STRUCTURES OF RANDOMNESS

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FIGURE 1. Pablo Picasso, *Guernica*, 1937

1. Randomness and Culture

There is no other concept that was more influential in the history of Modern Art of the post World War II era than "Randomness". Starting with the post war years, the concept of *randomness* penetrated the collective consciousness of European culture like never before. It profoundly altered its main modes of expression in both the arts and the sciences. This coming to terms with the concept of randomness caused a major shift in consciousness and world view.

It is not hard to see the defining moment when the notion of Randomness forcefully intruded in every conceivable description of the world. Europe at the end of World War II looked like an enormous pile of rubbles, an entire continent in ruins. City after city razed to the ground, nothing left standing, a whole civilization that appeared obliterated. A devastation on such a scale had never visited the continent before. Any sign or order, of structure, appeared lost and broken down into incomprehensible fragments

of its previous existence. Most of the destruction of European cities during the war was caused by aerial bombings. These were a novel form of warfare at the time. Although previously used in colonial wars, bombing was first experienced within Europe only a few years earlier, with the bombing of Guernica, by German and Italian war planes, during the Spanish Civil War in 1937, famously portrayed in Picasso's painting. Randomness determined what an aerial bombing would destroy or spare, which houses would be left standing, who lived and who died: a destruction that was so utterly complete and yet that was not aimed, not targeted, just random. It has been observed many times how World War II marked the end of Religion in Europe: the concept of a personal god ceased to make sense after the horrors and the destruction of the first half of the century. Einstein once said about god that only his non-existence can excuse him, and that certainly reflects how a large part of Europe felt at the time. It was not only organized religion that essentially disappeared in the wake of that enormous wave of destruction, however. A broader sense of the predictability of life had come to an end as well. The tradition of the European Enlightenment had already significantly moved away from religious thinking, embracing instead the vision of a mechanical and predictable universe, run by deterministic laws of physics, where motion is entirely specified by its initial conditions. Meditating upon the ruins of Europe at the end of World War II, people saw the determinist view of the Enlightenment crumble, replaced by a new understanding of the dominant force of nature: mindless blind Chance.

London and Warsaw

Berlin and Leningrad

FIGURE 2. Georges Mathieu, *Evanescence*, 1945

1.1. **Traces.** At the end of World War II, only traces of humanity were left amidst the desolation. European art began to come to terms with the need for an entirely new language of expression, articulating remnants and traces in the surrounding chaos, in the arbitrariness of destruction.

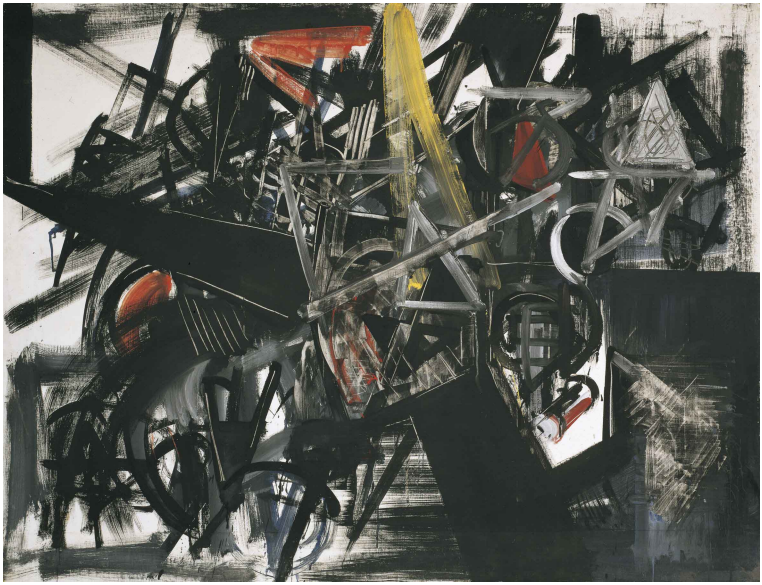


FIGURE 3. Emilio Vedova, *Sbarramento*, 1951

Several artistic movements began, for the first time in art history, to incorporate randomness in their modes of expression. The movements Arte Informale/Art Informel in Italy and France and Lyrical Abstraction in France and England, with artists like Alberto Burri, Emilio Vedova, Georges Mathieu, Antoni Tàpies, created a new language for artistic expression, and developed a new lyrical vision of hope for the future, literally growing and blooming over the ruins of Europe.

Even though America was largely spared the destruction of the war on its own territory, the Abstract Expressionism movement

developed a parallel investigation into the use of randomness as a mode of expression for the new era, with artists like Hans Hofmann, Janet Sobel, Jackson Pollock, Lee Krasner. In an often quoted statement, Pollock remarked that "new needs need new techniques, and the modern artists have found new ways and new means of making their statements: the modern painter cannot express this age, the airplane, the atom bomb, the radio, in the old forms of the Renaissance or of any other past culture". There is a clear sense, in this generation of artists, that the world they live in has gone through a dramatic change of perspective, and just as Adorno remarked that "to write poetry after Auschwitz is barbaric", these visual artists understood that using art as a representational image of a deterministic world was no longer possible after the experience of the second world war. Other artistic movements followed, away from the figurative, embracing both abstraction and the language of randomness: from the Conceptual Art of Sol LeWitt to the Generative Art of Ellsworth Kelly and Hiroshi Kawano. Randomness also developed, during the same period of time, as a new compositional form in music and literature.

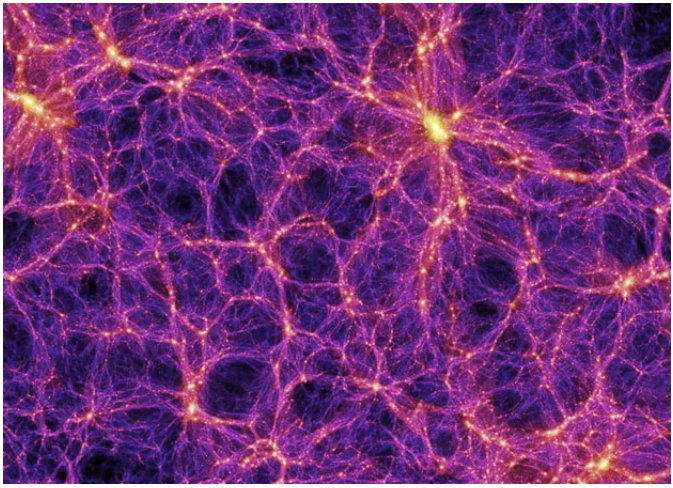
1.2. **Randomness and the Cosmos.** For the Postwar artists, Randomness became a poetic way of reconnecting a devastated humanity to the cosmos, in a universe where the illusion of a supernatural god was no longer sustainable. One sees themes explicitly hinting at this cosmic connection in the work of various artists. For example, in the painting "Milky Way" by the Abstract Expressionism artist Janet Sobel, painted in 1945, right at the end of the war years, the randomness of the painting is compared to a cosmic structure, the galaxy, the larger home of

FIGURE 4. Janet Sobel, *Milky Way*, 1945

humanity, our home in the universe, in a larger perspective away from the devastations of national states at war.

