SPECTRAL ACTION FOR BIANCHI TYPE-IX COSMOLOGICAL MODELS

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ABSTRACT. A rationality result previously proved for Robertson-Walker metrics is extended to a homogeneous anisotropic cosmological model, namely the Bianchi type-IX minisuperspace. It is shown that the Seeley-de Witt coefficients appearing in the expansion of the spectral action for the Bianchi type-IX geometry are expressed in terms of polynomials with rational coefficients in the cosmic evolution factors $w_1(t), w_2(t), w_3(t)$, and their higher derivates with respect to time. We begin with the computation of the Dirac operator of this geometry and calculate the coefficients a_0, a_2, a_4 of the spectral action by using heat kernel methods and parametric pseudodifferential calculus. An efficient method is devised for computing the Seeley-de Witt coefficients of a geometry by making use of Wodzicki's noncommutative residue, and it is confirmed that the method checks out for the cosmological model studied in this article. The advantages of the new method are discussed, which combined with symmetries of the Bianchi type-IX metric, yield an elegant proof of the rationality result.

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1. Introduction

Quantum cosmology studies the early universe where the energy scale is so high that one would need to incorporate into the theory both quantum gravity and the Standard Model with unbroken symmetries such as the hypothetical supersymmetry. This makes the exact solution of any quantum cosmological problem essentially impossible as it amounts to solving quantum fields genuinely interacting with quantized gravity, the nature of which we know little about. However, since we are only interested in the large-scale behavior of the early universe whose mass-energy distribution is highly homogeneous, we are encouraged to exploit this symmetry and focus only on a few long-wavelength degrees of freedom. In quantum cosmology, this common practice is known as the minisuperspace approximation, which can be rigorously justified under certain criteria [32, 45].

Well-known examples of minisuperspaces include the Robertson-Walker model and its anisotropic generalization to the Bianchi type-IX model. The Robertson-Walker metric is of the form

$$ds^2 = dt^2 + a(t)^2 d\sigma^2,$$

where $a(t)$ is a general cosmic factor of the expanding universe and $d\sigma^2$ is the round metric on the 3-sphere $\mathbb{S}^3$. The Bianchi type-IX model, which enjoys a reduced $SU(2)$ isometry group rather than the full $\mathbb{S}^3$ symmetry, is written as

$$(1) \quad ds^2 = w_1(t)w_2(t)w_3(t)dt^2 + \frac{w_2(t)w_3(t)}{w_1(t)}\sigma_1^2 + \frac{w_3(t)w_1(t)}{w_2(t)}\sigma_2^2 + \frac{w_1(t)w_2(t)}{w_3(t)}\sigma_3^2,$$

where σ_i are left-invariant 1-forms on $SU(2)$-orbits.

Beside trying to fit the Standard model into the early universe spacetime, another possible investigation of physical importance is to quantize supersymmetric systems in a Bianchi-type IX minisuperspace. Since the energy scale of the early universe is very likely to be higher than the supersymmetry breaking scale, it is of great interest to see how supersymmetry may possibly change the picture of the early universe. Since the Hamiltonians of supersymmetric systems can often be identified with well-behaving elliptic operators acting on spin bundles for Majorana fermions or fields of differential forms for Dirac fermions, the *spectral action* [5] approach appears to be a handy tool, where the celebrated Atiyah-Singer index theorem may be used to calculate the Witten indices of certain supersymmetric theories and thus to conclude whether the supersymmetry can be spontaneously broken [49, 2].

Noncommutative geometry [12] describes geometric spaces by spectral triples $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involutive algebra represented by bounded operators on a Hilbert space $\mathcal{H}$, and D is an unbounded self-adjoint operator in $\mathcal{H}$ that plays the role of the Dirac operator by encoding the metric information. This set up, which includes a great variety of noncommutative spaces, generalizes Riemannian geometry since Connes' reconstruction theorem [14] states that if $\mathcal{A}$ is commutative, then, under suitable regularity conditions, the triple consists of the algebra of smooth function on a spin^c manifold acting on the L^2-spinors and D is the Dirac operator.

Given a spectral triple, the spectral action principle [5, 16] considers the functional,

$$\text{Trace}\big(f(D/\Lambda)\big) \sim \sum_{\beta \in \Pi} f_\beta \Lambda^\beta \fint |D|^{-\beta} + f(0)\zeta_D(0) + \cdots,$$

2

as the fundamental action functional, where f is a positive even function on the real line and Λ is a real parameter that fixes the mass scale. The details of the above asymptotic expansion are explained in [16]. For a spin^c manifold M, the coefficients of this expansion are determined by the Seeley-de Witt coefficients a_n appearing in the heat expansion,

$$\mathrm{Trace}\big(e^{-tD^2}\big) \sim t^{-\dim(M)/2} \sum_{n=0}^{\infty} a_{2n}(D^2) t^n \qquad (t \to 0^+).$$

In addition to extracting local geometric information and recovering the Einstein-Hilbert action, the spectral action generalizes the theory to include noncommutative spaces which give rise to the Standard Model gauge fields with Yang-Mills actions as well as couplings to the modified gravity [8, 9, 13, 38, 40]. In this sense, this principle is able to unify the Standard Model with general relativity under a generalized notion of Riemannian geometry.

Considering the variety of noncommutative geometric spaces and the physical implications of the spectral action [4, 6, 7, 10], it is of great importance to use and develop different methods for computing this action. In particular, any interpretation of the full expansion of a spectral action is desirable. For example, using the Poisson summation formula, the spectral action for the Dirac operator on highly symmetric manifolds, such as products of spheres by tori, was computed in [8].

A considerable amount of work has been carried out on the cosmological implications of the spectral action in recent years [31, 37, 38, 39, 40, 41, 42, 43, 18]. For the Euclidean Robertson-Walker spacetime with a general cosmic factor $a(t)$, Chamseddine and Connes have devised an efficient method in [9] for computing the terms of the spectral action, which is based on making the use of the Euler-Maclaurin formula and the Feynman-Kac formula. They computed the terms up to a_{10} in the expansion and made a conjecture, which was addressed in [21] by using pseudodifferential operators and heat kernel techniques. That is, it was shown that a general term in the expansion is described by a polynomial with rational coefficients in $a(t)$ and its derivatives of a certain order.

The present paper is intended to study the spectral action for the Bianchi type-IX minisuperspace, which is a homogeneous *anisotropic* cosmological model. Since the spectral action of a geometry depends on the eigenvalues of the square of its Dirac operator, we explicitly compute the Dirac operator D of the Bianchi type-IX metric in §2 and derive the pseudodifferential symbol of D^2. In §3, following a brief review of the heat kernel method that uses pseudodifferential calculus for the computation of the Seeley-de Witt coefficients [26], we present the calculation of the terms a_0, a_2, a_4 in the expansion of the spectral action associated with D.

We devise a new method for calculating the Seeley de-Witt coefficients of a geometry in terms of noncommutative residues of operators, which extends the result on the realization of the Einstein-Hilbert action as the residue of a power of the Laplacian, see [28, 30, 25]. This method is explained in detail in §4 and it checks out to give the same result for the calculated terms a_0, a_2, a_4 for the Bianchi type-IX metric. Combining the symmetries of the metric with technical properties of pseudodifferential symbols of parametrices of the Laplacians, which significantly simplify in view of the new method using the Wodzicki residue, we prove a rationality result for a general term in the expansion of the spectral action for the Bianchi type-IX metric in §5. That is, we show that general terms of the

expansion are expressed by several variable polynomials with rational coefficients evaluated on $w_1(t), w_2(t), w_3(t)$, and their derivatives of certain orders. In §6, we discuss the gravitational instantons, which form an especially interesting class of Bianchi type-IX models, and elaborate on the significance of the rationality for the spectral action in relation to the arithmetic and number theoretic structures in mathematical physics. Finally, our main results and conclusions are summarized in §7.

2. The Dirac Operator of Bianchi Type-IX Metrics

The heat kernel method that uses pseudodifferential calculus for computing the Seeley-de Witt coefficients of an elliptic operator on a compact manifold relies on the pseudodifferential symbol of the operator in local charts [26]. Thus, in this section we compute the Dirac operator D of the Bianchi type-IX metric and thereby obtain the symbol of D^2.

