Lambda) = \mathrm{Tr}(f(D_{S^3}/\Lambda)) = \sum_{n\in\mathbb{Z}} n(n+1) f((n+\frac{1}{2})/\Lambda)$$

- zeta function

$$\zeta_{D_{S^3}}(s) = 2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2})$$

$\zeta(s,q) =$ Hurwitz zeta function

- by asymptotic expansion

$$\mathcal{S}_{S^3}(\Lambda) \sim \Lambda^3 f_3 - \frac{1}{4}\Lambda f_1$$

- can also compute using Poisson summation formula (Chamseddine–Connes): estimate error term $O(\Lambda^{-\infty})$

Example: round 3-sphere $S^3_a$ radius $a$

$$\zeta_{D_{S^3_a}}(s) = a^s(2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2}))$$

$$\mathcal{S}_{S^3_a}(\Lambda) \sim (\Lambda a)^3 f_3 - \frac{1}{4}(\Lambda a) f_1$$

Example: spherical space form $Y = S^3_a/\Gamma$ (Ćaćić, Marcolli, Teh)

$$\mathcal{S}_Y(\Lambda) \sim \frac{1}{\#\Gamma}\,\mathcal{S}_{S^3_a}(\Lambda)$$

## Why a model of (Euclidean) Gravity?

- $M$ compact Riemannian 4-manifold

$$\mathrm{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4$$

coefficients $a_0$, $a_2$ and $a_4$:

- cosmological term

$$f_4 \Lambda^4 \int\!\!\!\!\!\!- |D|^{-4} = \frac{48 f_4 \Lambda^4}{\pi^2} \int \sqrt{g}\, d^4x$$

- Einstein–Hilbert term

$$f_2 \Lambda^2 \int\!\!\!\!\!\!- |D|^{-2} = \frac{96\, f_2 \Lambda^2}{24\pi^2} \int R \sqrt{g}\, d^4x$$

- modified gravity terms (Weyl curvature and Gauss–Bonnet)

$$f(0)\, \zeta_D(0) = \frac{f_0}{10\pi^2} \int (\frac{11}{6} R^* R^* - 3 C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma}) \sqrt{g}\, d^4x$$

$C^{\mu\nu\rho\sigma}$ = Weyl curvature and $R^* R^* = \frac{1}{4} \epsilon^{\mu\nu\rho\sigma} \epsilon_{\alpha\beta\gamma\delta} R^{\alpha\beta}_{\mu\nu} R^{\gamma\delta}_{\rho\sigma}$

momenta: (effective) gravitational and cosmological constant

Spectral action on a fractal spacetime:

- $S^1_\beta \times \mathcal{P}$: Apollonian packing
- $S^1_\beta \times \mathcal{P}_Y$: fractal dodecahedral space

1. Construct a spectral triple for the geometries $\mathcal{P}$ and $\mathcal{P}_Y$
2. Compute the zeta function
3. Compute the asymptotic form of the spectral action
4. Effect of product with $S^1_\beta$

$\Rightarrow$ look for new terms in the spectral action (in additional to usual gravitational terms) that detect presence of fractality

## The spectral triple of a fractal geometry

- case of Sierpinski gasket: Christensen, Ivan, Lapidus
- similar case for $\mathcal{P}$ and $\mathcal{P}_Y$
- for $D$-dim packing

$$\mathcal{P}_D = \{S^{D-1}_{a_{n,k}} \,:\, n \in \mathbb{N},\, 1 \leq k \leq (D+2)(D+1)^{n-1}\}$$

$$(\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{\mathcal{P}_D}, \mathcal{D}_{\mathcal{P}_D}) = \oplus_{n,k}(\mathcal{A}_{\mathcal{P}_D}, \mathcal{H}_{S^{D-1}_{a_{n,k}}}, \mathcal{D}_{S^{D-1}_{a_{n,k}}})$$