Interestingly, the view of cosmic structures shown in Janet Sobel's painting is closer to our modern understanding of cosmology than one might have imagined at the time. Not on the scale of the Milky Way, a single galaxy, but on a much larger scale that we now call "The Cosmic Web".

Until recently, astronomers believed that galaxy clusters were uniformly distributed in the universe. In 1989 Margaret Geller and John Huchra discovered a larger scale structure, which they named the "Great Wall". This discovery releaved the surprising fact that galaxies accumulate along 2-dimensional structures. Since then walls and filaments of galaxy clusters have been observed, separated by large voids: the foam-like structure of the Cosmic Web.



2. What is Randomness?

What exactly is Randomness, this new language of our era? We will review some of the main aspects of the mathematics of Randomness and how these manifest themselves in the artistic movements of the Postwar years.

2.1. **Probabilities and Frequencies.** Consider a finite set, used as an alphabet, $A = \{\ell_1, \dots, \ell_N\}$. This could mean the labels on the faces of a dice, or a set of graphical symbols used to compose words, or to compose musical phrases. We will call the symbols in the alphabet A letters, even though they may be numbers, or any other type of signifier, as long as they are the minimal units out of which messages will be composed. We assign to each letter ℓ_i of the alphabet A a probability p_i, so that we have a list of probabilities associated to A,

$$\{p_1, \dots, p_N\} \quad p_i \geq 0 \quad p_1 + \cdots + p_N = 1$$

In the case of a fair coin, with an alphabet $A = \{1, 2\}$ labeling the two possible outcomes, the fairness means that the probability is uniform, $p_1 = 1/2 = p_2$. Over longer and longer sequences in the alphabet the *frequencies* of occurrence of the symbols approaches the *probability*: this is the *law of large numbers* of Probability Theory.

The Shannon Entropy, which we have already discussed in a previous chapter, measures the amount of information by the quantity

$$S = -\sum_{i=1}^{N} p_i \log(p_i).$$

FIGURE 5. Lee Krasner, *Composition*, 1949

This is the average amount of information (in bits, taking the base 2 logarithm) contained in an event drawn randomly, according to the assigned probability distribution.

FIGURE 6. Regina Valluzzi, *Structure Evolution*, ca. 2010

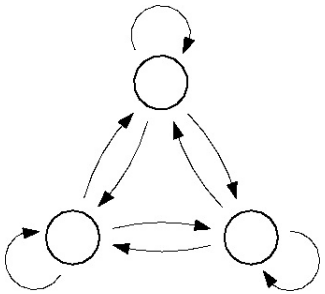
The Shannon Entropy is maximal for the uniform probability $p_i = 1/N$: it is very difficult to predict the outcome when the probability is uniform. The Entropy, on the contrary, is minimal for the least uniform probability, that assigns probability one to one of the outcomes and zero to all the other, $p_1 = 1$ and $p_i = 0$ for all i not equal to 1: in such a case it is very easy to predict the ourcome. Thus, Shannon Entropy can be seen as a measure of the level of "unpredictability" of a certain outcome.

Artists realized very clearly that, from an aesthetic viewpoint the chosen frequencies/probabilities are most "interesting" when

they are neither uniform nor extremal, that is, when they fall into an intermediate non-uniform range where the Shannon Entropy is neither maximal nor minimal. For example, consider the painting "Composition" by Lee Krasner shown in the figure. It shows a pattern created by a list of graphical symbols, where some of the symbols repeat identically or nearly identically in different positions. What makes the pattern interesting to observe, and the painting aesthetically appealing, is the fact that the pattern is neither fully uniformly random, nor it is just the repetition of a single or a small number of symbols in a very easily detectable, completely regular, identically repeating pattern. The painting captures exactly this "in between" region where Shannon complexity is not too small, but it is also not near its maximum. An interesting pattern should be neither entirely predictable nor entirely unpredictable. Another example is given by the painting "Structure Evolution" by Regina Valluzzi, where we see several different patterns, some very predictable, some seemingly a lot closer to uniform randomness, with transitions between different types of patterns in the painting creating regions of different Shannon Entropy and an overall effect of transition between randomness and order, which the artist related to formation of structures.

2.2. **Markov Chains.** Markov chains are random processes with one step memory (as in a chain, the last link added is connected only to its immediate predecessor). In a Markov process, there is a list of possible states the process can be in at a given time. Time ticks in discrete units, like seconds or minutes on a clock, and at each new step the system can move to a new state, or remain in the same state, with a certain probability. Let p_{ij} denotes the probability of going from state i to state j (or remain in the same

state for $i = j$). These satisfy $p_{ij} \geq 0$ and $\sum_j p_{ij} = 1$, which just means that they are indeed probabilities. This is usually illustrated with a graph as in the figure, where the circles represent the possible states of the system, and the arrows represent the possible transitions, each with a given probability.