The most efficient way of computing the Dirac operator of a geometric space is to use an orthonormal coframe $\{\theta^a\}$ for the metric, which, from the definition of D, yields

$$D = \sum_a \theta^a \nabla^S_{\theta_a},$$

where ∇^S is the spin connection of the spin bundle and $\{\theta_a\}$ is the predual of the coframe, see [24]. Since ∇^S is the lift of the Levi-Civita connection ∇ to the spin bundle, one starts with computing the matrix of 1-forms $\omega = (\omega^a_b)$ such that $\nabla = d + \omega$ in terms of θ^a in a local chart (x^μ), which can be lifted to the matrix of the spin connection 1-forms by making use of the Lie algebra isomorphism $\mu : \mathfrak{so}(m) \to \mathfrak{spin}(m)$ given by

$$\mu(A) = \frac{1}{4} \sum_{a,b} A^{ab} e_a e_b, \qquad A = (A^{ab}) \in \mathfrak{so}(m),$$

where m is the dimension of the manifold. We note that $\{e_a\}$ is the standard basis for $\mathbb{R}^m$ considered inside the Clifford algebra of $\mathbb{R}^m$ where $\mathfrak{spin}(m)$ is spanned linearly by $\{e_a e_b; a < b\}$.

The ω^a_b are found uniquely by writing

$$\nabla \theta^a = \sum_b \omega^a_b \otimes \theta^b,$$

and imposing the conditions that characterize the Levi-Civita connection, namely metric-compatibility and torsion-freeness which respectively imply that

$$\omega^a_b = -\omega^b_a, \qquad d\theta^a = \sum_b \omega^a_b \wedge \theta^b.$$

Therefore the Dirac operator is written as

$$D = \sum_{a,\mu} \gamma^a dx^\mu(\theta_a) \frac{\partial}{\partial x^\mu} + \frac{1}{4} \sum_{a,b,c} \gamma^c \omega^b_{ac} \gamma^a \gamma^b,$$

where ω^b_{ac} are defined by

$$\omega^b_a = \sum_c \omega^b_{ac} \theta^c,$$

and the matrices γ^a represent the Clifford action of θ^a on the spin bundle, namely that they satisfy the relations $(\gamma^a)^2 = -I$ and $\gamma^a \gamma^b + \gamma^b \gamma^a = 0$ for $a \neq b$.

We go through this process for the Bianchi type-IX metric,

$$
\begin{aligned}
ds^2 &= w_1 w_2 w_3 \, dt \, dt + \frac{w_1 w_2 \cos(\eta)}{w_3} d\phi \, d\psi + \frac{w_1 w_2 \cos(\eta)}{w_3} d\psi \, d\phi \\
&\quad + \left(\frac{w_2 w_3 \sin^2(\eta) \cos^2(\psi)}{w_1} + w_1 \left(\frac{w_3 \sin^2(\eta) \sin^2(\psi)}{w_2} + \frac{w_2 \cos^2(\eta)}{w_3} \right) \right) d\phi \, d\phi \\
&\quad + \frac{\left(w_1^2 - w_2^2\right) w_3 \sin(\eta) \sin(\psi) \cos(\psi)}{w_1 w_2} d\eta \, d\phi \\
&\quad + \frac{\left(w_1^2 - w_2^2\right) w_3 \sin(\eta) \sin(\psi) \cos(\psi)}{w_1 w_2} d\phi \, d\eta \\
&\quad + \left(\frac{w_2 w_3 \sin^2(\psi)}{w_1} + \frac{w_1 w_3 \cos^2(\psi)}{w_2} \right) d\eta \, d\eta + \frac{w_1 w_2}{w_3} d\psi \, d\psi,
\end{aligned}
$$

which is written in the local coordinates $(x^\mu) = (t, \eta, \phi, \psi)$, where $\mathbb{S}^3$ is parametrized by the map

$$
(\eta, \phi, \psi) \mapsto \left(\cos(\eta/2) e^{i(\phi+\psi)/2}, \sin(\eta/2) e^{i(\phi-\psi)/2} \right),
$$

with the parameter ranges $0 \le \eta \le \pi, 0 \le \phi < 2\pi, 0 \le \psi < 4\pi$. An orthonormal coframe for ds^2 is given by

$$
\begin{aligned}
\theta^0 &= \sqrt{w_1 w_2 w_3} \, dt, \\
\theta^1 &= \sin(\eta) \cos(\psi) \sqrt{\frac{w_2 w_3}{w_1}} \, d\phi - \sin(\psi) \sqrt{\frac{w_2 w_3}{w_1}} \, d\eta, \\
\theta^2 &= \sin(\eta) \sin(\psi) \sqrt{\frac{w_1 w_3}{w_2}} \, d\phi + \cos(\psi) \sqrt{\frac{w_1 w_3}{w_2}} \, d\eta, \\
\theta^3 &= \cos(\eta) \sqrt{\frac{w_1 w_2}{w_3}} \, d\phi + \sqrt{\frac{w_1 w_2}{w_3}} \, d\psi.
\end{aligned}
$$

By explicit calculations in this basis we find that the non-vanishing ω^b_{ac} are determined by the following terms:

$$
\begin{aligned}
\omega^0_{11} &= -\frac{w_2\left(w_1 w_3' - w_3 w_1'\right) + w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}}, & \omega^0_{22} &= -\frac{w_2\left(w_3 w_1' + w_1 w_3'\right) - w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}}, \\
\omega^0_{33} &= -\frac{w_2\left(w_3 w_1' - w_1 w_3'\right) + w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}}, & \omega^1_{23} &= -\frac{w_1^2 w_2^2 - w_3^2\left(w_1^2 + w_2^2\right)}{2(w_1 w_2 w_3)^{3/2}}, \\
\omega^1_{32} &= -\frac{w_1^2\left(w_2^2 - w_3^2\right) + w_2^2 w_3^2}{2(w_1 w_2 w_3)^{3/2}}, & \omega^2_{31} &= -\frac{w_2^2 w_3^2 - w_1^2\left(w_2^2 + w_3^2\right)}{2(w_1 w_2 w_3)^{3/2}}.
\end{aligned}
$$

Thus we achieve an explicit calculation of the pseudodifferential symbol of the Dirac operator which is written as

$$
\begin{aligned}
\sigma(D)(x,\xi) &= \sum_{a,\mu} i\gamma^a e_a^\mu \xi_{\mu+1} + \frac{1}{4\sqrt{w_1 w_2 w_3}} \left(\frac{w_1'}{w_1} + \frac{w_2'}{w_2} + \frac{w_3'}{w_3} \right) \gamma^1 \\
&\quad - \frac{\sqrt{w_1 w_2 w_3}}{4} \left(\frac{1}{w_1^2} + \frac{1}{w_2^2} + \frac{1}{w_3^2} \right) \gamma^2 \gamma^3 \gamma^4 \\
&= -\frac{i\gamma^2 \sqrt{w_1}\left(\csc(\eta)\cos(\psi)\left(\xi_4 \cos(\eta) - \xi_3 \right) + \xi_2 \sin(\psi) \right)}{\sqrt{w_2}\sqrt{w_3}} \\
&\quad + \frac{i\gamma^3 \sqrt{w_2}\left(\sin(\psi)\left(\xi_3 \csc(\eta) - \xi_4 \cot(\eta) \right) + \xi_2 \cos(\psi) \right)}{\sqrt{w_1}\sqrt{w_3}} \\
&\quad + \frac{i\gamma^1 \xi_1}{\sqrt{w_1}\sqrt{w_2}\sqrt{w_3}} + \frac{i\gamma^4 \xi_4 \sqrt{w_3}}{\sqrt{w_1}\sqrt{w_2}} \\
&\quad + \frac{1}{4\sqrt{w_1 w_2 w_3}} \left(\frac{w_1'}{w_1} + \frac{w_2'}{w_2} + \frac{w_3'}{w_3} \right) \gamma^1 \\
&\quad - \frac{\sqrt{w_1 w_2 w_3}}{4} \left(\frac{1}{w_1^2} + \frac{1}{w_2^2} + \frac{1}{w_3^2} \right) \gamma^2 \gamma^3 \gamma^4,
\end{aligned}
$$

where the following non-vanishing e_a^μ are used:

$$
\begin{aligned}
e_0^0 &= \frac{1}{\sqrt{w_1 w_2 w_3}}, & e_1^1 &= -\frac{\sqrt{w_1}\sin(\psi)}{\sqrt{w_2 w_3}}, & e_2^1 &= \frac{\sqrt{w_2}\cos(\psi)}{\sqrt{w_1 w_3}}, \\
e_1^2 &= \frac{\sqrt{w_1}\csc(\eta)\cos(\psi)}{\sqrt{w_2 w_3}}, & e_2^2 &= \frac{\sqrt{w_2}\csc(\eta)\sin(\psi)}{\sqrt{w_1 w_3}}, & e_1^3 &= -\frac{\sqrt{w_1}\cot(\eta)\cos(\psi)}{\sqrt{w_2 w_3}}, \\
e_2^3 &= -\frac{\sqrt{w_2}\cot(\eta)\sin(\psi)}{\sqrt{w_1 w_3}}, & e_3^3 &= \frac{\sqrt{w_3}}{\sqrt{w_1 w_2}}.
\end{aligned}
$$

