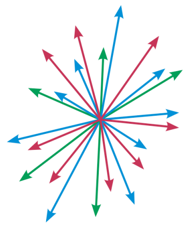
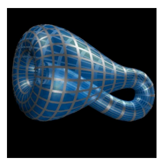
# The notion of Space in Mathematics through the lens of Modern Art

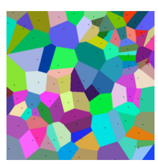
Matilde Marcolli

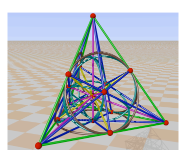
Century Books, Pasadena, July 2016

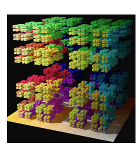
**"Space" is always decorated with adjectives** (like numbers: integer, rational, real, complex)

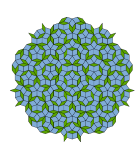
- Linear space 

- Topological space 

- Metric space 

- Projective space 

- Measure space 

- Noncommutative space 

**Space is a kind of structure**

Often (not always) a set (points of space) with some relational properties

- Operations: adding vectors, cutting and pasting, measuring size, intersections and unions

- Proximity relation: neighborhoods, closeness, convergence, distance

- Hierarchy of structures: Smooth ⇒ Topological ⇒ Measure space

- Symmetries: transformations preserving all the structure (very different symmetries for different kinds of spaces)

- Telling spaces apart: invariants (numerical, algebraic structures, discrete)

**Philosophical digression**: *Absolute* vs. *Relational* view of space

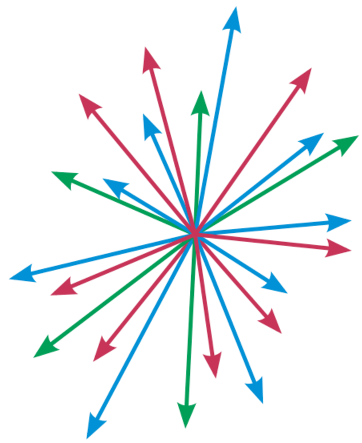
- Relational/Transformational viewpoint: Heraclitus, Democritus, Descartes, Leibniz, Bergson

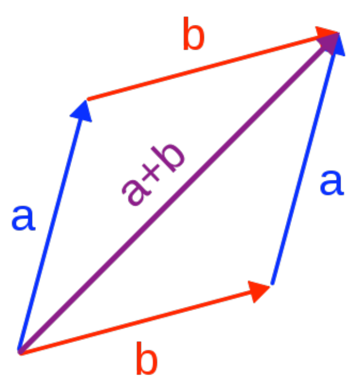
- Absolute view of space: Eleatic school (Parmenides, Zeno), Aristotle, Kant, Newton, Comte

Mathematical reconciliation of philosophical dichotomy:

- Felix Klein (Erlangen Program 1872): emphasis on transformation groups

**Linear spaces** (or vector spaces):

set of vectors  with dilations and composition of vectors



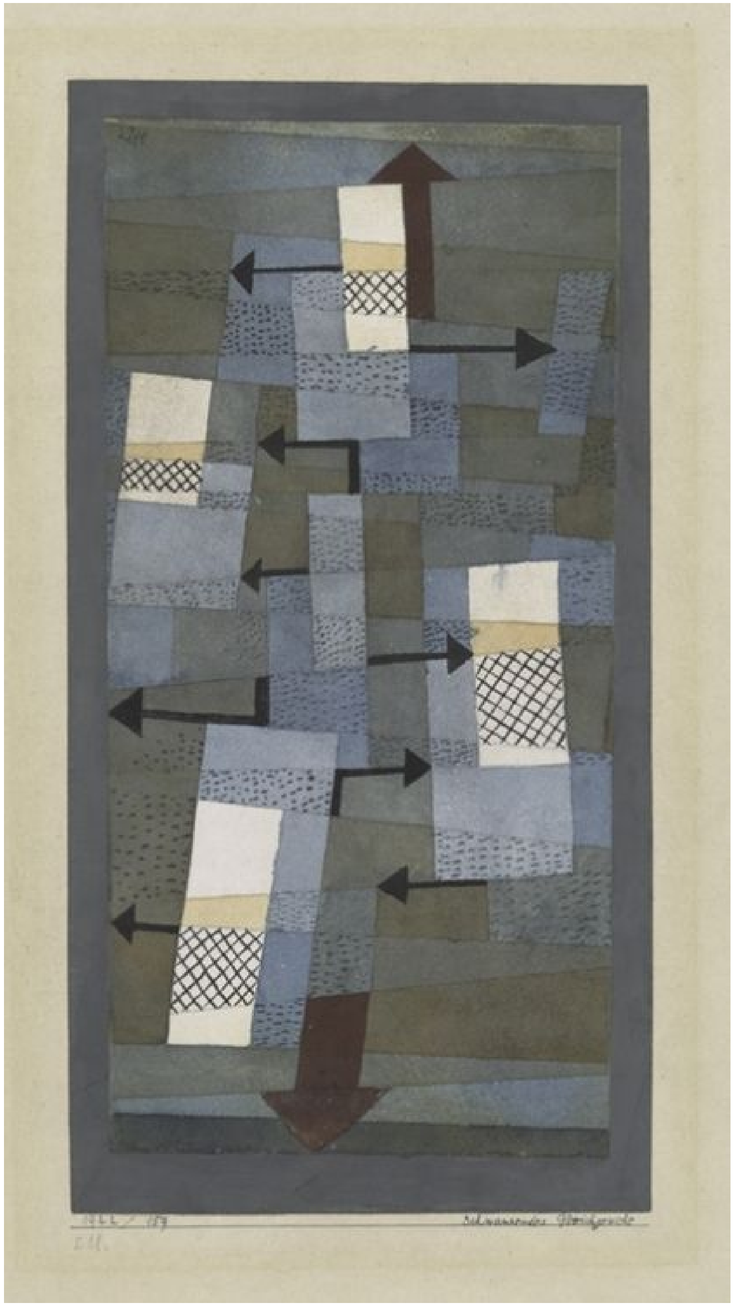

Examples: straight lines, planes, ...

Classical mechanics: equilibrium of forces

**Dimension:** Maximal number of linearly independent vectors

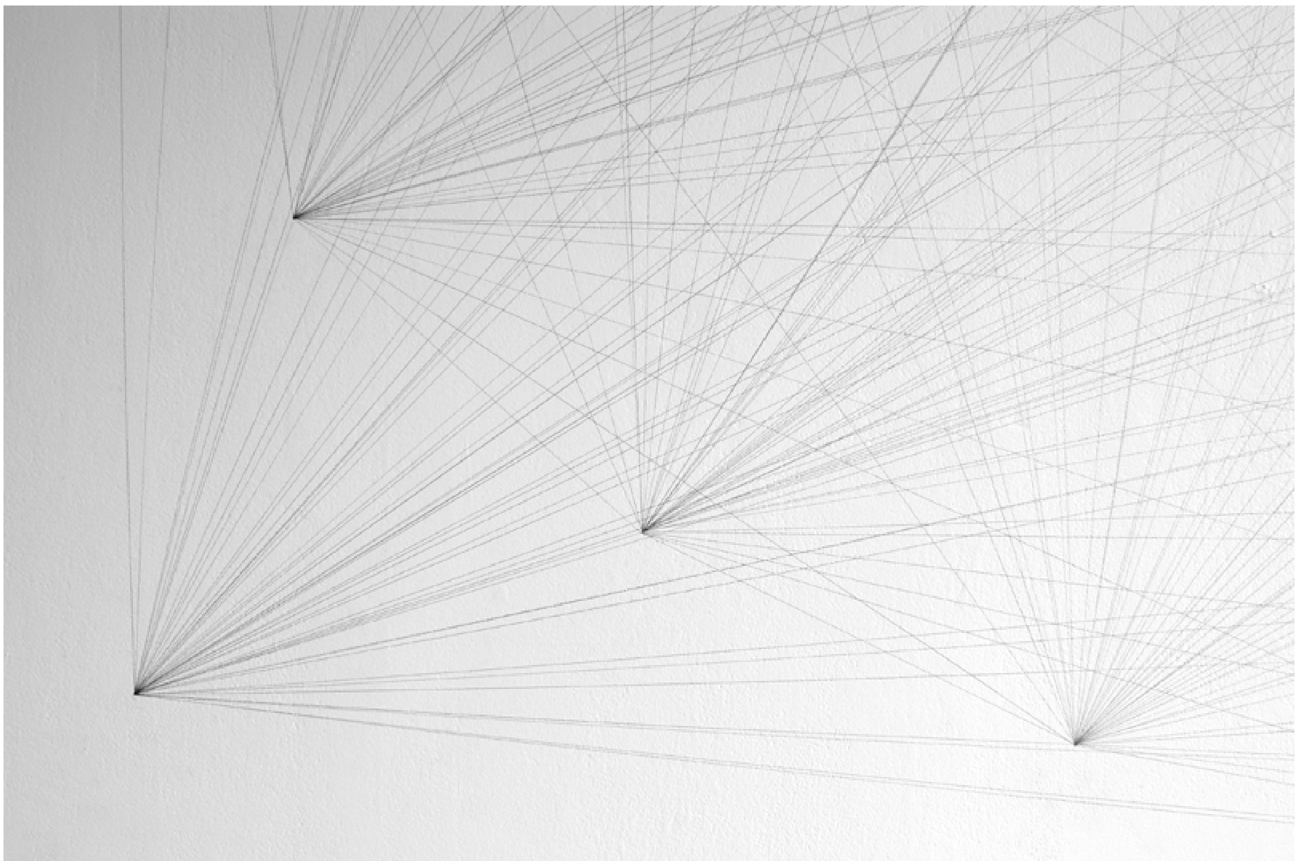
Paul Klee, *Mural from the Temple of Longing*, 1922