The initial state of the system is also likely to be one of the listed states with a certain probability, and we denote by π_i the probabilities of having i as initial state. Again, since these are also probabilities, they satisfy $\pi_i \geq 0$ and $\sum_i \pi_i = 1$. Moreover, the one step rule of the Markov chain states that the probabilities p_{ij} and π_i satisfy the relation

$$\sum_i \pi_i p_{ij} = \pi_j.$$

This means that the probability of the system being in a given state at the next step depends on where it was in the previous step with what probability and on the probabilities of the transitions. The sum is the sum over all the possible one-step histories of the system. The Entropy of a Markov process is

$$S = -\sum_i \pi_i \sum_j p_{ij} \log(p_{ij}).$$

FIGURE 7. Hiroshi Kawano, *Design 3-1: Color Markov Chain Pattern*, 1964

Generative Art was the first Postwar artistic movement that seriously incorporated these ideas. The mode of expression these artists adopted included art produced with the intervention of autonomous agents – computer programs, cellular automata, neural networks – and significantly involving random processes. Hiroshi Kawano was a leading early exponent of this tendency. We see

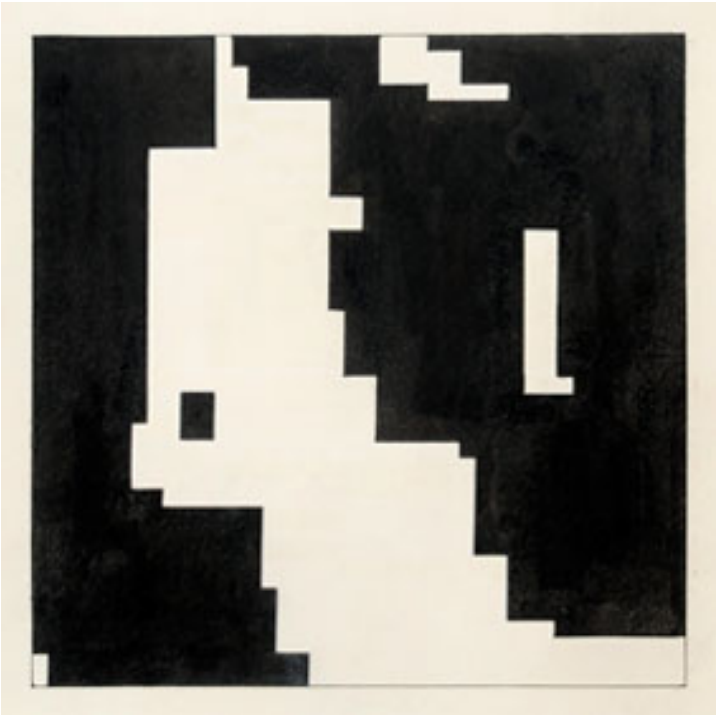
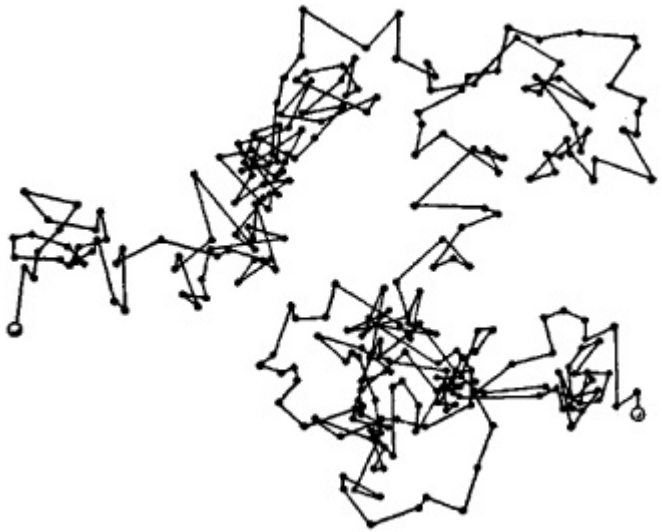


FIGURE 8. Hiroshi Kawano, *Design 2-3: Markov Chain Pattern*, 1964

in the figures two examples of Kawano paintings produced using a Markov chain as a way of determining the resulting pattern of shapes and colors.

2.3. **Brownian motion and stochastic processes.** A serious interest in investigating randomness in science began in the late XIX century with the study of Brownian motion, rigorously explained by Einstein in 1905.



Brownian motion is the random "dance" of particles in a fluid. It was first described by Lucretius in his "De Rerum Natura" for dust particles in the air (Book II, 113–140). Lucretius (correctly) proposed Brownian motion as a verification of the existence of atoms. In a typical trajectory of Brownian motion, as illustrated in the figure, one recognizes a characteristic lack of smoothness of the path: a particle would move along a smooth path, but its motion is diverted by imbalances of forces in the combined effect of all the collisions with the surrounding smaller liquid molecules in random thermal motion. This crucial aspect of Brownian motion, the non-smoothness of trajectories of motion, is captured in Camille McPhee's painting "Mutated Brownian Motion II".

FIGURE 9. Camille McPhee, *Mutated Brownian Motion II*, 2011

Unlike the Markov processes we discussed before, Brownian motion is a continuous time stochastic process. Einstein's model of Brownian motion is based on a diffusion equation for the density of Brownian particles, which gives solutions

$$\rho(x,t) = \frac{e^{-\frac{x^2}{4Dt}}}{\sqrt{4\pi Dt}}$$

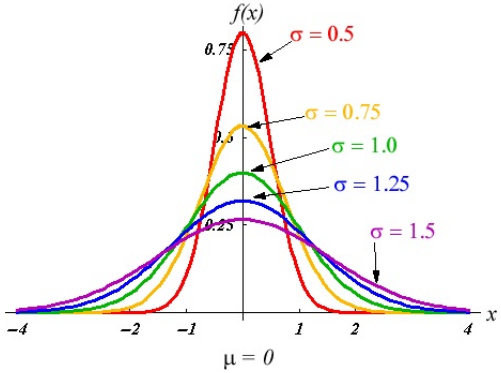
Norbert Wiener formulated Brownian motion as a continuous-time stochastic process

$$W_t - W_s \sim N(0, t-s)$$

with *normal distribution*

$$N(\mu, \sigma) = \frac{e^{-\frac{(x-\mu)^2}{2\sigma}}}{\sqrt{2\pi\sigma}},$$

which has the shape shown in the figure, for different values of the parameter σ.



The resulting stochastic process has $W_0 = 0$, W_t almost surely continuous, and the quantities $W_{t_1} - W_{s_1}$ and $W_{t_2} - W_{s_2}$ are independent random variables for non-overlapping time intervals $s_1 < t_1 \leq s_2 < t_2$. This last property reflects the randomness caused by collisions that change change direction to the Brownian motion of particles.