- for $\mathcal{P}_Y$ with $Y_a = S^3/\mathcal{I}_{120}$:

$$(\mathcal{A}_{\mathcal{P}_Y}, \mathcal{H}_{\mathcal{P}_Y}, \mathcal{D}_{\mathcal{P}_Y}) = (\mathcal{A}_{\mathcal{P}_Y}, \oplus_n \mathcal{H}_{Y_{a_n}}, \oplus_n D_{Y_{a_n}})$$

with $a_n = a(2+\phi)^{-n}$

Zeta functions for Apollonian packing of 3-spheres:

- Lengths zeta function (fractal string)

$$\zeta_{\mathcal{L}}(s) := \sum_{n\in\mathbb{N}} \sum_{k=1}^{6\cdot 5^{n-1}} a_{n,k}^s$$

with $\mathcal{L} = \mathcal{L}_4 = \{a_{n,k} \,|\, n \in \mathbb{N},\ k \in \{1, \dots, 6\cdot 5^{n-1}\}\}$

- zeta function of Dirac operator of the spectral triple

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \sum_{n=1}^{\infty} \sum_{k=1}^{6\cdot 5^{n-1}} \mathrm{Tr}(|D_{S^3_{a_{n,k}}}|^{-s})$$

each term $\mathrm{Tr}(|D_{S^3_{a_{n,k}}}|^{-s}) = a_{n,k}^s(2\zeta(s-2, \frac{3}{2}) - \frac{1}{2}\zeta(s, \frac{3}{2}))$ gives

$$\mathrm{Tr}(|\mathcal{D}_{\mathcal{P}}|^{-s}) = \left(2\zeta(s-2, \frac{3}{2}) - \frac{1}{2}\zeta(s, \frac{3}{2})\right) \sum_{n,k} a_{n,k}^s$$

$$= \left(2\zeta(s-2, \frac{3}{2}) - \frac{1}{2}\zeta(s, \frac{3}{2})\right) \zeta_{\mathcal{L}}(s)$$

Spectral action for Apollonian packing of 3-spheres:
(under good conditions on $\zeta_{\mathcal{L}}(s)$)

- Positive Dimension Spectrum: $\Sigma^+_{ST_{PSC}} = \{1, 3, \sigma_4(\mathcal{P})\}$

- asymptotic spectral action

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) \sim \Lambda^3 \, \zeta_{\mathcal{L}}(3) \, f_3 - \Lambda \, \frac{1}{4} \, \zeta_{\mathcal{L}}(1) \, f_1$$

$$+\Lambda^\sigma \left(\zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2})\right) \mathcal{R}_\sigma \, f_\sigma + \mathcal{S}^{osc}_\Lambda$$

$\sigma = \sigma_4(\mathcal{P})$ packing constant; residue $\mathcal{R}_\sigma = \mathrm{Res}_{s=\sigma}\zeta_{\mathcal{L}}(s)$, and momenta $f_\beta = \int_0^\infty v^{\beta-1} f(v) dv$

- additional term $\mathcal{S}^{osc}_\Lambda$ coming from series of contributions of poles of zeta function off the real line: oscillatory terms

## Oscillatory terms (fractals)

• zeta function $\zeta_{\mathcal{L}}(s)$ on fractals in general has additional poles off the real line (position depends on Hausdorff and spectral dimension: depending on how homogeneous the fractal)

• best case exact self-similarity: $s = \sigma + \frac{2\pi i m}{\log \ell}$, $m \in \mathbb{Z}$

• heat kernel on fractals has additional log-oscillatory terms in expansion

$$\frac{C}{t^{\sigma}}(1 + A\cos(\frac{2\pi}{\log \ell}\log t + \phi)) + \cdots$$

for constants $C, A, \phi$: series of terms for each complex pole

Log-oscillatory terms in expansion of the spectral action:

• G.V. Dunne, *Heat kernels and zeta functions on fractals*, J. Phys. A 45 (2012) 374016 [22p]