Vector spaces and sets of independent directions



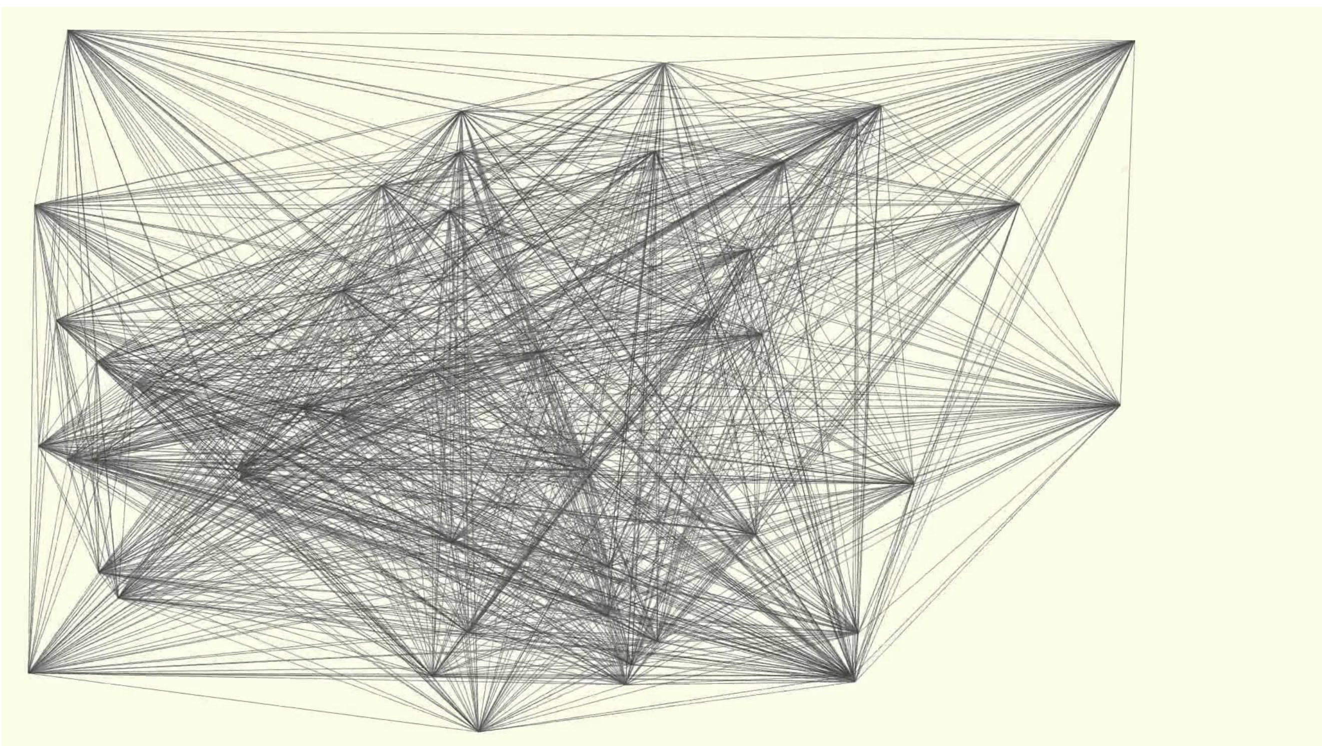
Paul Klee, *Unstable Equilibrium*, 1922

Vector calculus: equilibrium of forces



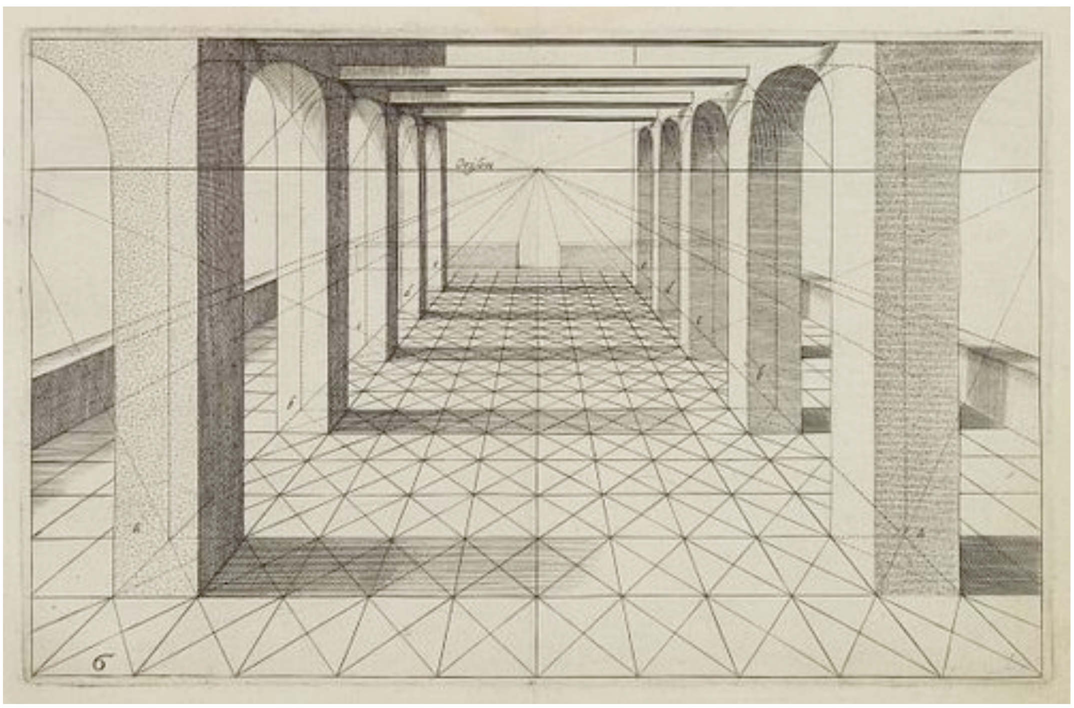
Eric Doeringer, *Sol LeWitt, Wall Drawing 118*, 2009

Vector calculus: vector compositions



Mitchell Chan, Studio F Minus, *Sol LeWitt, Wall Drawing 118*, 2013

**Projective spaces**:
non-zero vectors up to scaling:
(identify $v = (x_1, x_2, x_3)$ and $\lambda v = (\lambda x_1, \lambda x_2, \lambda x_3)$, scaled by nonzero $\lambda$)



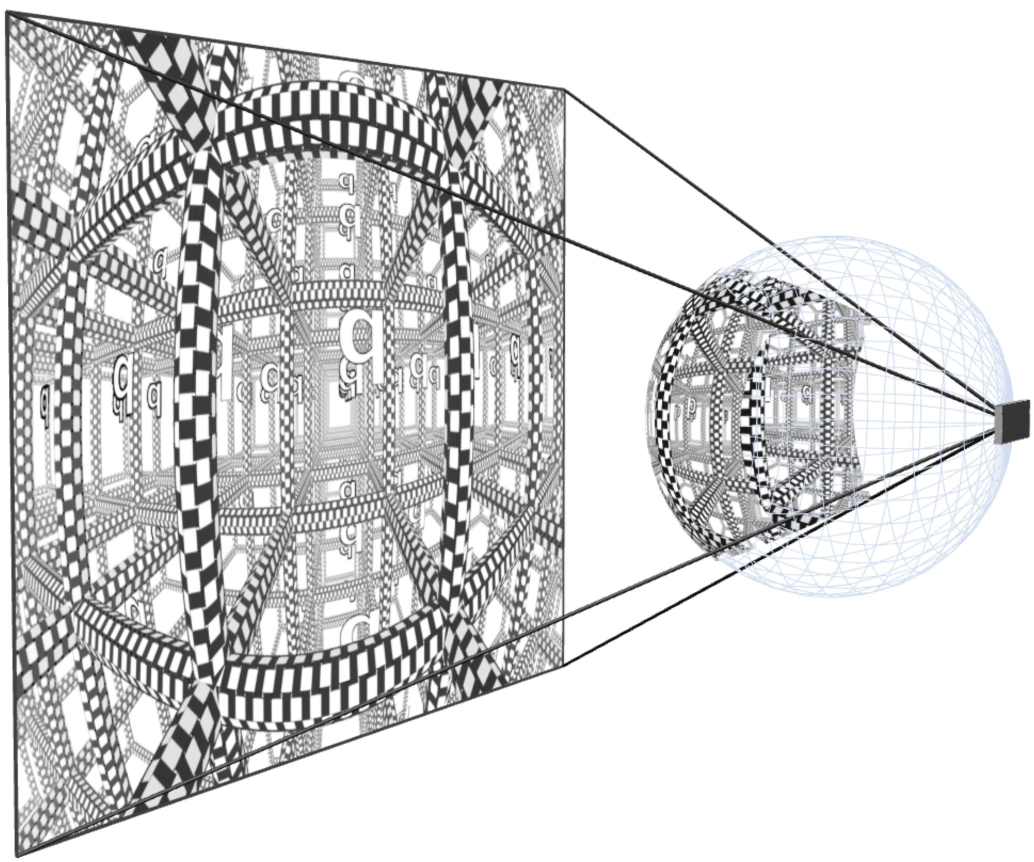
Renaissance perspective drawings

**Projective Transformations**
- map lines to lines
- preserve incidence relations
- do not preserve parallelism

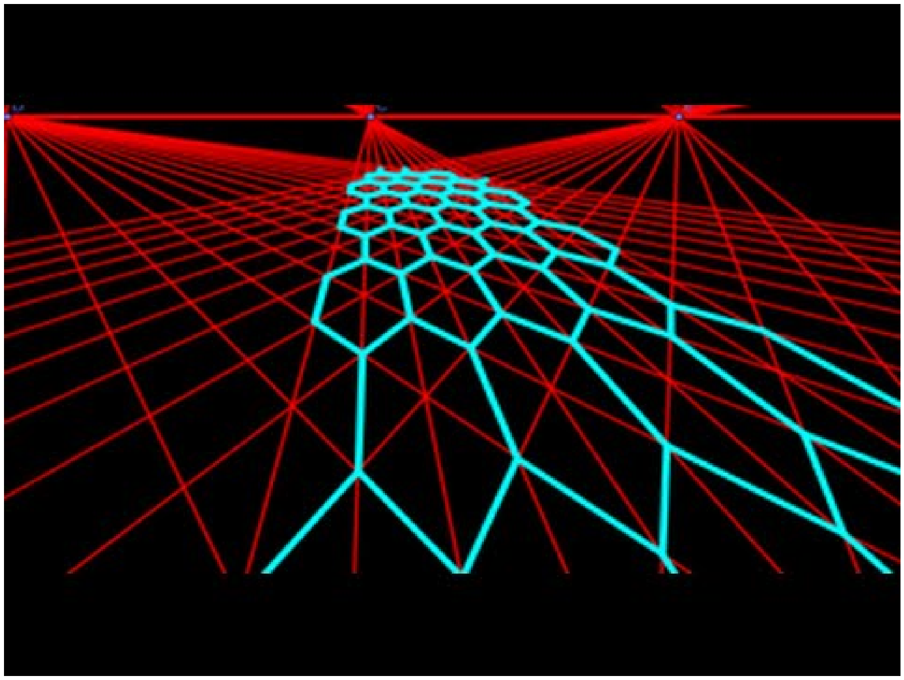
Projective geometry in Renaissance art:

- **Perspective**

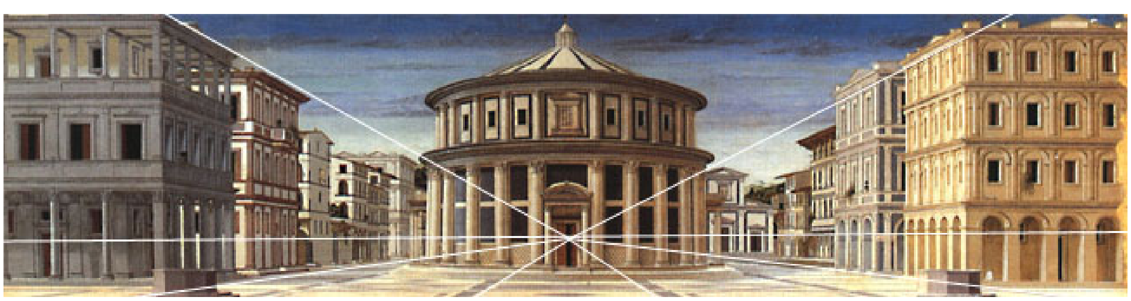
- **Anamorphosis**



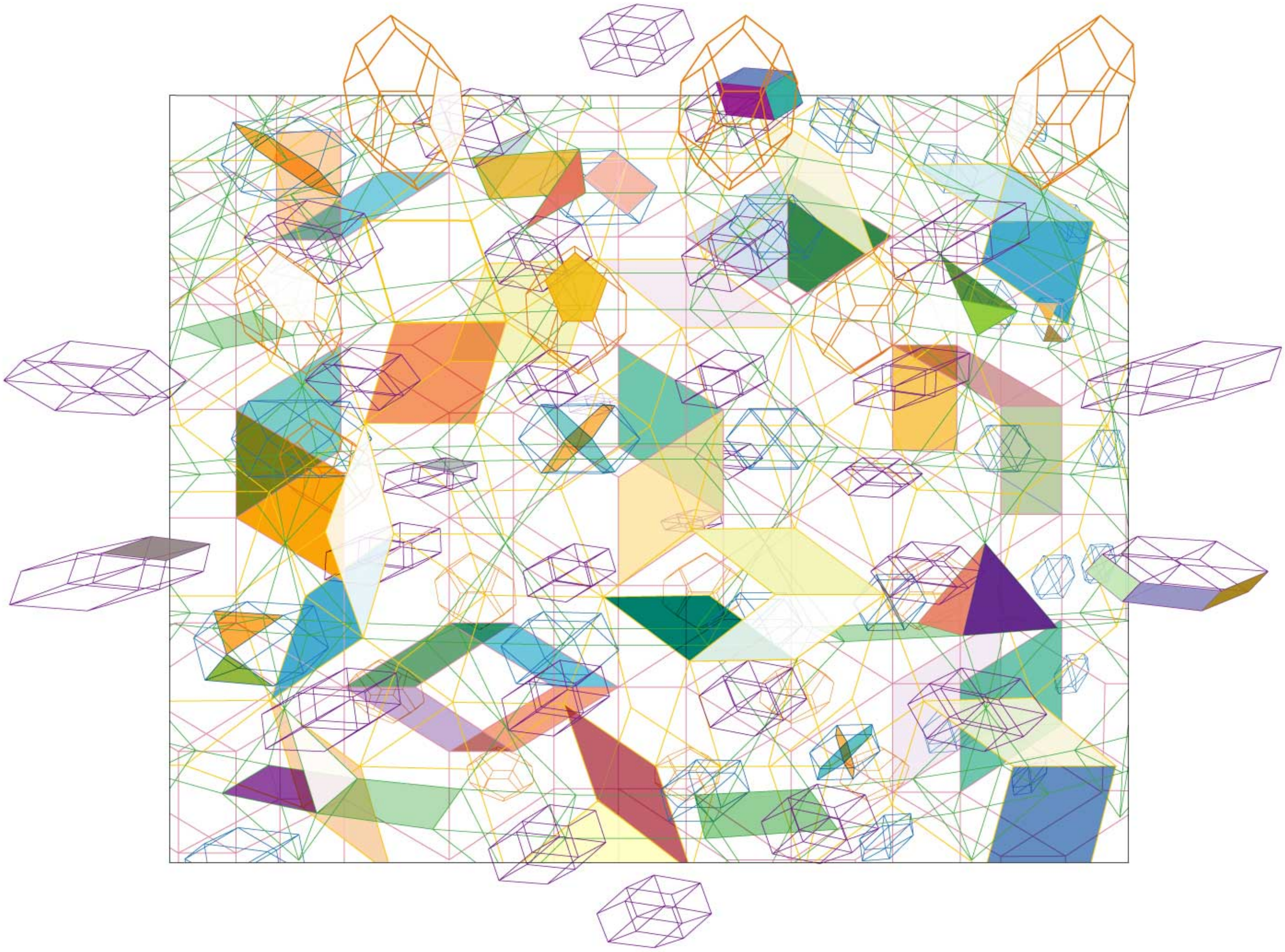
## Perspective



a computational device for drawing based on the axioms of projective geometry



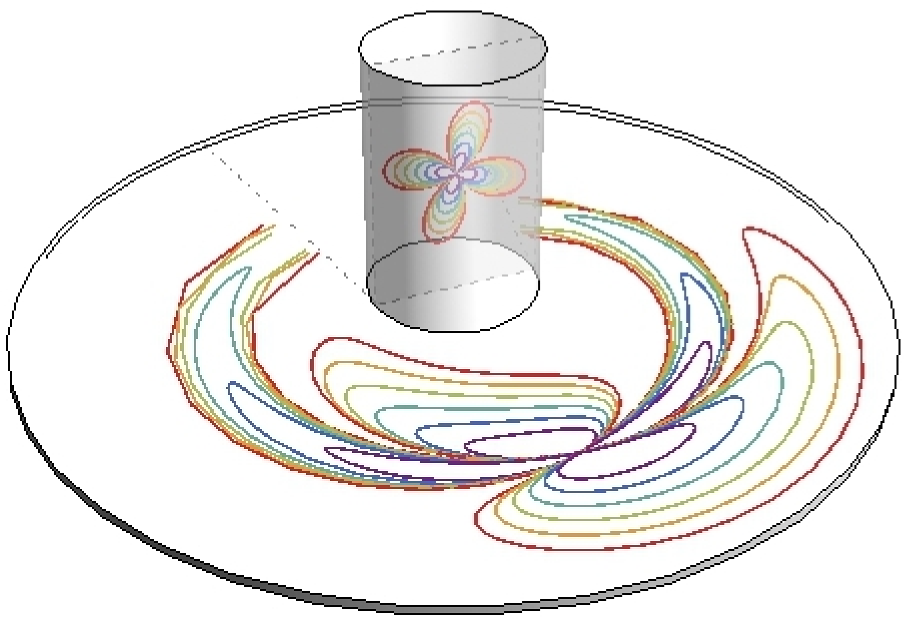
The Ideal City, 1470



Tony Robbins, *Drawing 50*, 2009

Rebecca Norton and Jeremy Gilbert-Rolfe,
*Awkward X2*, 2010

## Anamorphosis

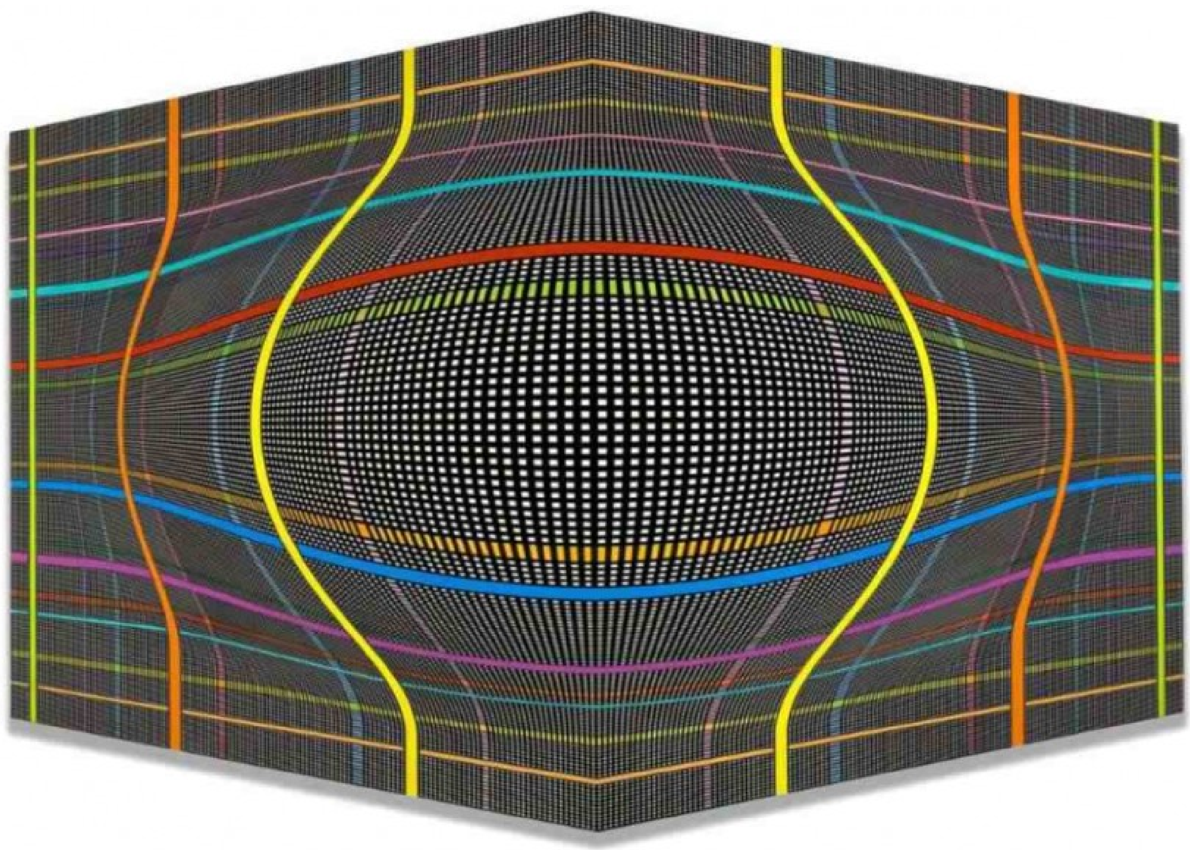


Cylindrical Anamorphosis

a computational device for deforming shapes according to projective transformations