Two other examples of pictorial renderings of Brownian motion by contemporary artists are given by Camille McPhee's, "Mutated Brownian Motion" and by Elena Kozhevnikova's, "Brownian Motion II". While McPhee's portraits of Brownian motion

FIGURE 10. Camille McPhee, *Mutated Brownian Motion*, 2011

tend to focus more on the typical non-differentiable shape of trajectories, Kozhevnikova's appears to focus more on representing a random diffusion process governed by the normal distribution.

FIGURE 11. Elena Kozhevnikova, *Brownian Motion II*, ca. 2013

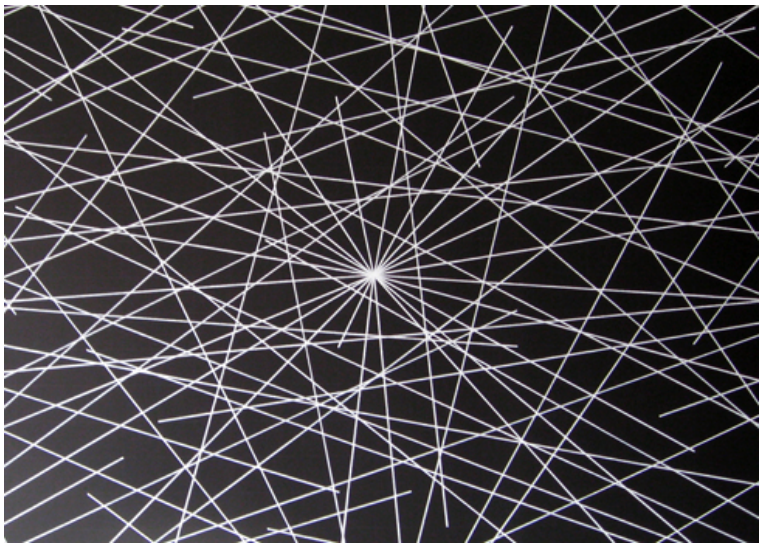
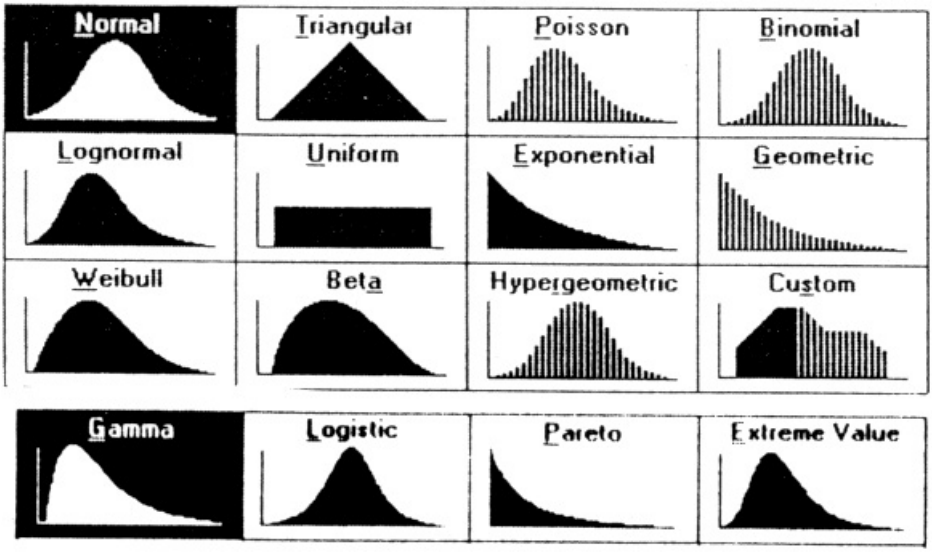


FIGURE 12. Sol LeWitt, *Wall Drawing 289*, 1978

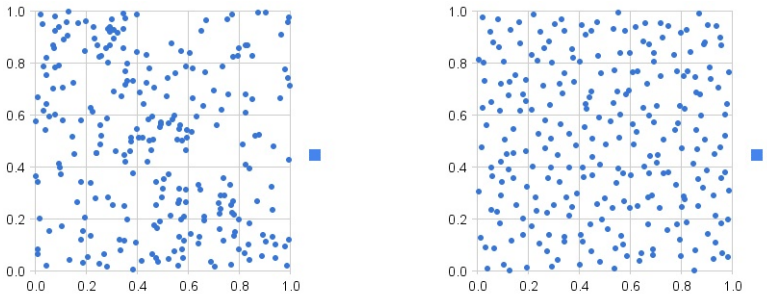
3. Kinds of Randomness

In discussing the Shannon Entropy, we noticed how it is maximized by the most uniform probability distribution. Artists working with randomness also discussed the idea of a "uniform" randomness. For example, Sol LeWitt wrote about his wall drawings: "Lines not short, not straight, crossing and touching, drawn at random, using four colors, uniformly dispersed with maximum density, covering the entire surface of the wall". What does *"at random ... uniformly dispersed with maximum density"* actually mean from the point of view of Probability Theory? What *kind of randomness* are we talking about here?

There are *many different kinds of randomness*! In the *continuous* case we have normal distribution, uniform distribution, triangular distribution, etc. In the *discrete case* we have Bernoulli distribution, Poisson distribution, hypergeometric distribution, etc. The different shapes of some of these distributions are depicted in the figure.



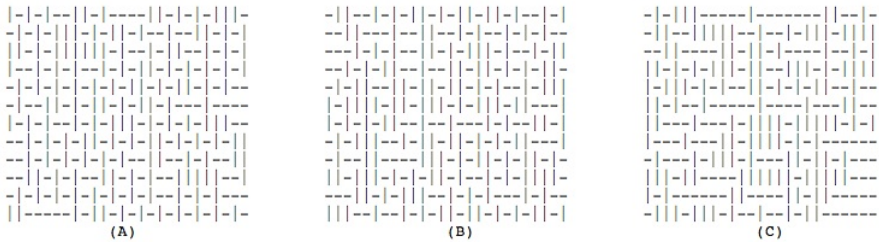
So what does it mean for something to be *truly* random? Does that correspond to our intuition of randomness? Consider the following test: which of the two plots in the figure below is obtained using the *uniform* distribution?



One would be tempted to say that the second figure looks more like "uniformly" random than the first one. However, what one perceives there as uniformity is in fact more regularity, that is, the existence of correlations. These regularities and correlations, as we discussed before, take us towards lower values of the Shannon Entropy, while uniform randomness, as in the first figure does not exhibit any correlation pattern between the positions of the points.