Having the symbol of D, a direct calculation yields

$$
\sigma(D^2)(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi),
$$

where the expressions for the $p_k(x,\xi)$ are recorded below. The principal symbol of D^2, which is homogeneous of order 2 in ξ, is given by

$$
\begin{aligned}
p_2(x,\xi) &= \frac{1}{w_1 w_2 w_3}(\xi_4^2 w_1^2 \cot^2(\eta)\cos^2(\psi) + \xi_3^2 w_1^2 \csc^2(\eta)\cos^2(\psi) \\
&\quad + \xi_2 \xi_4 w_1^2 \cot(\eta)\sin(2\psi) - \xi_2 \xi_3 w_1^2 \csc(\eta)\sin(2\psi) \\
&\quad - 2\xi_3 \xi_4 w_1^2 \cot(\eta)\csc(\eta)\cos^2(\psi) + \xi_2^2 w_1^2 \sin^2(\psi) \\
&\quad + \xi_4^2 w_2^2 \cot^2(\eta)\sin^2(\psi) - \xi_2 \xi_4 w_2^2 \cot(\eta)\sin(2\psi) \\
&\quad + \xi_3^2 w_2^2 \csc^2(\eta)\sin^2(\psi) + \xi_2 \xi_3 w_2^2 \csc(\eta)\sin(2\psi) \\
&\quad - 2\xi_3 \xi_4 w_2^2 \cot(\eta)\csc(\eta)\sin^2(\psi) + \xi_2^2 w_2^2 \cos^2(\psi) \\
&\quad + \xi_4^2 w_3^2 + \xi_1^2)I,
\end{aligned}
$$

where I is the 4×4 identity matrix. The component of $\sigma(D^2)$ that is homogeneous of order 1 has a lengthy expression:

$$
\begin{aligned}
p_1(x,\xi) \;=\; & \Big(-\frac{i\xi_2 w_1 \cot(\eta)\cos^2(\psi)}{w_2 w_3} - \frac{i\xi_2 w_2 \cot(\eta)\sin^2(\psi)}{w_1 w_3} \\
& -\frac{3i\xi_4 w_2 \csc^2(\eta)\sin(2\psi)}{4w_1 w_3} + \frac{3i\xi_4 w_1 \csc^2(\eta)\sin(2\psi)}{4w_2 w_3} \\
& -\frac{i\xi_4 w_2 \cos(2\eta)\csc^2(\eta)\sin(2\psi)}{4w_1 w_3} + \frac{i\xi_4 w_1 \cos(2\eta)\csc^2(\eta)\sin(2\psi)}{4w_2 w_3} \\
& +\frac{i\xi_3 w_2 \cot(\eta)\csc(\eta)\sin(2\psi)}{w_1 w_3} - \frac{i\xi_3 w_1 \cot(\eta)\csc(\eta)\sin(2\psi)}{w_2 w_3}\Big)I \\
& +\Big(\frac{i\xi_4 w_3}{2w_1^2} + \frac{i\xi_4 w_3}{2w_2^2} - \frac{i\xi_4}{2w_3}\Big)\gamma^2\gamma^3 \\
& +\Big(\frac{i\xi_4 w_2 \cot(\eta)\sin(\psi)}{2w_1^2} - \frac{i\xi_3 w_2 \csc(\eta)\sin(\psi)}{2w_1^2} - \frac{i\xi_2 w_2 \cos(\psi)}{2w_1^2} \\
& +\frac{i\xi_4 w_2 \cot(\eta)\sin(\psi)}{2w_3^2} - \frac{i\xi_3 w_2 \csc(\eta)\sin(\psi)}{2w_3^2} - \frac{i\xi_2 w_2 \cos(\psi)}{2w_3^2} \\
& -\frac{i\xi_4 \cot(\eta)\sin(\psi)}{2w_2} + \frac{i\xi_3 \csc(\eta)\sin(\psi)}{2w_2} + \frac{i\xi_2 \cos(\psi)}{2w_2}\Big)\gamma^2\gamma^4 \\
& +\Big(-\frac{i\xi_4 w_1 \cot(\eta)\cos(\psi)}{2w_2^2} + \frac{i\xi_3 w_1 \csc(\eta)\cos(\psi)}{2w_2^2} - \frac{i\xi_2 w_1 \sin(\psi)}{2w_2^2} \\
& -\frac{i\xi_4 w_1 \cot(\eta)\cos(\psi)}{2w_3^2} + \frac{i\xi_3 w_1 \csc(\eta)\cos(\psi)}{2w_3^2} - \frac{i\xi_2 w_1 \sin(\psi)}{2w_3^2} \\
& +\frac{i\xi_4 \cot(\eta)\cos(\psi)}{2w_1} - \frac{i\xi_3 \csc(\eta)\cos(\psi)}{2w_1} + \frac{i\xi_2 \sin(\psi)}{2w_1}\Big)\gamma^3\gamma^4 \\
& +\Big(-\frac{i\xi_4 w_1'}{2w_1^2 w_2} - \frac{i\xi_4 w_2'}{2w_1 w_2^2} + \frac{i\xi_4 w_3'}{2w_1 w_2 w_3}\Big)\gamma^1\gamma^4 \\
& +\Big(-\frac{i\xi_4 \cot(\eta)\cos(\psi)w_1'}{2w_1 w_2 w_3} + \frac{i\xi_3 \csc(\eta)\cos(\psi)w_1'}{2w_1 w_2 w_3} - \frac{i\xi_2 \sin(\psi)w_1'}{2w_1 w_2 w_3} \\
& +\frac{i\xi_4 \cot(\eta)\cos(\psi)w_2'}{2w_2^2 w_3} - \frac{i\xi_3 \csc(\eta)\cos(\psi)w_2'}{2w_2^2 w_3} + \frac{i\xi_2 \sin(\psi)w_2'}{2w_2^2 w_3} \\
& +\frac{i\xi_4 \cot(\eta)\cos(\psi)w_3'}{2w_2 w_3^2} - \frac{i\xi_3 \csc(\eta)\cos(\psi)w_3'}{2w_2 w_3^2} + \frac{i\xi_2 \sin(\psi)w_3'}{2w_2 w_3^2}\Big)\gamma^1\gamma^2 \\
& +\Big(\frac{i\xi_4 \cot(\eta)\sin(\psi)w_1'}{2w_1^2 w_3} - \frac{i\xi_3 \csc(\eta)\sin(\psi)w_1'}{2w_1^2 w_3} - \frac{i\xi_2 \cos(\psi)w_1'}{2w_1^2 w_3} \\
& -\frac{i\xi_4 \cot(\eta)\sin(\psi)w_2'}{2w_1 w_2 w_3} + \frac{i\xi_3 \csc(\eta)\sin(\psi)w_2'}{2w_1 w_2 w_3} + \frac{i\xi_2 \cos(\psi)w_2'}{2w_1 w_2 w_3} \\
& +\frac{i\xi_4 \cot(\eta)\sin(\psi)w_3'}{2w_1 w_3^2} - \frac{i\xi_3 \csc(\eta)\sin(\psi)w_3'}{2w_1 w_3^2} - \frac{i\xi_2 \cos(\psi)w_3'}{2w_1 w_3^2}\Big)\gamma^1\gamma^3.
\end{aligned}
$$