Hans Holbein the Younger, *The Ambassadors*, 1533

Holbein skull undistorted



Linda Besemer, *Big Corner Bulge*, 2008

Sara Willett, *Anamorphosis*, 2010

Jean-Max Albert, *Reflet Anamorphose*,1985

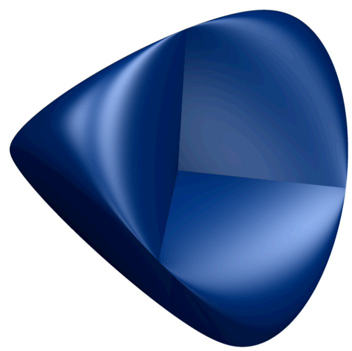
**Projective spaces**

1-dimensional real projective space = circle

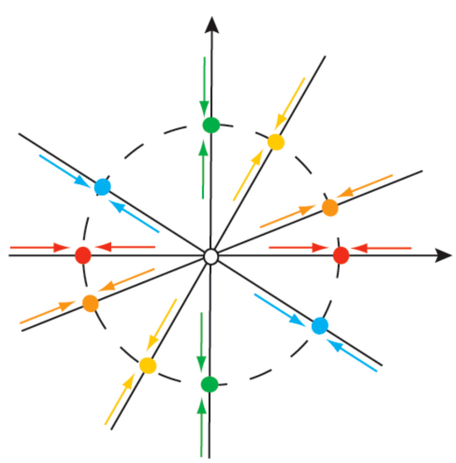
1-dimensional complex projective space = sphere

More interesting shapes:

2-dimensional real projective space:

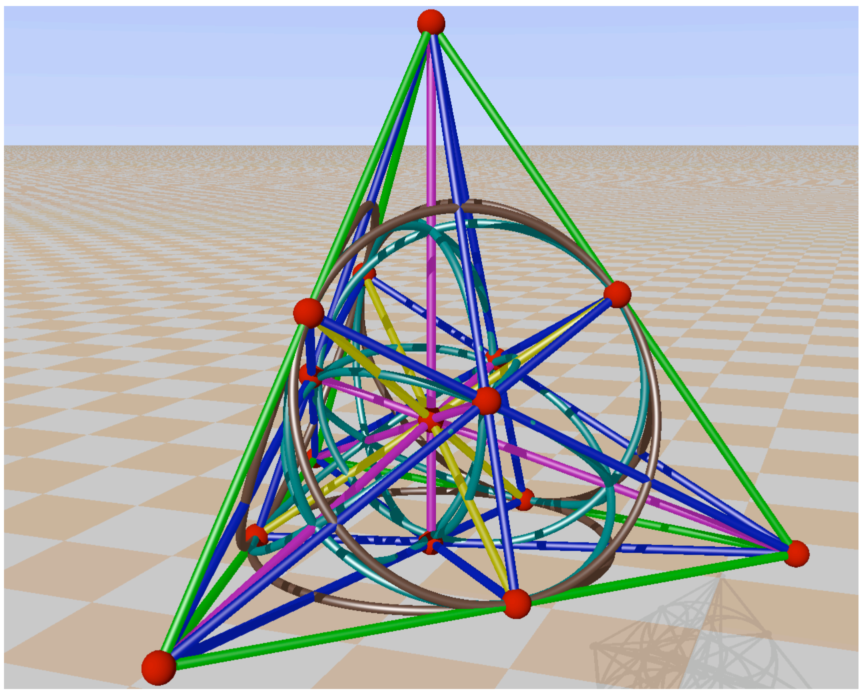


Identifying diametrically opposite points on the boundary of a disk



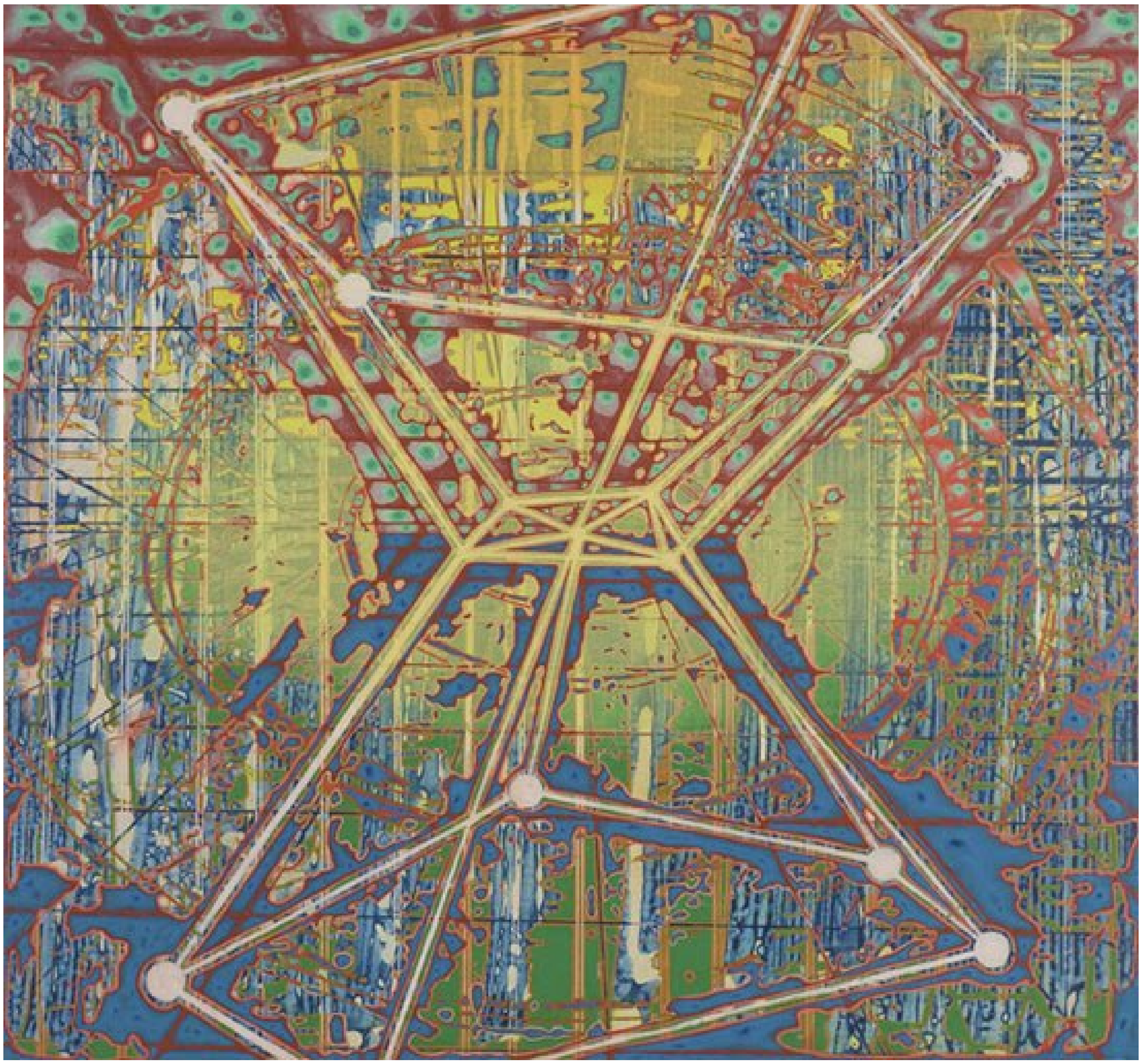
## Projective Spaces over Finite Fields

Different kinds of numbers (fields) $\Rightarrow$ different kinds of projective spaces



Finite projective spaces
(discrete versus continuum in geometry)

Relational properties: lines through given points, lines intersecting, planes containing lines, ...