A similar example[1] occurs in the following figure, where we can interpret each pattern as a string of text composed on an alphabet of two characters, a vertical and a horizontal bar. We want to identify which of these three patterns follows a uniform Bernoulli distribution, which means that both letters of the alphabet occur with probability 1/2 and the next choice is independent of the previous ones.

[1]The examples of random and non-random patterns reported here are from `http://norvig.com/experiment-design.html`



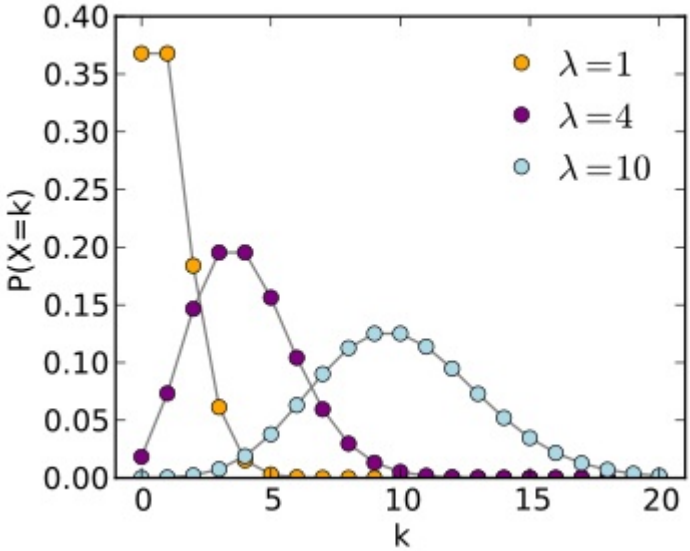
Intuitively, people generally perceive (A) or (B) to be more "uniformly random" than (C). However, in (A) the lines flip between the two positions horizontal and vertical 3/4 of the time, and in (B) they flip 2/3 of the time, while in (C) they flip 1/2 of the time, which is what one expects in the uniform case.

Randomness is not always what meets the eyes. This is an interesting point when one wants to use randomness in the visual arts. Our brain naturally tends to perceive as "patterns" those irregular clusters that form in a random arrangement, while it is more likely to perceive an evenly distributed set of points as "random" even when that "evenness" is in fact an indicator of non-random structures and correlations.

3.1. **The Poisson Distribution.** For a discrete, integer valued, random variable X, the Poisson distribution assigns as the probability of having $X = k$ the value

$$P_k = \frac{\lambda^k}{k!} e^{-\lambda},$$

where λ is an adjustable parameter. The shape of the Poisson distribution, for different values of the parameter λ is illustrated in the figure.



For small λ, the distribution is more peaked at smaller values, while for large λ is is more spread out and it peaks at larger values.

The Poisson distribution was famously used by the musician Iannis Xenakis, to compose his work *Achorripsis* (the name coming from the Greek $A\chi o\varsigma$, sound, and $\rho\iota\psi\eta$, cast, as in throwing dice). In this musical composition, Xenakis distributed different "sound densities" over the 28×7 "cells" with probabilities generated by the Poisson distribution with $\lambda = 0.6$. Versions of "Achorripsis" with different values of the parameter of the Poisson distribution create different moods and feelings to the sound pattern and a different musical experience. A detailed analysis

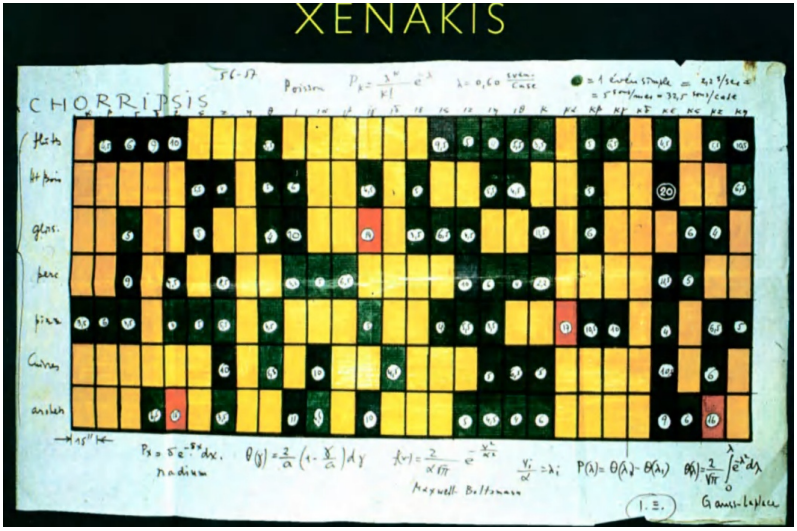


FIGURE 13. Iannis Xenakis, *Musical score of Achorripsis*, 1957

of "Achorripsis" can be found in the article of Linda Arsenault listed in the references.

Iannis Xenakis was an engineer and architect, who worked with Le Corbusier, before dedicating himself to music and becoming one of the most famous music composers of postwar Europe. The essay by S.Kanach and C.Lovelace in the bibliography analyze his work, and Xenakis' own writings "Formalized Music" on mathematics and music provide a wealth of information on his approach and especially on his use of stochastic processes as an integral part of musical composition. Xenakis thought of different probability distributions as different instruments one can create music

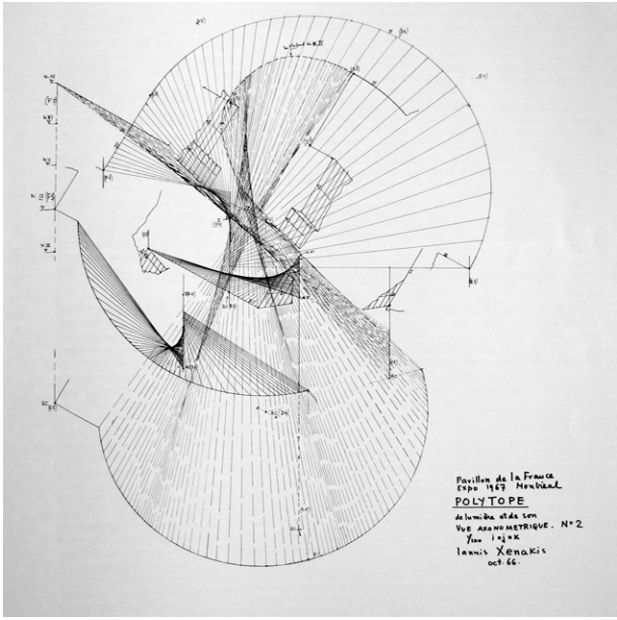


FIGURE 14. Iannis Xenakis, *Polytope*, 1971

with, each with their own color and their own special feeling. He wrote: "*Every probability function* is a particular stochastic variation, which *has its own personality*... Poisson, exponential, normal, uniform, Cauchy, logistic distributions".