Finally we have the zero order part of $\sigma(D^2)$:

$$
\begin{aligned}
p_0(x,\xi) &= (-\frac{w_1'}{8w_1w_2^2} - \frac{w_1'}{8w_1w_3^2} + \frac{3w_1'}{8w_1^3} - \frac{w_2'}{8w_1^2w_2} - \frac{w_3'}{8w_1^2w_3} - \frac{w_2'}{8w_2w_3^2} \\
&\quad + \frac{3w_2'}{8w_2^3} - \frac{w_3'}{8w_2^2w_3} + \frac{3w_3'}{8w_3^3})\gamma^1\gamma^2\gamma^3\gamma^4 + \\
&\quad (-\frac{w_1''}{4w_1^2w_2w_3} + \frac{w_1'w_2'}{8w_1^2w_2^2w_3} + \frac{w_1'w_3'}{8w_1^2w_2w_3^2} + \frac{5w_1'^2}{16w_1^3w_2w_3} - \frac{w_2''}{4w_1w_2^2w_3} \\
&\quad + \frac{w_2'w_3'}{8w_1w_2^2w_3^2} + \frac{5w_2'^2}{16w_1w_2^3w_3} - \frac{w_3''}{4w_1w_2w_3^2} + \frac{5w_3'^2}{16w_1w_2w_3^3} + \frac{w_2w_3}{16w_1^3} \\
&\quad + \frac{w_3}{8w_1w_2} + \frac{w_1w_3}{16w_2^3} + \frac{w_2}{8w_1w_3} + \frac{w_1}{8w_2w_3} + \frac{w_1w_2}{16w_3^3})I.
\end{aligned}
$$

3. CALCULATION OF THE TERMS UP TO a_4 IN THE SPECTRAL ACTION

Calculation of the Seeley-de Witt coefficients associated with an elliptic positive differential operator on an m-dimensional compact manifold M can be achieved by using the Cauchy integral formula and parametric pseudodifferential calculus, see [26] for a detailed discussion. Let us review this method briefly for the operator D^2, where D is the Dirac operator acting on a spin bundle on M. In order to derive a small time asymptotic expansion of the form

$$
\text{Trace}\big(e^{-tD^2}\big) \sim t^{m/2} \sum_{n=0}^{\infty} a_{2n}(D^2)t^n \qquad (t \to 0^+),
$$

one can start with the Cauchy integral formula to write

$$
e^{-tD^2} = \frac{1}{2\pi i}\int_\gamma e^{-t\lambda}(D^2 - \lambda)^{-1}\,d\lambda,
$$

where the contour γ in the complex plane goes around the non-negative real numbers clockwise. Then, the idea is to approximate $(D^2 - \lambda)^{-1}$ by pseudodifferential operators and to derive the above expansion by computing the trace of the corresponding approximation of the heat kernel.

The symbol of D^2 is of the form $p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$ where each p_i is homogeneous of order i in ξ. Since D^2 is an elliptic differential operator of order 2, the inverse of $D^2 - \lambda$ is approximated by its parametrix R_λ with

$$
\sigma(R_\lambda) \sim \sum_{j=0}^{\infty} r_j(x,\xi,\lambda),
$$

where each $r_j(x,\xi,\lambda)$ is a parametric pseudodifferential symbol of order $-2 - j$, in the sense that

$$
r_j(x, t\xi, t^2\lambda) = t^{-2-j}r_j(x,\xi,\lambda).
$$

The equation

$$
\sigma((D^2 - \lambda)R_\lambda) \sim ((p_2(x,\xi) - \lambda) + p_1(x,\xi) + p_0(x,\xi)) \circ \left(\sum_{j=0}^{\infty} r_j(x,\xi,\lambda)\right) \sim 1
$$

can be solved recursively by comparing the homogeneous terms on the both sides after expanding the left hand side using the asymptotic composition rule for the

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symbols. Indeed, one finds that

$$r_0(x, \xi, \lambda) = (p_2(x, \xi) - \lambda)^{-1},$$

and for any $n > 1$, the term $r_n(x, \xi, \lambda)$ is found to be expressed in terms of $r_0(x, \xi, \lambda), \ldots, r_{n-1}(x, \xi, \lambda)$ by the formula

$$r_n(x, \xi, \lambda) = -\sum \frac{1}{\alpha!} \partial_\xi^\alpha r_j(x, \xi, \lambda) \, D_x^\alpha p_k(x, \xi) \, r_0(x, \xi, \lambda),$$

where the summation is over all $\alpha \in \mathbb{Z}_{\geq 0}^4, j \in \{0, 1, \ldots, n-1\}, k \in \{0, 1, 2\}$, such that $|\alpha| + j + 2 - k = n$.

The coefficients $a_{2n}(D^2)$ in the small time asymptotic expansion are then computable by integrating the invariantly defined functions

$$a_{2n}(x, D^2) = \frac{(2\pi)^{-m}}{2\pi i} \int \int_\gamma e^{-\lambda} \operatorname{tr}(r_{2n}(x, \xi, \lambda)) \, d\lambda \, d^m\xi$$

over the manifold against the volume form, which shows that these coefficients are local invariants of the geometry. We note that the odd coefficients vanish since for any odd j, the term $r_j(x, \xi, \lambda)$ is an odd function of the variable $\xi \in \mathbb{R}^m$, whose integral over $\mathbb{R}^m$ vanishes.

Applying this method to Bianchi type-IX metric we compute the corresponding a_0, a_2, a_4, which are recorded below (without writing the integral with respect t). The volume term is simply

$$a_0(D^2) \;=\; 4w_1 w_2 w_3,$$

and the scalar curvature term, after remarkable cancellations, is found to be

$$
\begin{aligned}
a_2(D^2) \;=\; & -\frac{w_1^2}{3} - \frac{w_2^2}{3} - \frac{w_3^2}{3} + \frac{w_1^2 w_2^2}{6w_3^2} + \frac{w_1^2 w_3^2}{6w_2^2} + \frac{w_2^2 w_3^2}{6w_1^2} - \frac{(w_1')^2}{6w_1^2} - \frac{(w_2')^2}{6w_2^2} \\
& -\frac{(w_3')^2}{6w_3^2} - \frac{w_1' w_2'}{3w_1 w_2} - \frac{w_1' w_3'}{3w_1 w_3} - \frac{w_2' w_3'}{3w_2 w_3} + \frac{w_1''}{3w_1} + \frac{w_2''}{3w_2} + \frac{w_3''}{3w_3}.
\end{aligned}
$$

Although it seems lengthy, after an enormous amount of cancellations, the next coefficient is expressed as:

$$
\begin{aligned}
a_4(D^2) \;=\; & -\frac{w_1^3 w_2^3}{15w_3^5} - \frac{w_1^3 w_3^3}{15w_2^5} - \frac{w_2^3 w_3^3}{15w_1^5} + \frac{w_1^3 w_2}{15w_3^3} + \frac{w_1 w_2^3}{15w_3^3} + \frac{w_1^3 w_3}{15w_2^3} + \frac{w_2^3 w_3}{15w_1^3} + \frac{w_1 w_3^3}{15w_2^3} \\[4pt]
& +\frac{w_2 w_3^3}{15w_1^3} - \frac{w_1 w_2}{15w_3} - \frac{w_1 w_3}{15w_2} - \frac{w_2 w_3}{15w_1} - \frac{w_2 (w_1')^2}{15w_1 w_3^3} - \frac{w_3 (w_1')^2}{15w_1 w_2^3} - \frac{w_3 (w_2')^2}{15w_1^3 w_2} \\[4pt]
& -\frac{w_1 (w_2')^2}{15w_2 w_3^3} - \frac{w_1 (w_3')^2}{15w_2^3 w_3} - \frac{w_2 (w_3')^2}{15w_1^3 w_3} + \frac{2(w_1')^2}{15w_1 w_2 w_3} + \frac{2(w_2')^2}{15w_1 w_2 w_3} \\[4pt]
& +\frac{2(w_3')^2}{15w_1 w_2 w_3} - \frac{w_2 (w_1')^2}{18w_1^3 w_3} - \frac{w_3 (w_1')^2}{18w_1^3 w_2} - \frac{w_1 (w_2')^2}{18w_2^3 w_3} - \frac{w_3 (w_2')^2}{18w_1 w_2^3} \\[4pt]
& -\frac{w_1 (w_3')^2}{18w_2 w_3^3} - \frac{w_2 (w_3')^2}{18w_1 w_3^3} - \frac{w_2 w_3 (w_1')^2}{18w_1^5} - \frac{w_1 w_3 (w_2')^2}{18w_2^5} - \frac{w_1 w_2 (w_3')^2}{18w_3^5} \\[4pt]
& -\frac{31(w_1')^4}{90w_1^5 w_2 w_3} - \frac{31(w_2')^4}{90w_1 w_2^5 w_3} - \frac{31(w_3')^4}{90w_1 w_2 w_3^5} - \frac{7w_1' w_2'}{60w_3^3} - \frac{7w_1' w_3'}{60w_2^3} - \frac{7w_2' w_3'}{60w_1^3}
\end{aligned}
$$