Sarah Walker, *Extrapolator*, 2010

Sarah Walker, *Everywhere is Always*, 2010

**Topological spaces** formalize the relation of "being near" a point

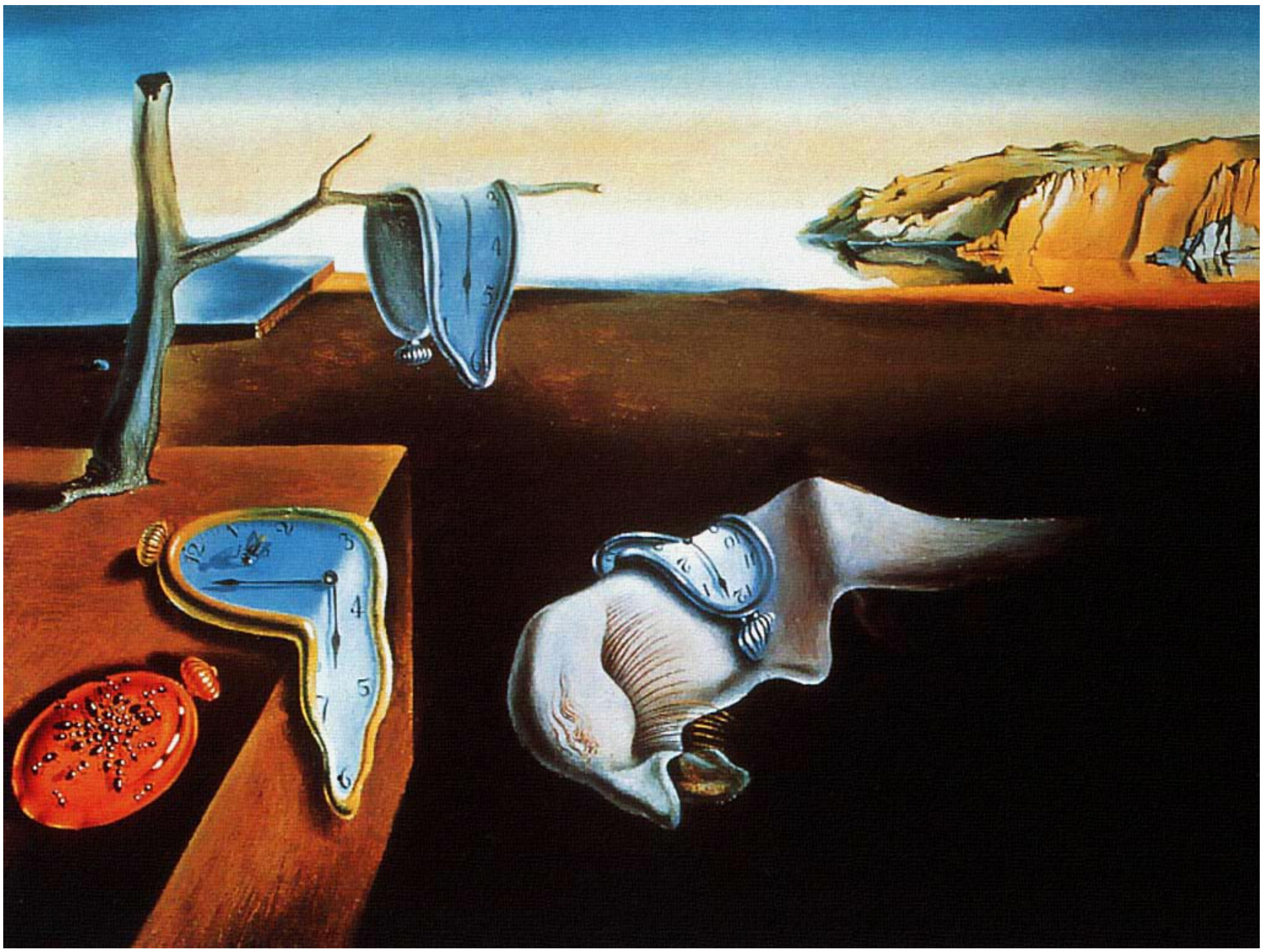
(qualitative: does not quantify how near)

Open condition: stable under small variations

(close condition: being on the border of two regions)

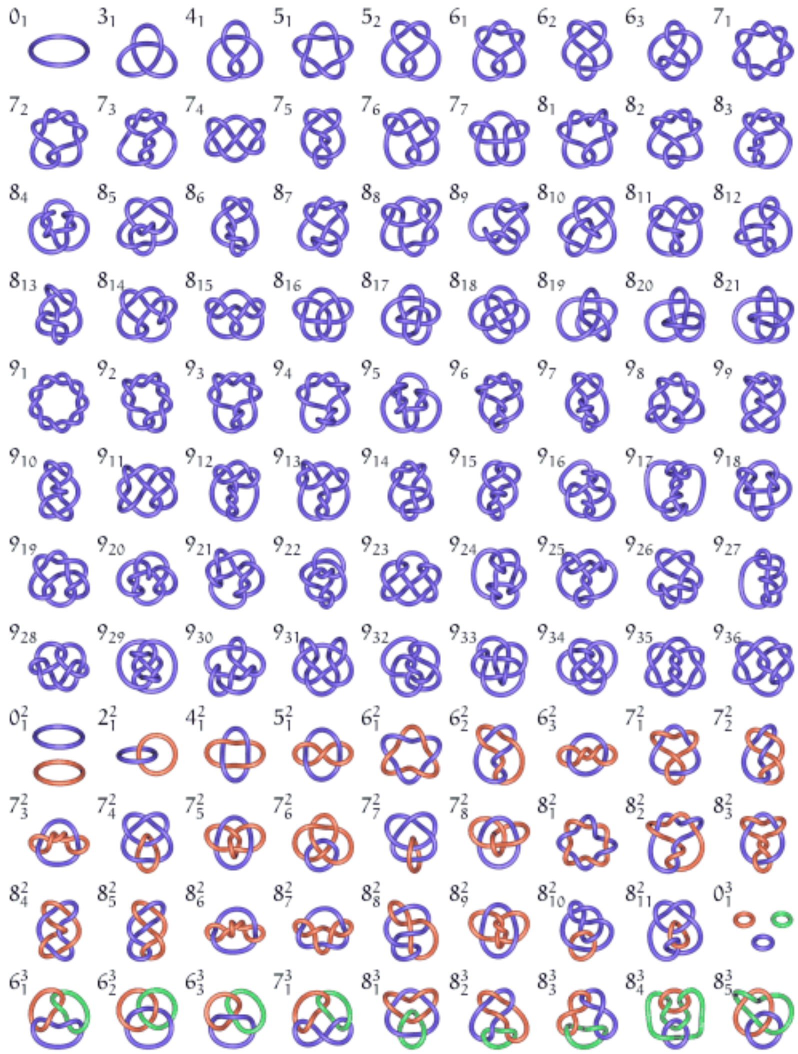
Transformations: continuous deformations

a donut is topologically the same as a cup of coffee

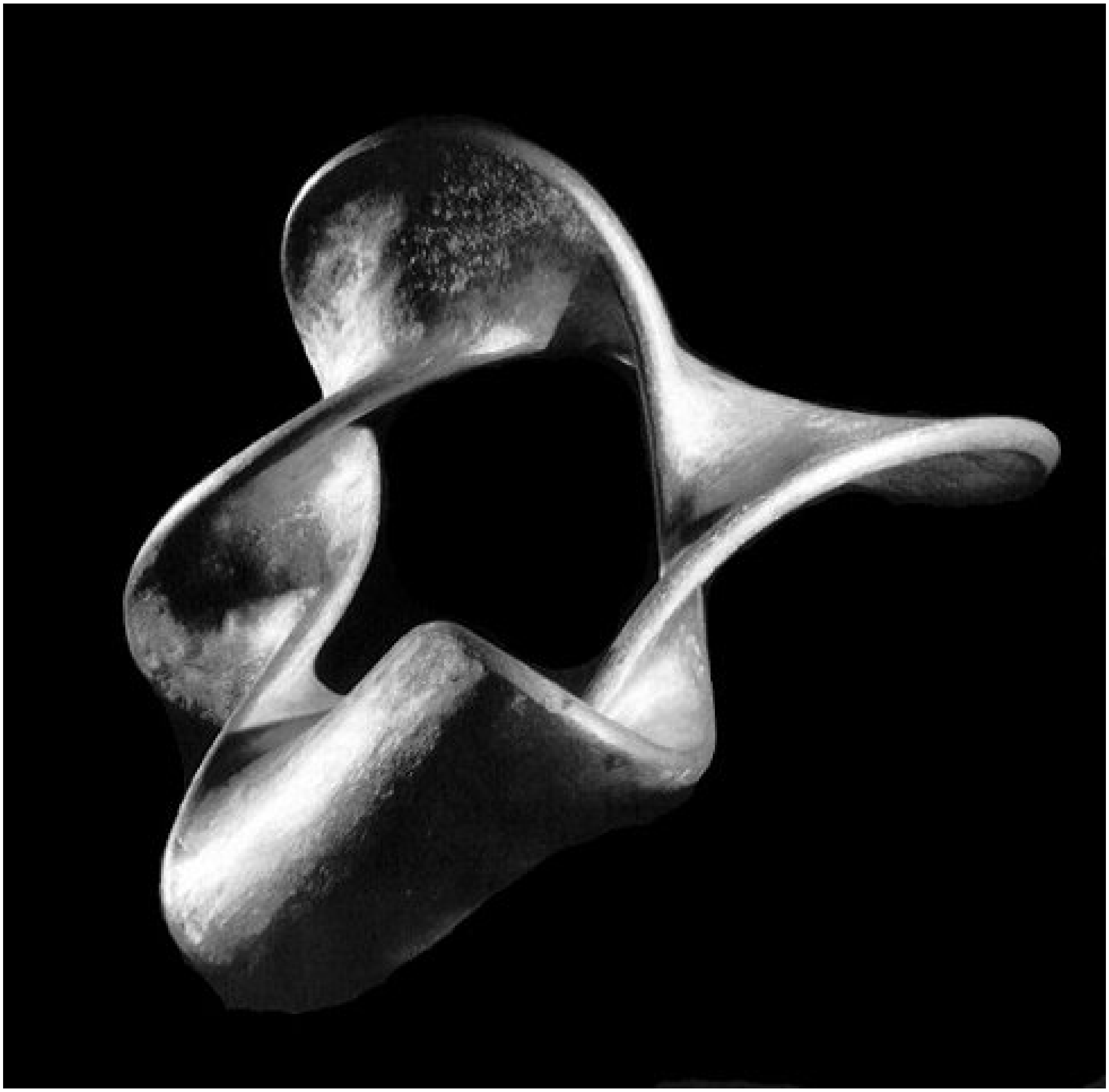


Salvador Dalí, *The Persistence of Memory*, 1931

# Knots and links

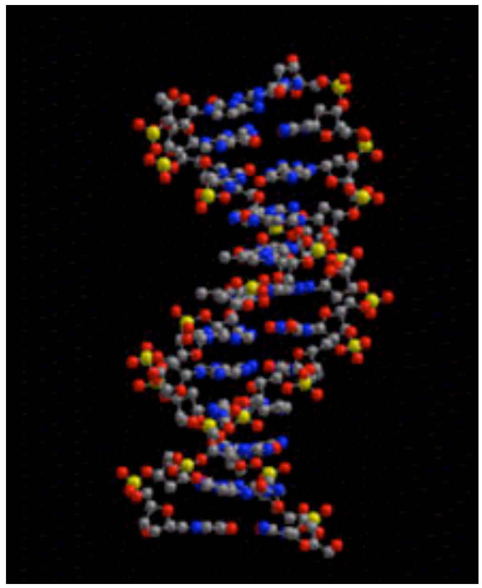



Topologically different: cannot be deformed one into the other without cutting
- *Invariants of knots*