4. All the Colors of Noise

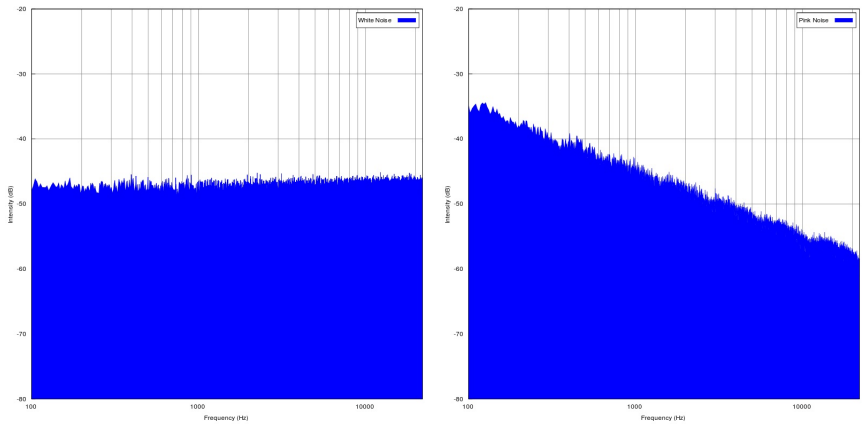
A random signal $x(t)$ can be decomposed into frequencies, using a truncated Fourier transform

$$\hat{x}_T(\omega) = \frac{1}{\sqrt{T}} \int_0^T x(t) e^{-i\omega t} dt.$$

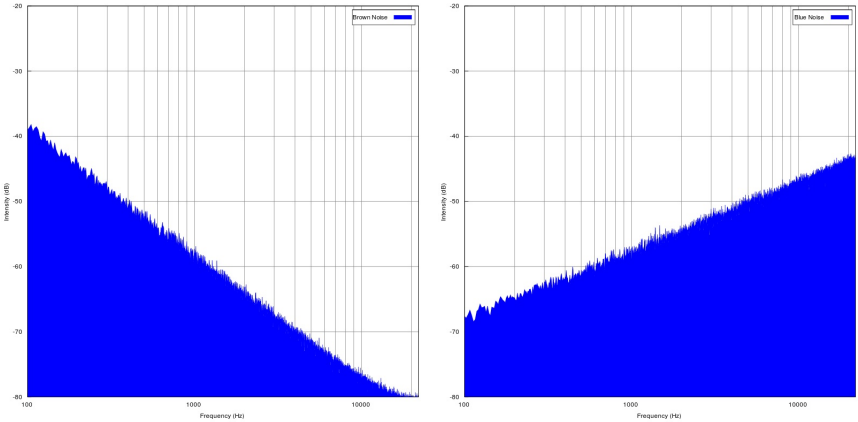
The spectral density (expectation value) $\hat{x}_T(\omega)$ describe the different distribution of frequencies in the random noise. Again, one can distinguish between several kinds of randomness in noise. *Power-law noise* refers to spectral densities of the form $1/\omega^\alpha$, for some exponent α. The best known examples of different forms of power-law noise are given by

- White noise: $\alpha = 0$
- Pink noise: $\alpha = 1$
- Brown noise (Brownian): $\alpha = 2$
- Blue noise: $\alpha = -1$

The corresponding distributions are shown in the figures.



White Noise and Pink Noise



Brown Noise and Blue Noise

White noise occurs, for example, in the "snow effect" of analog TV screens. In pink noise, also referred to as $1/f$ noise, the lower frequencies are louder than the higher frequencies, but it is usually perceived as being more "uniform" than white noise, because it results in equal power per octave. In blue noise, on the contrary, the energy of the signal increases with the frequency, increasing by three decibels in each successive octave. Brown noise is the signal noise of Brownian motion. Like pink noise, and more so, it has more energy at lower frequencies. It can be produced as an integral of white noise.

A white noise interference on a background landscape is visible in the painting "White Noise" by Andy Mercer. An elaborate use of various types of noise incorporated in paintings and installations can be seen in the work of the artist and movie director Laura Poitras in her "Astro Noise" series. This is an exhibition centered on the theme of mass surveillance and state violence,

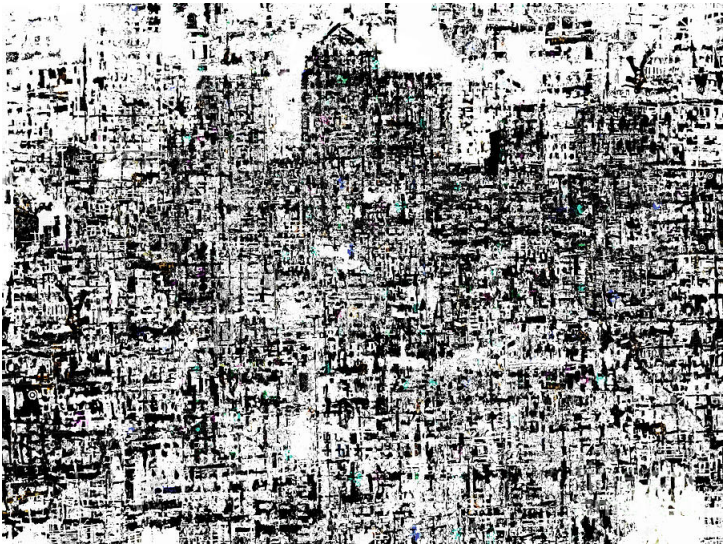


FIGURE 15. Andy Mercer, *White Noise*, 2011

also discussed in her movie *Citizenfour* dedicated to the Snowden case. The term "Astro Noise" is a reference to the cosmic microwave background noise, and at the same time a reference to the name given by Snowden to the file containing the evidence of NSA mass surveillance. In the piece of the Astro Noise series reproduced here, it is interesting how the term "Anarchist" is referring to the intercepting of government data (in this case satellite tracking), while at the same time it appears to be a reference to the noise filters, a kind of obfuscation action, an interference with the structures of power of society.