• M. Eckstein, B. Iochum, A. Sitarz, *Heat kernel and spectral action on the standard Podlés sphere*, Comm. Math. Phys. 332 (2014) 627–668

• M. Eckstein, A. Zajaç, *Asymptotic and exact expansion of heat traces*, arXiv:1412.5100 effect of product with $S^1_\beta$ (leading term without oscillations)

- case of $S^1_\beta \times S^3_a$ (Chamseddine–Connes)

$$D_{S^1_\beta \times S^3_a} = \begin{pmatrix} 0 & D_{S^3_a} \otimes 1 + i \otimes D_{S^1_\beta} \\ D_{S^3_a} \otimes 1 - i \otimes D_{S^1_\beta} & 0 \end{pmatrix}$$

Spectral action

$$\mathrm{Tr}(h(D^2_{S^1_\beta \times S^3_a}/\Lambda)) \sim 2\beta\Lambda \mathrm{Tr}(\kappa(D^2_{S^3_a}/\Lambda)),$$

test function $h(x)$, and test function

$$\kappa(x^2) = \int_{\mathbb{R}} h(x^2 + y^2) dy$$

- Case of $S^1_\beta \times \mathcal{P}$:

$$\mathcal{S}_{S^1_\beta \times \mathcal{P}}(\Lambda) \sim 2\beta \left( \Lambda^4 \, \zeta_{\mathcal{L}}(3) \, \mathfrak{h}_3 - \Lambda^2 \, \frac{1}{4} \, \zeta_{\mathcal{L}}(1) \, \mathfrak{h}_1 \right)$$

$$+2\beta \, \Lambda^{\sigma+1} \, \left( \zeta(\sigma - 2, \frac{3}{2}) - \frac{1}{4} \zeta(\sigma, \frac{3}{2}) \right) \, \mathcal{R}_\sigma \, \mathfrak{h}_\sigma$$

with momenta

$$\mathfrak{h}_3 := \pi \int_0^\infty h(\rho^2) \rho^3 d\rho, \quad \mathfrak{h}_1 := 2\pi \int_0^\infty h(\rho^2) \rho d\rho$$

$$\mathfrak{h}_\sigma = 2 \int_0^\infty h(\rho^2) \rho^\sigma d\rho$$

Interpretation:

- Term $2\Lambda^4\beta a^3\mathfrak{h}_3 - \frac{1}{2}\Lambda^2\beta a\mathfrak{h}_1$, cosmological and Einstein–Hilbert terms, replaced by

$$2\Lambda^4\beta\zeta_{\mathcal{L}}(3)\mathfrak{h}_3 - \frac{1}{2}\Lambda^2\beta\zeta_{\mathcal{L}}(1)\mathfrak{h}_1$$

  zeta regularization of divergent series of spectral actions of 3-spheres of packing

- Additional term in gravity action functional: corrections to gravity from fractality

$$2\beta\,\Lambda^{\sigma+1}\left(\zeta(\sigma-2,\frac{3}{2}) - \frac{1}{4}\zeta(\sigma,\frac{3}{2})\right)\mathcal{R}_\sigma\mathfrak{h}_\sigma$$

## Case of fractal dodecahedral space $\mathcal{P}_Y$

- Zeta functions

$$\zeta_{\mathcal{L}(\mathcal{P}_Y)}(s) = \sum_{n\geq 0} 20^n (2+\phi)^{-ns}$$

$$\zeta_{\mathcal{D}_{\mathcal{P}_Y}}(s) = \frac{a^s}{120}\left(2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2})\right)\zeta_{\mathcal{L}(\mathcal{P}_Y)}(s)$$

- Spectral action:

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}_Y}/\Lambda)) \sim (\Lambda a)^3 \frac{\zeta_{\mathcal{L}(\mathcal{P}_Y)}(3)}{120} f_3 - \Lambda a \frac{\zeta_{\mathcal{L}(\mathcal{P}_Y)}(1)}{120} f_1$$

$$+(\Lambda a)^\sigma \frac{\zeta(\sigma-2,\frac{3}{2}) - \frac{1}{4}\zeta(\sigma,\frac{3}{2})}{120\log(2+\phi)} f_\sigma + \mathcal{S}^{osc}_{Y,\Lambda}$$