$$-\frac{w'_1 w'_2}{45 w_1^2 w_3} - \frac{w'_1 w'_2}{45 w_2^2 w_3} - \frac{w'_2 w'_3}{45 w_1 w_3^2} + \frac{5 w_3 w'_1 w'_2}{36 w_1^4} + \frac{5 w_3 w'_1 w'_2}{36 w_2^4} + \frac{5 w_2 w'_1 w'_3}{36 w_1^4}$$

$$+ \frac{5 w_2 w'_1 w'_3}{36 w_3^4} + \frac{5 w_1 w'_2 w'_3}{36 w_2^4} + \frac{5 w_1 w'_2 w'_3}{36 w_3^4} + \frac{7 w_3 w'_1 w'_2}{90 w_1^2 w_2^2} + \frac{7 w_2 w'_1 w'_3}{90 w_1^2 w_3^2} + \frac{7 w_1 w'_2 w'_3}{90 w_2^2 w_3^2}$$

$$- \frac{41 \, (w'_1)^{\,3} w'_2}{180 w_1^4 w_2^2 w_3} - \frac{41 w'_1 \, (w'_2)^{\,3}}{180 w_1^2 w_2^4 w_3} - \frac{41 \, (w'_1)^{\,3} w'_3}{180 w_1^4 w_2 w_3^2} - \frac{41 w'_1 \, (w'_3)^{\,3}}{180 w_1^2 w_2 w_3^4} - \frac{41 w'_2 \, (w'_3)^{\,3}}{180 w_1 w_2^2 w_3^4}$$

$$- \frac{41 \, (w'_2)^{\,3} w'_3}{180 w_1 w_2^4 w_3^2} - \frac{23 \, (w'_1)^{\,2} \, (w'_2)^{\,2}}{90 w_1^3 w_2^3 w_3} - \frac{23 \, (w'_1)^{\,2} \, (w'_3)^{\,2}}{90 w_1^3 w_2 w_3^3} - \frac{23 \, (w'_2)^{\,2} \, (w'_3)^{\,2}}{90 w_1 w_2^3 w_3^3}$$

$$- \frac{w'_1 w'_3}{45 w_1^2 w_2} - \frac{w'_1 w'_3}{45 w_2 w_3^2} - \frac{w'_2 w'_3}{45 w_1 w_2^2} - \frac{91 \, (w'_1)^{\,2} w'_2 w'_3}{180 w_1^3 w_2^2 w_3^2} - \frac{91 w'_1 \, (w'_2)^{\,2} w'_3}{180 w_1^2 w_2^3 w_3^2}$$

$$- \frac{91 w'_1 w'_2 \, (w'_3)^{\,2}}{180 w_1^2 w_2^2 w_3^3} + \frac{w_2 w''_1}{24 w_3^3} + \frac{w_3 w''_1}{24 w_2^3} + \frac{w_1 w''_2}{24 w_3^3} + \frac{w_3 w''_2}{24 w_1^3} + \frac{w_1 w''_3}{24 w_2^3} + \frac{w_2 w''_3}{24 w_1^3}$$

$$- \frac{w''_1}{12 w_2 w_3} - \frac{w''_2}{12 w_1 w_3} - \frac{w''_3}{12 w_1 w_2} + \frac{w_2 w''_1}{36 w_1^2 w_3} + \frac{w_3 w''_1}{36 w_1^2 w_2} + \frac{w_1 w''_2}{36 w_2^2 w_3}$$

$$- \frac{5 w_2 w_3 w''_1}{72 w_1^4} - \frac{5 w_1 w_3 w''_2}{72 w_2^4} - \frac{5 w_1 w_2 w''_3}{72 w_3^4} + \frac{5 \, (w'_1)^{\,2} w''_1}{8 w_1^4 w_2 w_3} + \frac{5 \, (w'_2)^{\,2} w''_2}{8 w_1 w_2^4 w_3}$$

$$+ \frac{5 \, (w'_3)^{\,2} w''_3}{8 w_1 w_2 w_3^4} + \frac{71 w'_1 w'_2 w''_1}{180 w_1^3 w_2^2 w_3} + \frac{71 w'_1 w'_2 w''_2}{180 w_1^2 w_2^3 w_3} + \frac{71 w'_1 w'_3 w''_1}{180 w_1^3 w_2 w_3^2} + \frac{71 w'_1 w'_3 w''_3}{180 w_1^2 w_2 w_3^3}$$

$$+ \frac{71 w'_2 w'_3 w''_3}{180 w_1 w_2^2 w_3^3} + \frac{71 w'_2 w'_3 w''_2}{180 w_1 w_2^3 w_3^2} + \frac{41 \, (w'_2)^{\,2} w''_1}{360 w_1^2 w_2^3 w_3} + \frac{41 \, (w'_3)^{\,2} w''_1}{360 w_1^2 w_2 w_3^3} + \frac{41 \, (w'_2)^{\,2} w''_3}{360 w_1 w_2^3 w_3^2}$$

$$+ \frac{41 \, (w'_3)^{\,2} w''_2}{360 w_1 w_2^2 w_3^3} + \frac{41 \, (w'_1)^{\,2} w''_2}{360 w_1^3 w_2^2 w_3} + \frac{41 \, (w'_1)^{\,2} w''_3}{360 w_1^3 w_2 w_3^2} + \frac{11 w'_2 w'_3 w''_1}{36 w_1^2 w_2^2 w_3^2} + \frac{11 w'_1 w'_3 w''_2}{36 w_1^2 w_2^2 w_3^2}$$

$$+ \frac{11 w'_1 w'_2 w''_3}{36 w_1^2 w_2^2 w_3^2} - \frac{(w''_1)^{\,2}}{6 w_1^3 w_2 w_3} - \frac{(w''_2)^{\,2}}{6 w_1 w_2^3 w_3} - \frac{(w''_3)^{\,2}}{6 w_1 w_2 w_3^3} + \frac{w_3 w''_2}{36 w_1 w_2^2}$$

$$+ \frac{w_1 w''_3}{36 w_2 w_3^2} + \frac{w_2 w''_3}{36 w_1 w_3^2} - \frac{w''_1 w''_2}{15 w_1^2 w_2^2 w_3} - \frac{w''_2 w''_3}{15 w_1 w_2^2 w_3^2} - \frac{w''_1 w''_3}{15 w_1^2 w_2 w_3^2} - \frac{w'_1 w_1^{(3)}}{6 w_1^3 w_2 w_3}$$

$$- \frac{w'_2 w_2^{(3)}}{6 w_1 w_2^3 w_3} - \frac{w'_3 w_3^{(3)}}{6 w_1 w_2 w_3^3} - \frac{w'_2 w_1^{(3)}}{10 w_1^2 w_2^2 w_3} - \frac{w'_3 w_1^{(3)}}{10 w_1^2 w_2 w_3^2} - \frac{w'_1 w_2^{(3)}}{10 w_1^2 w_2^2 w_3} - \frac{w'_3 w_2^{(3)}}{10 w_1 w_2^2 w_3^2}$$

$$- \frac{w'_1 w_3^{(3)}}{10 w_1^2 w_2 w_3^2} - \frac{w'_2 w_3^{(3)}}{10 w_1 w_2^2 w_3^2} + \frac{w_1^{(4)}}{30 w_1^2 w_2 w_3} + \frac{w_2^{(4)}}{30 w_1 w_2^2 w_3} + \frac{w_3^{(4)}}{30 w_1 w_2 w_3^2}.$$

Following the conjecture of Chamseddine and Connes for Robertson-Walker metrics [9], which was addressed in [21], the crucial observation to make at this stage is that all of the coefficients appearing in the above terms are rational numbers. This indicates that the rationality result holds for the Bianchi type-IX metric, which is proved in §5.

4. Heat Coefficients and the Wodzicki Residue

In this section we introduce a method for computing the Seeley-de Witt coefficients by making use of Wodzicki's noncommutative residue [47, 48]. The advantage of this method is that it involves significantly less complexity in computations, thus, for instance, it illuminates the structure of the heat expansion of the Bianchi type-IX metric more elegantly.