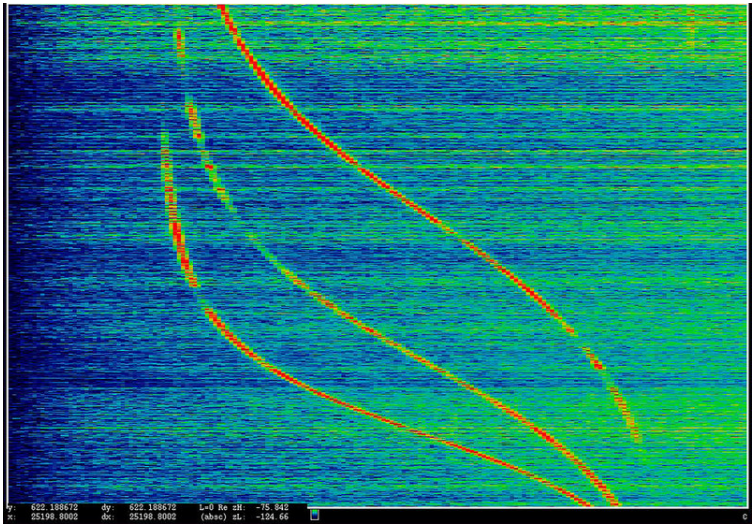
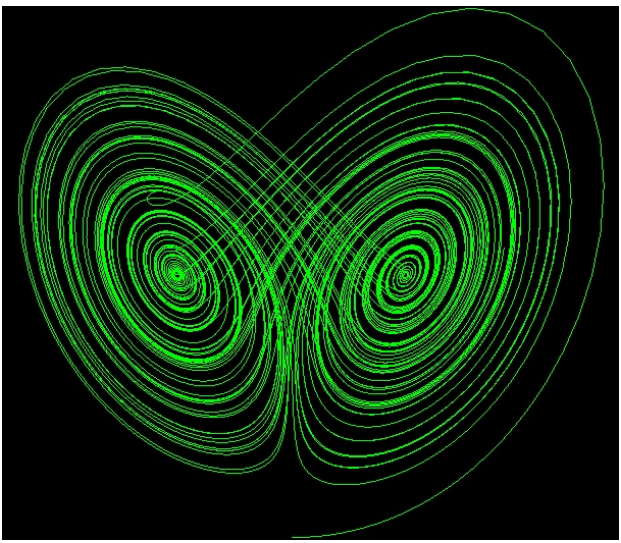


FIGURE 16. Laura Poitras, *ANARCHIST: Data Feed with Doppler Tracks from a Satellite (Intercepted May 27, 2009)*, Astro Noise, 2016

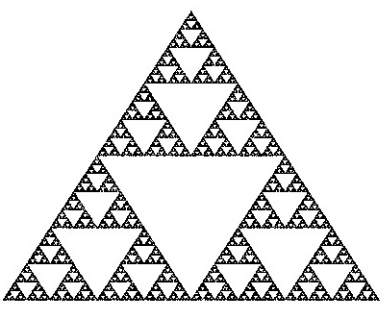
5. Chaos versus Randomness

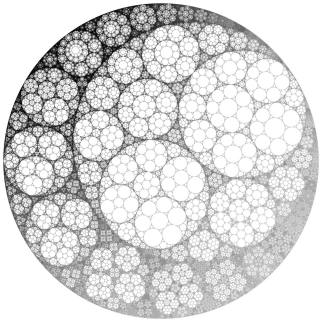
Chaos theory developed from early ideas of Poincaré (1880's) on non-periodic orbits of dynamical systems. As a mathematical theory, it came to prominence in the 1960's, after Edward Lorenz discovered a *strange attractor* in the trajectories of fluid dynamics equations used in meteorology. The strange attractor has the property that trajectories that start near each other will typically end up moving around and in the proximity of the attractor, but their trajectories may vary significantly after a period of time and they can end up in different regions of the attractor. Typical trajectories, moreover, as they get close to the attractor, jump seemingly unpredictably from one to the other of the two main spiral regions of the attractor. The attractor itself has a very complicated structure, which is representative of a fractal geometry. The Lorentz attractor is shown in the figure.



It seems at first, by observing the trajectories of a system like the Lorentz attractor, that the motion is governed by a random process. This is in fact not the case. It is a *chaotic* motion, but chaos does *not* mean randomness. Indeed, in most cases chaos is deterministic. What chaos means is *sensitive dependence on the initial condition*, that is, trajectories that start nearby will end up diverging fast from each other. This in particular means that small errors in the determination of the initial conditions are amplified by the dynamics of the system, making it effectively unpredictable, even though the system itself remains fully deterministic.

As in the case of the Lorentz attractor, the strange attractor sets of chaotic dynamical systems are typically *fractals*. This creates a close connection between Chaos Theory and Fractal Geometry. As we already discussed in a previous chapter, fractals are spaces of non-integer dimension: curves of dimension between one and two, surfaces of dimension between two and three, etc. of the type shown in the figures.





The fractal dimension detects how the measure of the set scales with its size. Fractals typically exhibit a self-similarity property

FIGURE 17. Jackson Pollock, *One: N.31*, 1950

that describes this scaling: the space looks identical to a union of scaled copies of itself.

5.1. **The fractal dimension of Pollock's drip paintings.**

A series of paper appeared in famous research journals at the end of the '90s and during the first decade of this century proposed an intriguing possibility of identifying a specific "signature" in Jackson Pollock's paintings by considering their fractal dimension (see the references). It was observed that the fractal dimension of Pollock's paintings increased from $D \sim 1$ in 1943 to $D = 1.72$ in 1952. This seems to coincide with the increased complexity of the paintings over that same period of time. This observation could lead to a possible reliable method for dating

FIGURE 18. One of the disputed Pollock paintings

Pollock's paintings, in the absence of other more specific information. More intriguingly, and certainly more controversially, it may also provide a way to detect the authenticity (or lack thereof) of paintings attributed to Pollock.