$\sigma = \dim_H(\mathcal{P}_Y) = \frac{\log(20)}{\log(2+\phi)} = 2.3296...$

- on product geometry $S^1_\beta \times \mathcal{P}_Y$

$$\mathcal{S}_{S^1_\beta \times \mathcal{P}_Y}(\Lambda) \sim 2\beta \left( \Lambda^4 \frac{a^3 \zeta_{\mathcal{L}(\mathcal{P}_Y)}(3)}{120} \mathfrak{h}_3 - \Lambda^2 \frac{a \zeta_{\mathcal{L}(\mathcal{P}_Y)}(1)}{120} \mathfrak{h}_1 \right)$$

$$+2\beta\, \Lambda^{\sigma+1} \frac{a^\sigma (\zeta(\sigma-2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2}))}{120 \log(2+\phi)} \mathfrak{h}_\sigma + \mathcal{S}^{osc}_{S^1_\beta \times Y, \Lambda}$$

- Note: correction term now at different $\sigma$ than Apollonian $\mathcal{P}$
- oscillatory terms $\mathcal{S}^{osc}_{Y,\Lambda}$ more explicit than in the Apollonian case

## Oscillatory terms: dodecahedral case

• zeros of zeta function $\zeta_{\mathcal{L}}(s)$

$$s_m = \sigma + \frac{2\pi i m}{\log(2+\phi)}, \qquad m \in \mathbb{Z}$$

with $\sigma = \log(20)/\log(2+\phi)$

• contribution to heat kernel expansion of non-real zeros:

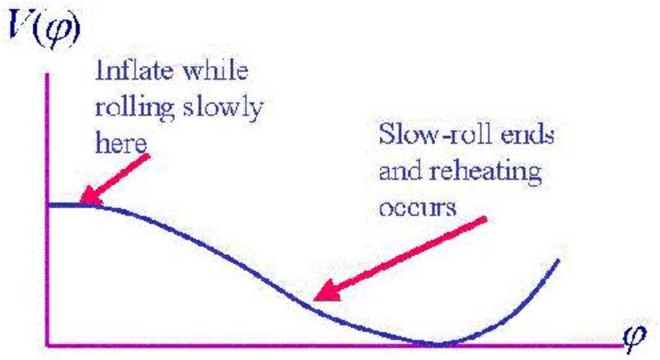
$$\frac{C}{t^\sigma}(a_0 + 2\Re(a_1 t^{-2\pi i/\log(2+\phi)}) + \cdots)$$

with coefficients $a_m$ proportional to $\Gamma(s_m)$: for fixed real part $\sigma$ decays exponentially fast along vertical line

• oscillatory terms are small

# Slow-roll inflation potential from the spectral action

• perturb the Dirac operator by a scalar field $D^2 + \phi^2 \Rightarrow$ spectral action gives potential $V(\phi)$




• shape of $V(\phi)$ distinguishes most cosmic topologies: spherical forms and Bieberbach manifolds (Marcolli, Pierpaoli, Teh)

## Fractality corrections to potential $V(\phi)$

- additional term in potential

$$\mathcal{U}_\sigma(x) = \int_0^\infty u^{(\sigma-1)/2}(h(u+x) - h(u))du$$

depends on $\sigma$ fractal dimension

- size of correction depends on (leading term)

$$(\zeta(\sigma-2, \frac{3}{2}) - \frac{1}{4}\zeta(\sigma, \frac{3}{2}))\mathcal{R}_\sigma$$

- further corrections to $\mathcal{U}_\sigma$ come from the oscillatory terms

$\Rightarrow$ presence of fractality (in this spectral action model of gravity) can be read off the slow-roll potential (hence the slow-roll coefficients, which depend on $V$, $V'$, $V''$)