Given a closed m-dimensional manifold M, the Wodzicki residue is the unique trace functional on the algebra of classical pseudodifferential operators acting on the smooth sections of a vector bundle over M (up to multiplication by a constant). The local symbol σ of a classical pseudodifferential operator P_σ of order $d \in \mathbb{Z}$ has an asymptotic expansion of the form

$$\sigma(x, \xi) \sim \sum_{j=0}^{\infty} \sigma_{d-j}(x, \xi) \qquad (\xi \to \infty),$$

where each σ_{d-j} is positively homogeneous of order $d-j$ in ξ. The noncommutative residue of the operator P_σ is defined by

$$\mathrm{Res}(P_\sigma) = \int_{S^*M} \mathrm{tr}\left(\sigma_{-m}(x, \xi)\right) d^{m-1}\xi \, d^m x,$$

where $S^*M = \{(x, \xi) \in T^*M; \|\xi\|_g = 1\}$ is the cosphere bundle of M and the integral is in fact the integral of the corresponding Wodzicki residue density over M, see [47, 48, 29] for more details.

An alternative definition for Res, which is quite spectral, provides a link between the Seeley-de Witt coefficients and the noncommutative residue. That is, for any pseudodifferential operator P_σ, the map that sends a complex number s with a large enough real part to $\mathrm{Trace}(P_\sigma \Delta^{-s})$, where Δ is a Laplacian, has a meromorphic extension to the complex plane with at most simple poles at its singularities. The noncommutative residue can be defined as the linear functional

$$P_\sigma \mapsto \mathrm{res}_{s=0} \mathrm{Trace}(P_\sigma \Delta^{-s}),$$

which turns out to be a trace functional. Thus, there is a constant c_m such that for any classical P_σ, we have

$$\mathrm{Res}(P_\sigma) = \int_{S^*M} \sigma_{-m}(x, \xi) \, d^{m-1}\xi \, d^m x = c_m \left(\mathrm{res}_{s=0} \mathrm{Trace}(P_\sigma \Delta^{-s})\right).$$

The constant c_m can be computed easily as follows. The operator $\mathrm{Res}(\Delta^{-m/2})$ is of order $-m$ and its principal symbol is given by

$$\sigma_P(\Delta^{-m/2}) = \sigma_P(\Delta^{-1})^{m/2} = \sigma_P(\Delta)^{-m/2},$$

which yields

$$\mathrm{Res}(\Delta^{-m/2}) = \int_{S^*M} \mathrm{tr}\left(\sigma_P(\Delta)^{-m/2}\right) d^{m-1}\xi \, d^m x.$$

On the other hand, writing $\sigma_P(\Delta) = \left(\sum_{i,j} g^{ij} \xi_i \xi_j\right) I$, we have

$$\begin{aligned}
\mathrm{res}_{s=0} \mathrm{Trace}(\Delta^{-m/2} \Delta^{-s}) \quad &= \quad \mathrm{res}_{s=m/2} \mathrm{Trace}(\Delta^{-s}) \\
&= \quad \frac{1}{\Gamma(m/2)} \frac{(2\pi)^{-m}}{2\pi i} \int \int \int_\gamma e^{-\lambda} \mathrm{tr}\left((\sigma_P(\Delta) - \lambda)^{-1}\right) d\lambda \, d^m\xi \, d^m x \\
&= \quad \frac{(2\pi)^{-m}}{\Gamma(m/2)} \mathrm{rk}(V) \int \int e^{-\sum_{i,j} g^{ij} \xi_i \xi_j} d^m\xi \, d^m x \\
&= \quad \frac{(2\pi)^{-m}}{\Gamma(m/2)} \mathrm{rk}(V) \int \sqrt{\frac{\pi^m}{\det(g^{ij})}} \, d^m x \\
&= \quad \frac{2^{-m} \pi^{-m/2}}{\Gamma(m/2)} \mathrm{rk}(V) \int \det^{-1/2}(g^{ij}) \, d^m x,
\end{aligned}$$

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where for the second identity, we have used the formula (2) proved below, for $n = 0$. Therefore, we have

$$c_m = 2^m \pi^{m/2} \Gamma(m/2) \frac{\int \left(\sum_{i,j} g^{ij} \xi_i \xi_j\right)^{-m/2} d^{m-1}\xi \, d^m x}{\int \det^{-1/2}(g^{ij}) \, d^m x} = 2^{m+1} \pi^m.$$

Using the Mellin transform, $\lambda^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty e^{-t\lambda} t^{s-1} \, dt, \lambda > 0$, one has

$$\text{Trace}(\Delta^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty \left(\text{Trace}(e^{-t\Delta}) - \dim \ker(\Delta)\right) t^s \, \frac{dt}{t}.$$

By breaking the interval of the integration in the latter to $[0, 1]$ and $(1, \infty)$, and by substituting the small time asymptotic expansion,

$$\text{Trace}(e^{-t\Delta}) = t^{-m/2} \sum_{n=0}^N a_{2n} t^n + O(t^{-m/2+N+1}),$$

in the first part, one finds that

$$\text{res}_{s=m/2-n} \text{Trace}(\Delta^{-s}) = \frac{a_{2n}(\Delta)}{\Gamma(m/2 - n)},$$

for any non-negative integer $n \leq m/2 - 1$, see [26]. In particular we have

$$(2) \qquad\qquad \text{res}_{s=1} \text{Trace}(\Delta^{-s}) = a_{m-2}(\Delta).$$

This observation yields the following assertion, which is used crucially in the sequel.

Lemma 4.1. *If Δ is a Laplacian acting on the smooth sections of a vector bundle over an m-dimensional manifold, then*

$$a_{m-2}(\Delta) = \frac{1}{c_m} \text{Res}(\Delta^{-1}) = \frac{1}{2^{m+1}\pi^m} \text{Res}(\Delta^{-1}).$$

Proof. It follows from the identity (2) and the fact that

$$\text{res}_{s=1} \text{Trace}(\Delta^{-s}) = \text{res}_{s=0} \text{Trace}(\Delta^{-1}\Delta^{-s}) = \frac{1}{c_m} \text{Res}(\Delta^{-1}).$$

$\square$

Since we are mainly concerned with studying the spectral action for the Bianchi type-IX metric in this article, let us assume that D is the Dirac operator on a 4-dimensional manifold. By applying Lemma (4.1) to $\Delta = D^2$, we have

$$a_2(D^2) = \frac{1}{c_4} \text{Res}(D^{-2}) = \frac{1}{32\pi^4} \int_{S^*M} \text{tr}\left(\sigma_{-4}(D^{-2})\right) d^3\xi \, d^4x,$$

where $\sigma_{-4}(D^{-2})$ is the homogeneous component of order -4 in the expansion of the symbol of the parametrix of D^2. In the following theorem, we show that the next coefficients $a_{2n}(D^2), n \geq 2$, can similarly be expressed as noncommutative residues of Laplacians.

Theorem 4.1. *Let D be the Dirac operator on a 4-dimensional manifold. For any non-negative even integer r, we have*

$$a_{2+r}(D^2) = \frac{1}{2^5 \, \pi^{4+r/2}} \text{Res}(\Delta^{-1}),$$

where

$$\Delta = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^r},$$

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in which $\Delta_{\mathbb{T}^r}$ is the flat Laplacian on the r-dimensional torus $\mathbb{T}^r = (\mathbb{R}/\mathbb{Z})^r$.

Proof. It follows from Lemma 4.1 that

$$a_{2+r}(\Delta) = \frac{1}{c_{4+r}} \mathrm{Res}(\Delta^{-1}),$$

where $\sigma_{-4-r}(\Delta^{-1})$ is the homogeneous term of order $-4-r$ in the expansion of the symbol of the parametrix of Δ.

Since the metric on $\mathbb{T}^r$ is chosen to be flat, its volume term is evidently the only non-zero heat coefficient, which combined with the Künneth formula (cf. [26]) implies that

$$
\begin{aligned}
a_{2+r}((x, x'), \Delta) &= a_{2+r}(x, D^2) a_0(x', \mathbb{T}^r) \\
&= 2^{-r} \pi^{-r/2} a_{2+r}(x, D^2).
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
a_{2+r}(D^2) &= \frac{2^r \pi^{r/2}}{c_{4+r}} \mathrm{Res}(\Delta^{-1}) \\
&= \frac{1}{2^5 \pi^{4+r/2}} \mathrm{Res}(\Delta^{-1}).
\end{aligned}
$$

$\square$

A direct consequence of this theorem provides an efficient method for computing the Seeley-de Witt coefficients with significantly less complexities in the calculations. It also yields an elegant proof of the rationality result for the Bianchi type-IX metric, which is presented in the following section.