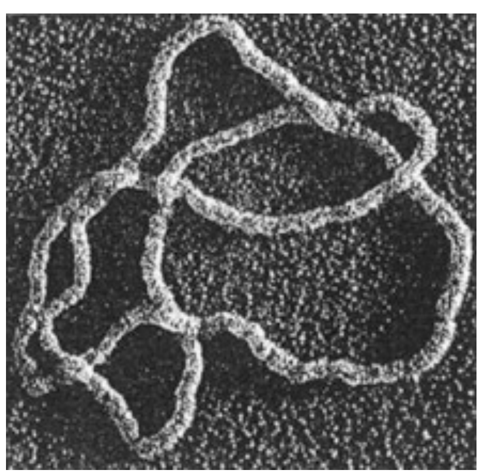


Attilio Pierelli, *Theory of the Universes: Knots*, 1986

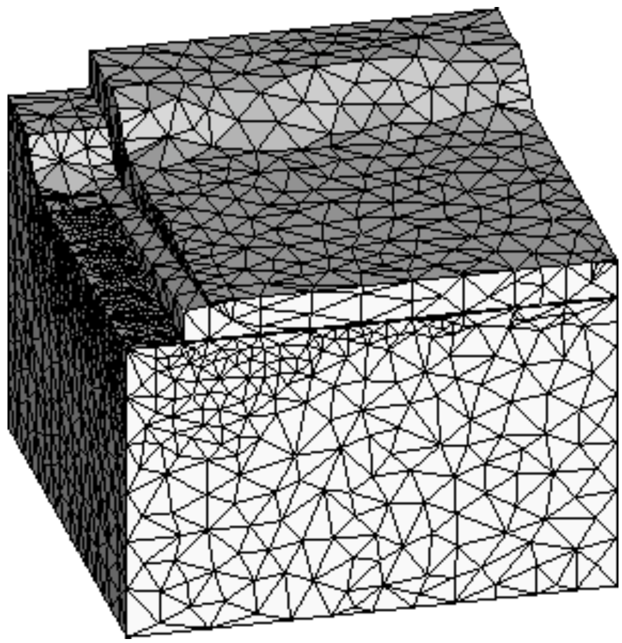
## Topology of knots and DNA



Topoisomerases enzymes act on the topology: unknotting DNA prior to replication

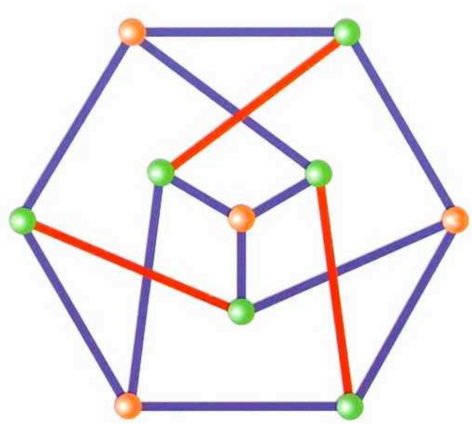


## Nice topological spaces: triangulations



Essential to computer graphics

## Graphs: simplest class of "piecewise linear" spaces

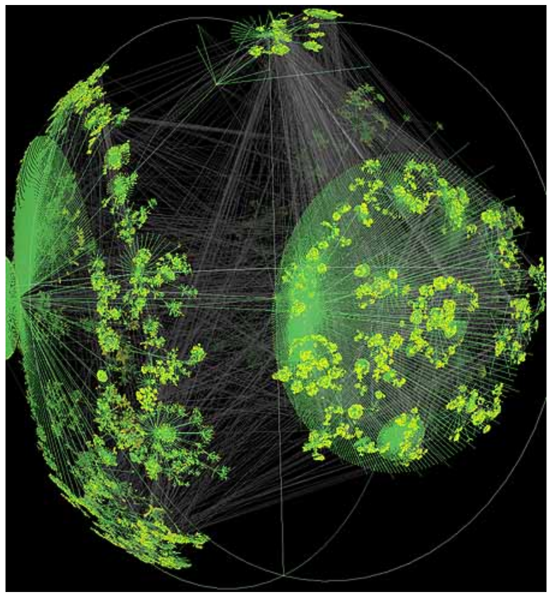


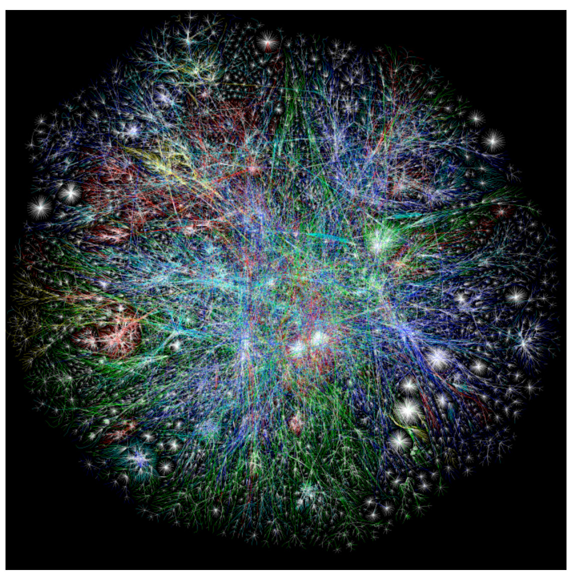
# Examples of graphs:


San Francisco subway system

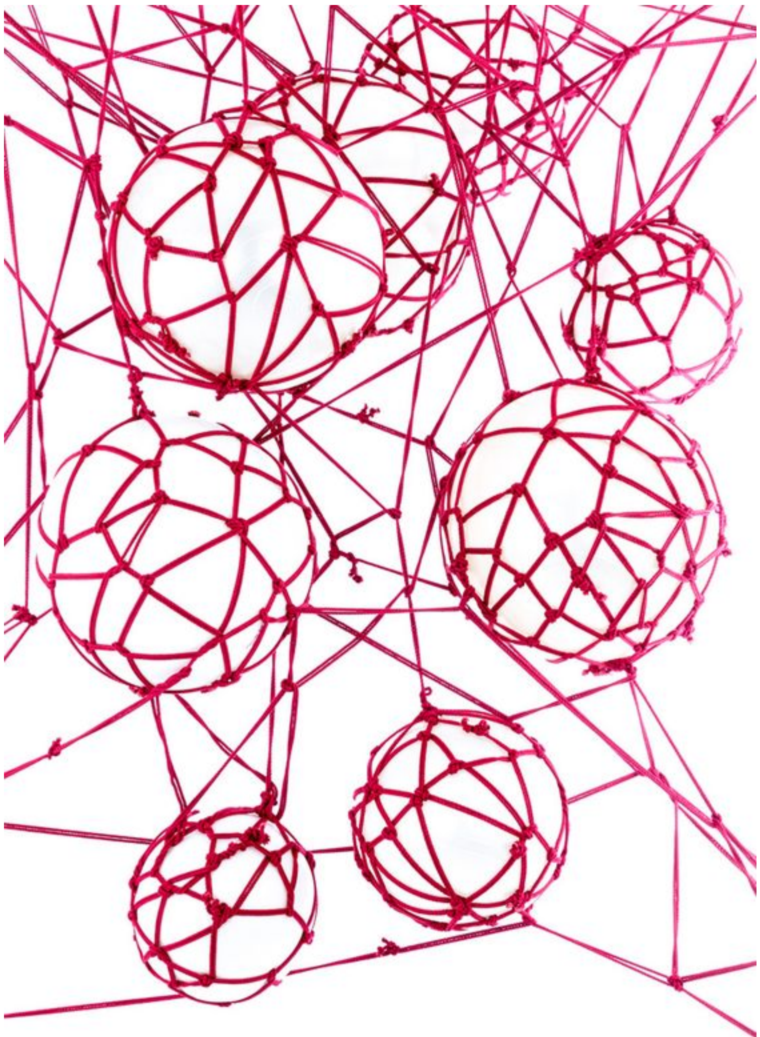
Moscow subway system

The most interesting graph of today:
the world wide web





Methods of topology for internet connectivity

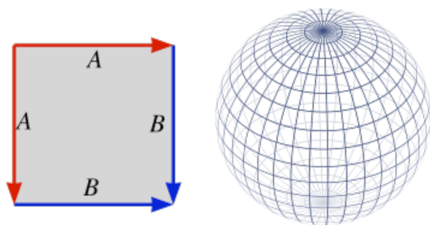


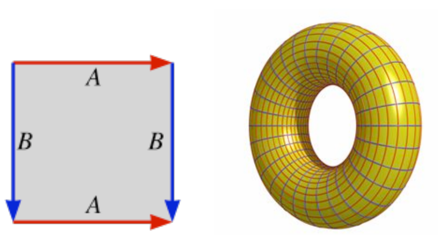
Hashime Kinoko, *Cell*, 2014

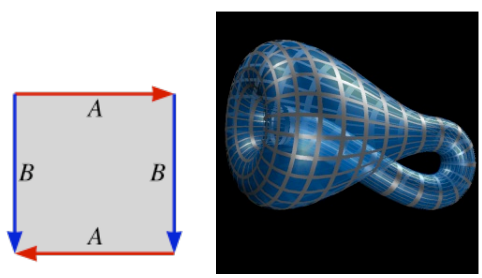
Marcus Sendlinger, *Tristar*, 2008

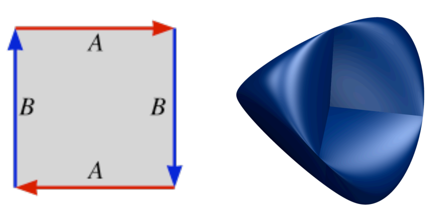
Matjuska Teja Krasek, *Double Star*, 2005

More examples of topological spaces:

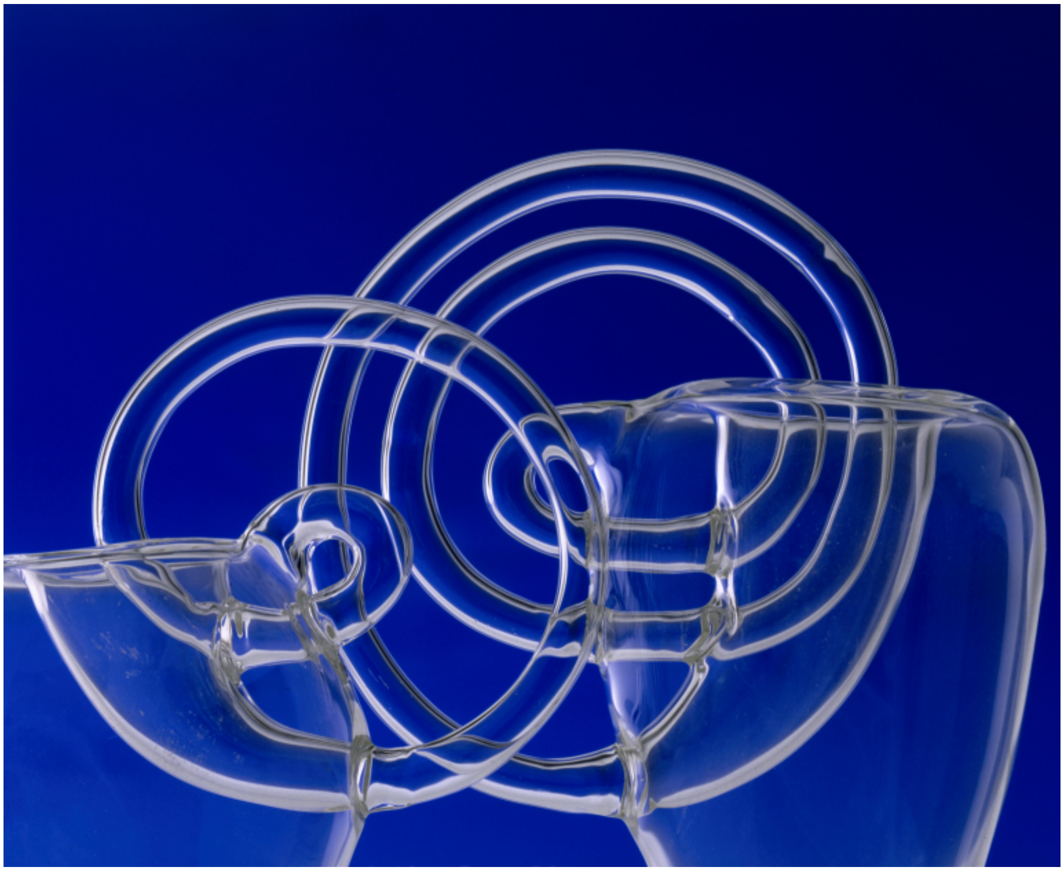
- Sphere: 


- Torus: 


- Klein bottle: 


- Real projective plane: 

Alan Bennett, *Klein Bottle*, 1995



Alan Bennett, *Glass Vessel*, 1995

How to distinguish topological spaces?

• Euler characteristic

$$\chi = \#\text{Faces} - \#\text{Edges} + \#\text{Vertices}$$

is a topological invariant

- Sphere: $\chi = 2$, orientable

- Real projective space: $\chi = 1$, non-orientable

- Klein bottle: $\chi = 0$, non-orientable

- Torus: $\chi = 0$, orientable

- Genus $g$ surface: $\chi = 2 - 2g$, orientable

- Orientability

Max Bill: Möbius band sculpture

Maurits Cornelis Escher: Möbius band

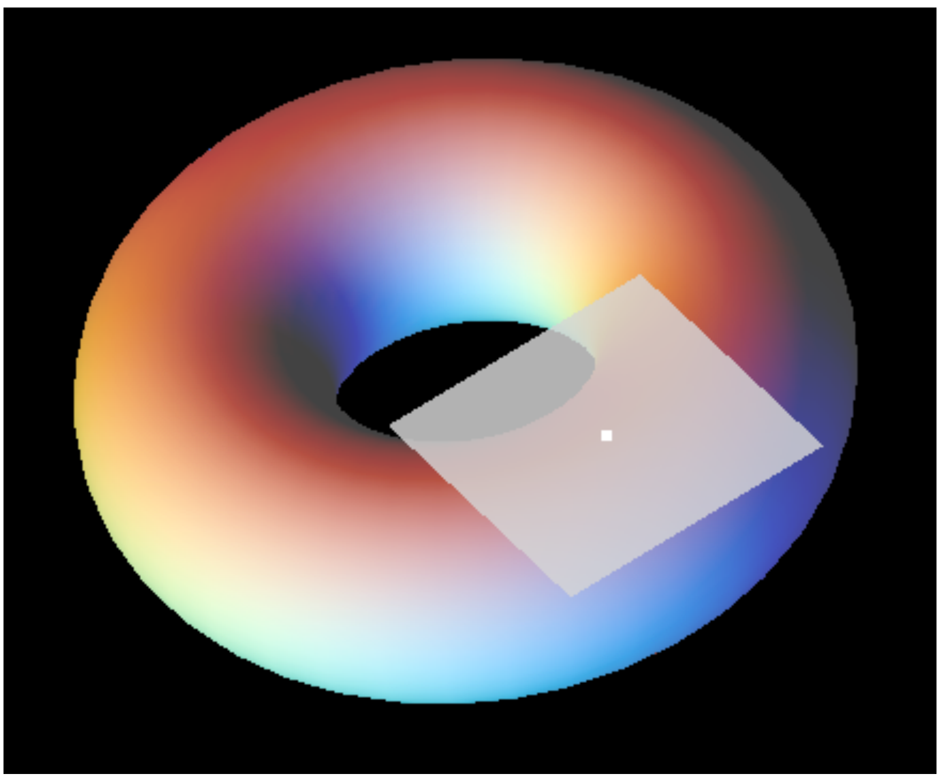
Alan Bennett, *Single Surface*, 1995

Higher genus non-orientable surface

**Smooth spaces** (or smooth manifolds):
Topological spaces locally indistinguishable from a vector space

Example: the Earth from ground level looks flat

Tangent space



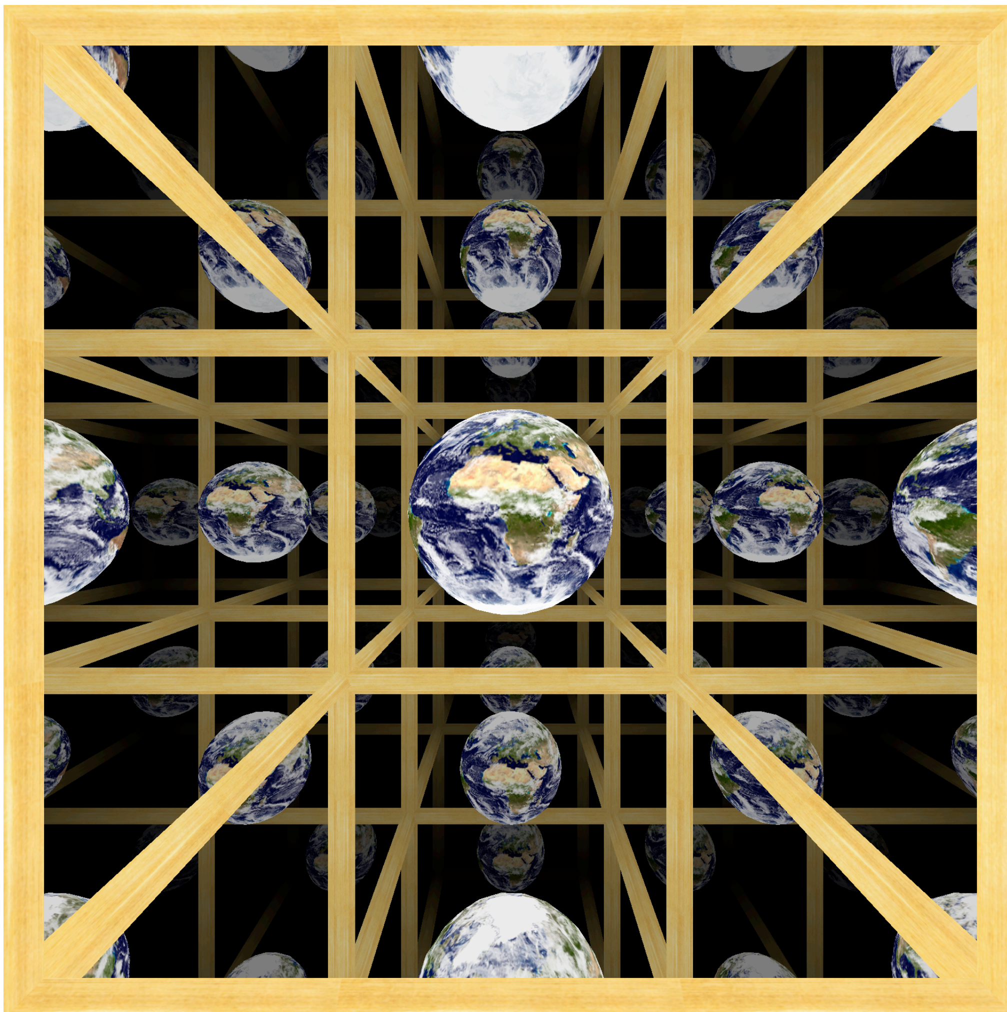
Local coordinates: number of independent parameters describing a physical system

- Dimension from tangent space (linear space)

Local versus global properties:

locally like flat space (linear space)

but globally: nontrivial topology



View from inside a 3-torus

(Jeff Weeks "The shape of space")

Smooth space $\Rightarrow$ Topological space
but beware ...