In 2003 Taylor, one of the authors of the first study on Pollock paintings and fractal dimension, claimed that 24 putative Pollock paintings are fakes, based on fractal analysis. Naturally, this generated considerable controversy in the art world. Two other physicists, Jones-Smith and Mathur, later criticizes

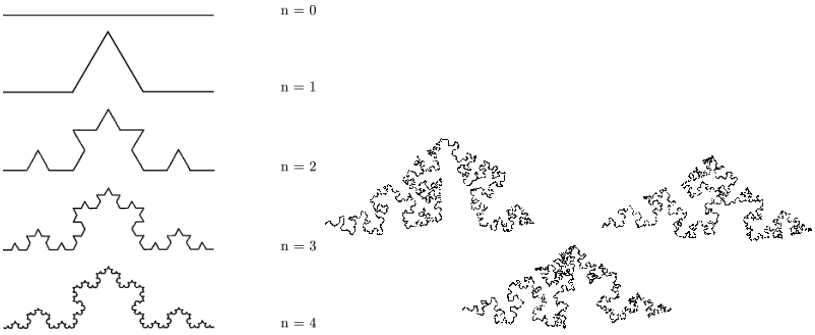


FIGURE 19. Deterministic and Random Fractal

the methodology used by Taylor. In 2008 Coddington, Elton, Rockmore, and Wang analyzed Pollock's paintings using the *entropy dimension*, *local dimension* and *multifractal analysis* as a way of measuring "structured randomness" . Their results confirmed Taylor's claim that the disputed paintings are fake. The main point of this approach is the fact that *deterministic* fractals are very homogeneous and have the same fractal dimension everywhere. Moreover, for such fractals, different notions of fractal dimension agree. On the contrary *random* fractals have varying local dimension and are best described via a stratification by subfractals, that is, by *multifractal* analysis.

Here is where Fractal Geometry, usually associated with deterministic systems with chaotic dynamics, merges with Randomness, giving rise to a class of "random fractals" that combines the qualities of both randomness and chaos. The Pollock paintings certainly have a random structure, due to the drip technique

FIGURE 20. Jackson Pollock, *One: N.31*, 1950, detail

itself, hence an analysis based on fractal geometry and fractal dimensions, is best achieved using the setting of random fractals.

This type of fractals, as observed above, do not necessarily have the same fractal dimension everywhere and are better understood in terms of a *local* dimension.

The *box counting dimension* of a fractal set X is obtained by covering the shape with grid boxes of small sizes ϵ, and computing

$$\dim_b(X) = -\lim_{\epsilon\to 0} \frac{\log N(\epsilon)}{\log(\epsilon)},$$

where $N(\epsilon)$ is the number of boxes of size ϵ that are needed to cover the set. The box counting dimension is usually only an approximation from above to the fractal dimension.

The *Entropy Dimension* is a refinement of the box counting dimension. Take a measure μ supported on the fractal X. For example, if X is an image at a certain resolution, μ may be counting how many pixels are in a certain region of the image. Then, when covering X by grid boxes of size ϵ as before, one computes the Entropy

$$S_\epsilon(\mu, X) = -\sum_{i=1}^{N(\epsilon)} \mu(B_i) \log \mu(B_i),$$

where B_i are the grid boxes of size ϵ used to cover X. Then one obtains the Entropy Dimension as

$$\dim_e(\mu, X) = -\lim_{\epsilon\to 0} \frac{S_\epsilon(\mu, X)}{\log(\epsilon)}.$$

The *local dimension* at a point x of a fractal X is comparing the growth of the measure μ of balls $B_r(x)$ of radius r centered at x with the growth of the radius,

$$D(\mu, x) = \lim_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}.$$

If the local dimension is constant (or constant with probability one) then the Entropy Dimension and the fractal dimension are

FIGURE 21. Another of the disputed Pollock paintings

the same. However, in the case of random fractals where typically the local dimension is varying, the Entropy Dimension is a different invariant of the fractal geometry.

In the case of Pollock's paintings, as one can see by examining them in closer detail, as in Figure 20, there is an intricate interplay of several overlapping structures with their own different local dimensions of fractality. The *multifractal analysis* corresponds to separating out "strata" of constant local dimension and computing the Entropy Dimension of each. This method

provide more detailed information than just computing a single average value of the fractal dimension.

Other mathematical methods were applied to the analysis of Pollock paintings. In particular, the work of Herczyński, Cernuschi, and Mahadevan used *fluid dynamics*, which allows for a description of how patterns in fluids form as they fall, depending on viscosity and speed. This is a way of capturing specific signatures in the drip paint techniques of Pollock that would be particular to his own personal style.

There is no clear conclusion, no silver bullet that would uniquely identify the specific style of the artist in a measurable mathematical way that can completely resolve questions of authenticity.

However, this is not surprising, for complicated structures it is hard to find a single complete invariant that characterizes them. Even if mathematical analysis alone is not sufficient to authenticate Pollock's paintings with complete certainty, it still provides interesting methods for studying them.

References

- Don S. Lemons, *An Introduction to Stochastic Processes in Physics*, Johns Hopkins University Press, 2002.
- Kenneth Falconer, *Fractal Geometry: Mathematical Foundations and Applications*, John Wiley, 2013.
- Linda M. Arsenault, *Iannis Xenakis's Achorripsis: the matrix game*, Computer Music Journal, Vol.26 (2002) N.1, 58–72.
- S. Kanach, C. Lovelace, *Iannis Xenakis: composer, architect, visionary*, The Drawing Center, 2011.
- Iannis Xenakis, *Formalized music: thought and mathematics in music*, Pendragon Press, 1992.
- Richard P. Taylor, Adam Micolich and David Jonas, *Fractal analysis of Pollock's drip paintings*, Nature, Vol. 399 (1999) 422-423.
- Kate Jones-Smith and Harsh Mathur, *Fractal Analysis: Revisiting Pollock's drip paintings*, Nature, Vol. 444 (2006) E9
- Jim Coddington, John Elton, Daniel Rockmore, Yang Wang, *Multifractal analysis and authentication of Jackson Pollock paintings*, MOMA
- D.-J. Feng and Y. Wang, *A class of self-affine measures*, J. Fourier Anal. and Appl. 11, (2005) 107–124
- Andrzej Herczyński, Claude Cernuschi, L. Mahadevan, *Painting with drops, jets, and sheets*, Phys. Today, Vol.64 N.6 (2011) 31–40
- Stanislaw Lem, *Filozofia Przypadku* (The Philosophy of Chance), 1968