Corollary 4.1. *Assuming the conditions and notations of Theorem 4.1 we have*

$$a_{2+r}(D^2) = \frac{1}{2^5 \pi^{4+r/2}} \int \mathrm{tr}\left(\sigma_{-4-r}(\Delta^{-1})\right) d^{3+r}\xi \, d^4 x.$$

Proof. It follows from the fact that if $\sigma(D^2) = p_2(x, \xi) + p_1(x, \xi) + p_0(x, \xi)$, where each p_i is homogeneous of order i in ξ, then $\sigma(\Delta) = p_2'(x, \xi) + p_1(x, \xi) + p_0(x, \xi)$, with $p_2'(x, \xi) = p_2(x, \xi) + (\xi_5^2 + \cdots + \xi_{4+r}^2)I$. Therefore the homogeneous terms $\sigma_{-2-j}(\Delta^{-1})$ of order $-2-j$ in the expansion $\sigma(\Delta^{-1}) \sim \sum_{j=0}^{\infty} \sigma_{-2-j}(\Delta^{-1})$ are independent of the coordinates of $\mathbb{T}^r$.

$\square$

We confirm the validity of the coefficients a_0, a_2, a_4, calculated for the Bianchi type-IX metric in §3 by noting that the method devised in the present section produces the same expressions. We stress that in practice the new method is significantly more convenient since the expression that leads to a Seeley-de Witt coefficient simplifies when one considers its restriction to the corresponding cosphere bundle in order to compute the noncommutative residue.

The noncommutative residue was originally discovered in the 1-dimensional case by Adler [1] and Manin [34]. Its coincidence with the Dixmier trace [11] on pseudo-differential operators of order $-m$ on an m-dimensional closed manifold indicates its applicability for explicit and convenient computations. It is also worth mentioning that a noncommutative residue developed for noncommutative tori [23, 22]

simplified a purely noncommutative heat kernel computation significantly and clarified in [20] the reason for mysterious and remarkable cancellations that occur in this type of computations.

5. Rationality of the Spectral Action for Bianchi Type-IX Metrics

The Seeley-de Witt coefficients a_{2n} appearing in the expansion of the spectral action for the Bianchi type-IX metric are expressed in terms of several variable polynomials with *rational* coefficients evaluated on the cosmic evolution factors $w_1(t), w_2(t), w_3(t)$, and their derivatives of certain orders. This extends the statement conjectured in [9] and addressed in [21] for Robertson-Walker metrics to a homogeneous *anisotropic* cosmological model.

In order to prove the rationality result for the Bianchi type-IX metric, similar to the treatment in [21], let us start with the crucial observation that the local forms $a_{2n}(x, D^2)d^3x$, where D is the Dirac operator of this geometry, are invariant over the spatial manifold $\mathbb{S}^3$. This can be seen from the defining formula (1) for the metric, in which the left invariance of the 1-forms $\sigma_1, \sigma_2, \sigma_3$, implies that the metric is invariant under any diffeomorphism arising from left multiplication by an element of $SU(2)$. Since the action is transitive and left multiplication by an element of $SU(2)$ is an isometry, any isometry-invariant function on $\mathbb{S}^3$ is independent of the spatial coordinates. In particular, the restriction of the kernel of e^{-tD^2} to the diagonal and consequently the differential forms $a_{2n}(x, D^2) d^4x$ are invariant, and if we set $a_{2n}(x, D^2) d^3x = \tilde{a}_{2n}(x, D^2) dvol_g$, where $dvol_g$ is the volume form, then $\tilde{a}_{2n}(x, D^2)$ is independent of the spatial coordinates for any n.

Furthermore, we can easily determine the general form of $\tilde{a}_{2n}(x, D^2)$ by applying the method devised in §4, which is based on making use of the noncommutative residue, combined with the Künneth formula and restricting the computations to the cosphere bundle. In fact, writing $\sigma(\Delta^{-1}) \sim \sum_{j=-2}^{-\infty} \sigma_j(x, \xi)$, where each σ_j is homogeneous of order j, one finds recursively that

$$
\begin{aligned}
\sigma_{-2}(x, \xi) &= p_2(x, \xi)^{-1}, \\
\sigma_{-2-n}(x, \xi) &= -\sum \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_j(x, \xi) D_x^\alpha p_k(x, \xi) \sigma_{-2}(x, \xi) \qquad (n > 0),
\end{aligned}
$$

where the summation is over all multi-indices of non-negative integers α, $-2 - n < j \leq -2, 0 \leq k \leq 2$, such that $|\alpha| - j - k = n$.

Thus, if we define $\zeta_{\mu+1} = \sum_\nu e_\mu^\nu \xi_{\nu+1}$, then it can be shown by induction that

$$
(3) \qquad \sigma_{-2-n}(x, \xi)|_{S^*(M \times \mathbb{T}^{n-2})} = \sigma_{-2-n}(x, \xi(\zeta))|_{\zeta \in \mathbb{S}^{n+1}} = (w_1 w_2 w_3)^{-\frac{3}{2}n} P_n(\zeta),
$$

for any integer $n \geq 2$, where $P_n(\zeta)$ is a polynomial in $\zeta_1, \ldots, \zeta_{n+2}$, with the coefficients being matrices whose entries are in the algebra generated by rational numbers, trigonometric functions of the spatial coordinates, and $W_i^{(p)}$ where $i \in \{1, 2, 3\}$, $p \in \{0, 1, \ldots, n\}$. This fact leads to the following statement about the general form of the coefficients $a_{2n}(D^2)$.

Theorem 5.1. *For any non-negative integer n, the coefficient $a_{2n}(D^2)$ in the expansion of the spectral action for the Bianchi-type IX metric is of the form*

$$
a_{2n}(D^2) = (w_1 w_2 w_3)^{1-3n} Q_{2n}\left(w_1, w_2, w_3, w_1', w_2', w_3', \ldots, w_1^{(2n)}, w_2^{(2n)}, w_3^{(2n)}\right),
$$

where Q_{2n} is a polynomial with rational coefficients.

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Proof. It follows from Corollary 4.1 that

$$
\begin{aligned}
a_{2+r}(x, D^2) &= \frac{1}{2^5 \pi^{4+r/2}} \int_{S^*(M \times \mathbb{T}^r)} \mathrm{tr}(\sigma_{-4-r}(\Delta^{-1})) \, d^{3+r}\xi \\
&= \frac{1}{2^5 \pi^{4+r/2}} \int_{\mathbb{S}^{3+r}} \mathrm{tr}(\sigma_{-4-r}(\Delta^{-1})) \, dvol_g \, d^{3+r}\zeta,
\end{aligned}
$$

where $\zeta_{\mu+1} = \sum_\nu e_\mu^\nu \xi_{\nu+1}$ so that the Jacobian of the coordinate transformation is just $dvol_g$. This implies that

$$
\tilde{a}_{2n}(x, D^2) = \frac{1}{2^5 \pi^{n+3}} \int_{\mathbb{S}^{2n+1}} \mathrm{tr}(\sigma_{-2-2n}(\Delta^{-1})) \, d^{2n+1}\zeta,
$$

which, as shown above, is independent of the spatial coordinates. Thus, we have

$$
\begin{aligned}
a_{2n}(D^2) &= \int \tilde{a}_{2n}(x, D^2) \, dvol_g \\
&= \mathrm{Vol} \cdot \tilde{a}_{2n}(D^2) \\
&= 16\pi^2 w_1 w_2 w_3 \, \tilde{a}_{2n}(D^2) \\
&= \frac{w_1 w_2 w_3}{2\pi^{n+1}} \int_{\mathbb{S}^{2n+1}} \mathrm{tr}(\sigma_{-2-2n}(\Delta^{-1})) \, d^{2n+1}\zeta.
\end{aligned}
$$

The equation (3) allows us to write $\sigma_{-2-2n}(x, \xi(\zeta)) = (w_1 w_2 w_3)^{-3n} P_{2n}(\zeta)$, which yields

$$
\begin{aligned}
a_{2n}(D^2) &= \frac{(w_1 w_2 w_3)^{1-3n}}{2\pi^{n+1}} \int_{\mathbb{S}^{2n+1}} \mathrm{tr}\left(P_{2n}(\zeta)(\Delta^{-1})\right) d^{2n+1}\zeta \\
&= (w_1 w_2 w_3)^{1-3n} Q_{2n}\left(w_1, w_2, w_3, w_1', w_2', w_3', \ldots, w_1^{(2n)}, w_2^{(2n)}, w_3^{(2n)}\right),
\end{aligned}
$$

Note that $\mathrm{tr}(P_{2n}(\zeta))$ is a polynomial in $\zeta_1, \zeta_2, ..., \zeta_{2n+2}$, with the coefficients in the algebra generated by the rational numbers, trigonometric functions of the spatial coordinates, and $w_i^{(p)}$ where $i \in \{1, 2, 3\}$, $p \in \{0, 1, \ldots, 2n\}$.

The integral of a monomial $m_\alpha(\zeta) = c_\alpha \zeta_1^{\alpha_1} \cdots \zeta_{2n+2}^{\alpha_{2n+2}}$ over $\mathbb{S}^{2n+1}$ is either 0, or can be written as

$$
\int_{\mathbb{S}^{2n+1}} m_\alpha(\zeta) \, d^{2n+1}\zeta = \frac{2c_\alpha \prod_j \Gamma(\frac{\alpha_j+1}{2})}{\Gamma(n+1+\frac{|\alpha|}{2})},
$$

if each α_j is an even non-negative integer. Also, recall that $\Gamma(\frac{n}{2}) = q\pi^{\frac{1}{2}}$ for some $q \in \mathbb{Q}$ when $n \in 2\mathbb{N}+1$, and $\Gamma(\frac{n}{2}) \in \mathbb{Z}$ when $n \in 2\mathbb{N}$. Therefore we have

$$
\int_{\mathbb{S}^{2n+1}} m_\alpha(\zeta) \, d^{2n+1}\zeta = q\pi^{\frac{2n+2}{2}} = q\pi^{n+1},
$$

for some $q \in \mathbb{Q}$ if $c_\alpha \in \mathbb{Q}$. Since $a_{2n}(D^2) = (w_1 w_2 w_3)^{1-3n} Q_{2n}$ has no spatial dependence, we conclude that

$$
Q_{2n} = \frac{1}{2\pi^{n+1}} \int_{\mathbb{S}^{2n+1}} \mathrm{tr}(P_{2n}(\zeta)(\Delta^{-1})) \, d^{2n+1}\zeta
$$

belongs to the algebra generated by the $w_i^{(p)}$ and rational numbers. $\qquad\square$

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6. Gravitational Instantons, Modular forms, and Rationality

Among the Euclidean Bianchi type-IX models, an especially interesting class consists of the Bianchi IX gravitational instantons. A gravitational instanton is both self-dual (that is, the Weyl curvature tensor is self-dual) and an Einstein metric (the Ricci tensor is proportional to the metric). A remarkable feature of Bianchi IX gravitational instantons with $SU(2)$ symmetry is that they can be completely classified in terms of solutions to Painlevé VI integrable systems, [27, 44, 46]. The latter are a 4-parameter family of singular ordinary differential equations of the form

$$
\frac{d^2 X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2 - \left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt}
$$
$$
+ \frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right).
$$

The self-dual equation for the $SU(2)$ Bianchi IX metrics is written in [44] as an ordinary differential equation in the w_i and in additional functions α_i, $i = 1, 2, 3$ that arise as the components of the connection 1-form in a basis of anti-self-dual 2-forms, see [46]. In terms of the conformally invariant variable $x = (\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)^{-1}$ the self-dual equations for the Riemannian Bianchi IX metric can be rephrased as a system of equations

$$
w_i = \Omega_i x'(x(1-x))^{-1/2},
$$

$$
\Omega_1' = -\frac{\Omega_2\Omega_3}{x(1-x)}, \quad \Omega_2' = -\frac{\Omega_3\Omega_1}{x}, \quad \Omega_3' = -\frac{\Omega_1\Omega_2}{1-x}.
$$

These in turn can then be reduced to a case of the Poincaré VI equation with parameters

$$
(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8}),
$$

see [44, 46]. In [3], the solutions to this equation are given explicitly in terms of a parameterization involving theta functions and theta characteristics

$$
\vartheta[p,q](z, i\mu) := \sum_{m\in\mathbb{Z}} \exp\left(-\pi(m+p)^2\mu + 2\pi i(m+p)(z+q)\right).
$$

Namely, with the notation $\vartheta[p,q] := \vartheta[p,q](0, i\mu)$, and

$$
\vartheta_2 := \vartheta[1/2, 0], \qquad \vartheta_3 := \vartheta[0,0], \qquad \vartheta_4 := \vartheta[0, 1/2],
$$

one finds $\alpha_i = 2\,\partial_\mu \log\vartheta_{i+1}$ and

$$
w_1 = -\frac{i}{2}\vartheta_3\vartheta_4\frac{\frac{\partial}{\partial q}\vartheta[p, q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]}, \quad w_2 = \frac{i}{2}\vartheta_2\vartheta_4\frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2}, q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]},
$$

$$
w_3 = -\frac{1}{2}\vartheta_2\vartheta_3\frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2}, q]}{\vartheta[p,q]}.
$$

The asymptotics of these solutions were analyzed in [35], where it is shown that, for large μ, they approximate Eguchi-Hanson type gravitational instantons with $w_2 = w_3 \neq w_1$, [17].

It is clear that, for the Bianchi IX gravitational instantons, using the parameterization of [3], the Seeley-de Witt coefficients a_{2n} of the spectral action are rational

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functions, with $\mathbb{Q}$-coefficients, in the $\vartheta_2, \vartheta_3, \vartheta_4, \vartheta[p, q], \partial_q \vartheta[p, q]$ and $e^{i\pi p}$ and derivatives, hence they belong to the field generated, over $\mathbb{Q}$, by these functions. We will return in a second part of this work [19] to discuss in detail the arithmetic properties of the spectral action for Bianchi IX gravitational instantons.

In this perspective, one can view the rationality question about the spectral action in a similar light to analogous questions that occur whenever arithmetic and number theoretic structures arise in theoretical physics. For example, when Feynman integrals are interpreted as periods (see [36] for an overview of that setting), the fact that the relevant amplitude forms and domains of integration are algebraic over $\mathbb{Q}$ (or $\mathbb{Z}$) has direct implications on the class of numbers that arise as periods. Another such instance of arithmetic structures in physics, where rational coefficients play an important role, is in the zero temperature KMS states of quantum statistical mechanical systems: in the case constructed in [15] (see also Chapter 3 of [16]) for instance, the construction of an arithmetic algebra of observables, defined over $\mathbb{Q}$, is linked to modular functions and makes it possible to have KMS states with values in the modular field. The relation between the spectral action of Bianchi IX gravitational instantons and modular forms will be discussed in [19].

7. Conclusions

We have shown that the Seeley-de Witt coefficients $a_{2n}(D^2)$ associated with the Dirac operator D of the Bianchi type-IX metric, which appear in the expansion of the spectral action [5], are expressed by polynomials with rational coefficients evaluated on the cosmic evolution factors $w_1(t), w_2(t), w_3(t)$, and their derivatives of certain orders. It is quite interesting that although this metric provides a homogeneous anisotropic cosmological model, after remarkable cancellations, only rational coefficients appear in the final expression for each $a_{2n}(D^2)$. Such a rationality result was first conjectured in [9] for Robertson-Walker metrics, which was addressed in [21].

Our proof of the rationality statement for the Bianchi type-IX model, similar to the argument given in [21], begins with the crucial observation that the kernel of e^{-tD^2} is restricted to have no spatial dependence on the diagonal. We then take a novel approach to proceed the argument. That is, we have devised a general method that expresses the Seeley-de Witt coefficients of a geometry as noncommutative residues of operators. This is an efficient method that allows explicit calculations with significantly less complexities, compared to the method of using parametric pseudodifferential calculus [26]. More importantly, it leads to an elegant proof of the rationality result for the Bianchi type-IX metric. To be more explicit, the Wodzicki residue [47, 48] involves an integration over the cosphere bundle of a manifold, and the expression for computing $a_{2n}(D^2)$ simplifies to our favor when restricted to the cosphere bundle, in the view of our method.

Acknowledgments

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