Exotic smoothness:
4-dimensional flat space has infinitely many different smooth structures (Donaldson)

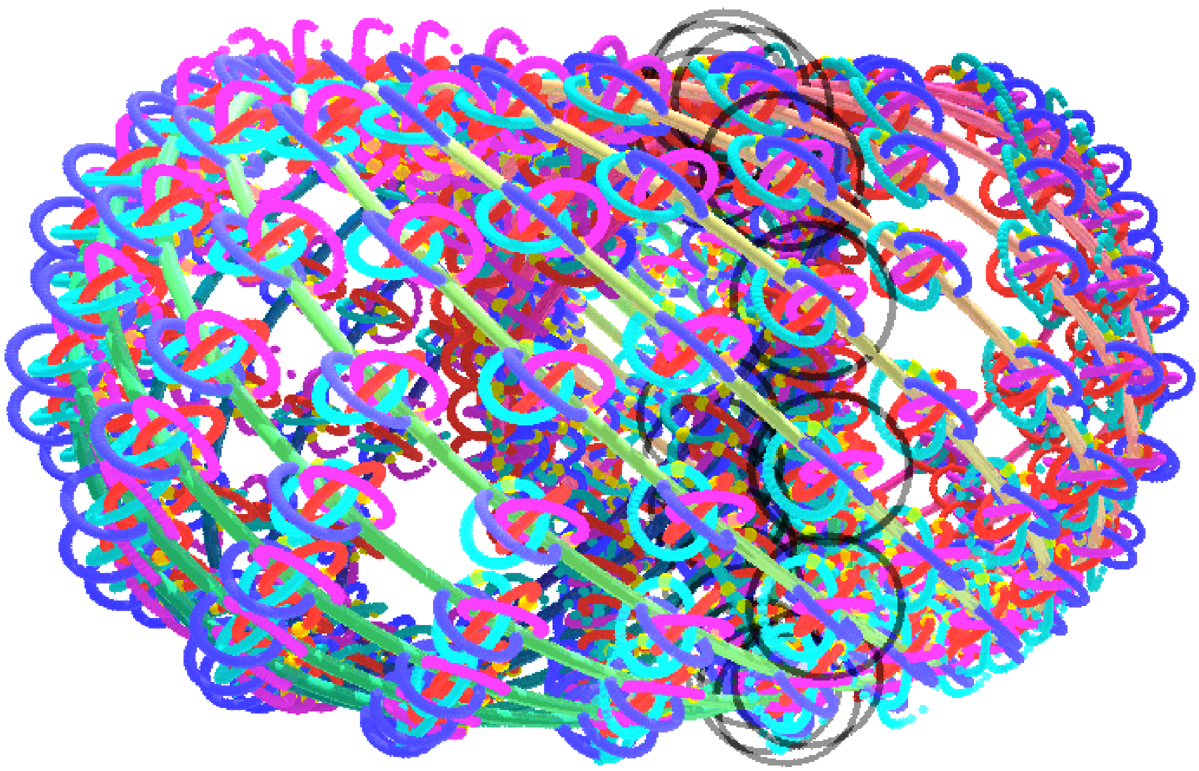
- small: contained inside ordinary flat space
- large: do not fit in ordinary space

Dimension 3 and 4 are the most complicated!!

Poincaré conjecture (Perelman):
there is only one type of 3-dimensional sphere

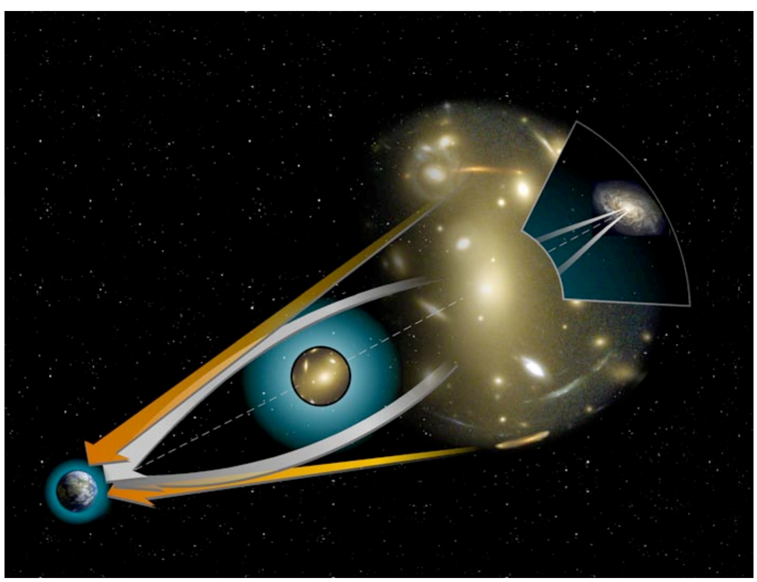
Smooth 4-dimensional sphere?? Unknown

... but exotic spheres in dimension 7 (Milnor)



Niles Johnson, *A slice of* $S^7$

Exotic smoothness can affect our understanding of the distant universe (gravitational lensing)

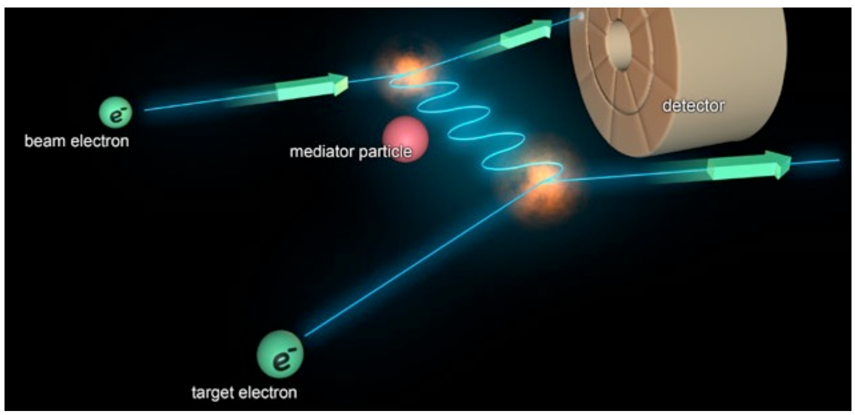


passing through a small exotic space changes lensing

## What detects exotic smoothness?

Not topological invariants (Euler characteristic etc)

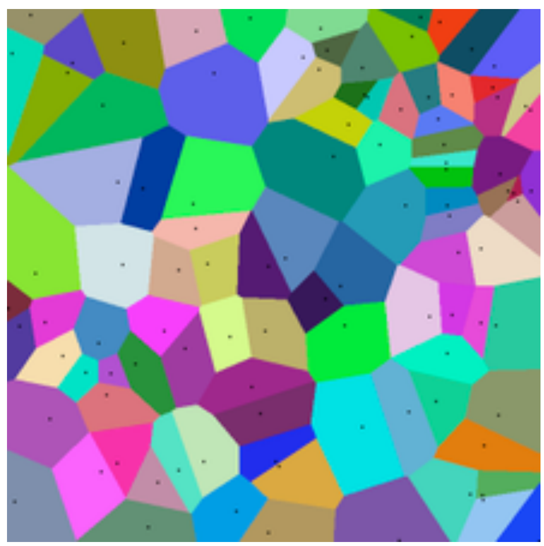
*Different properties of particle physics!*




Compare solutions of equations of motion for elementary particles:
- Donaldson invariants (1980s)
from electroweak forces
- Seiberg–Witten invariants (1990s)
from string theory

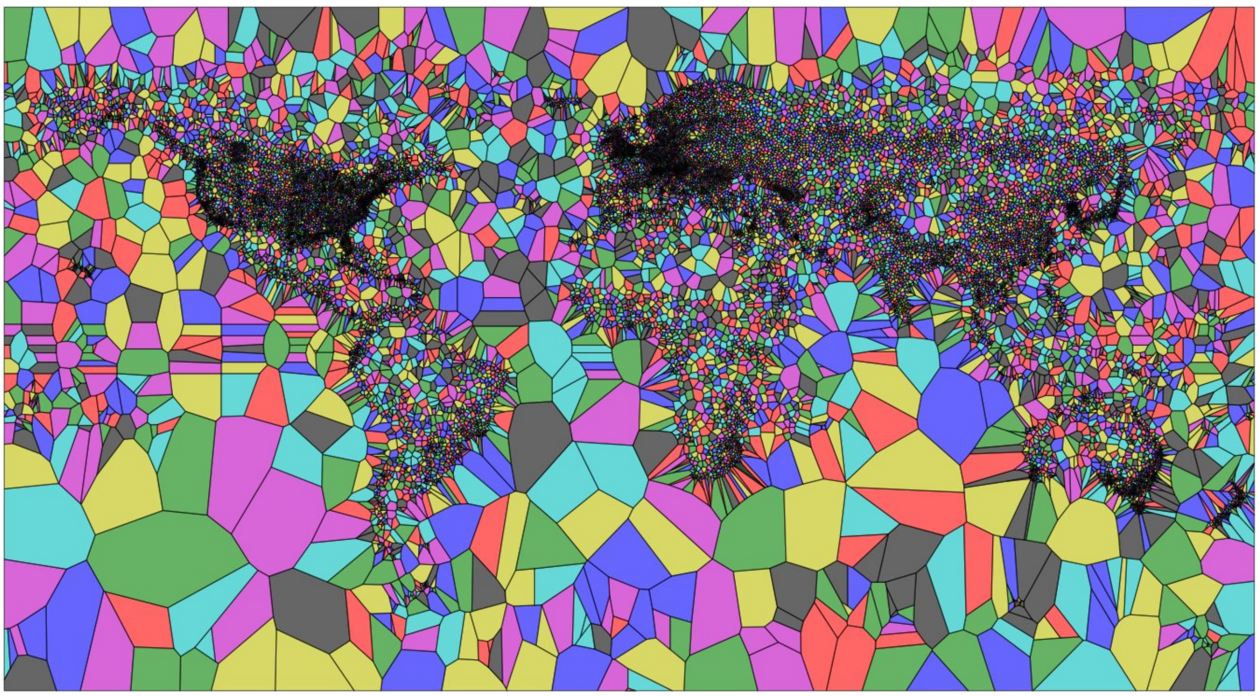
Dawn Meson, *Collisions II*, ca.2000

**Metric spaces** topological space where can measure distance between points
(Not just near but how near)



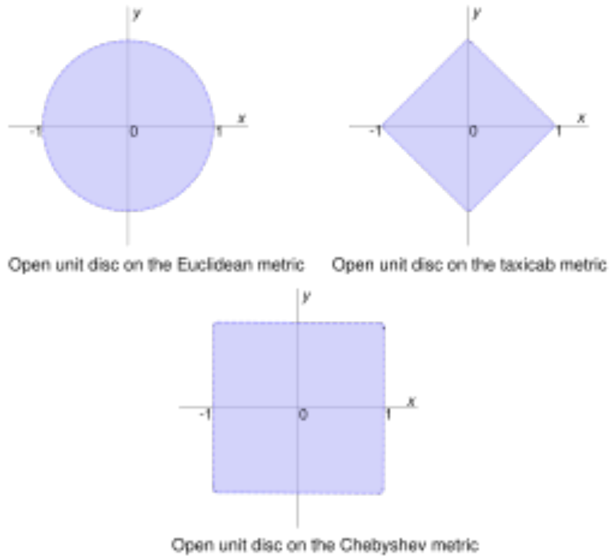
Voronoi cells: points closer to one of the “centers”

Metric space $\Rightarrow$ topological space
but not all topological spaces can be metric



Hohyon Ryu, *Voronoi map of World Weather Stations*, 2011

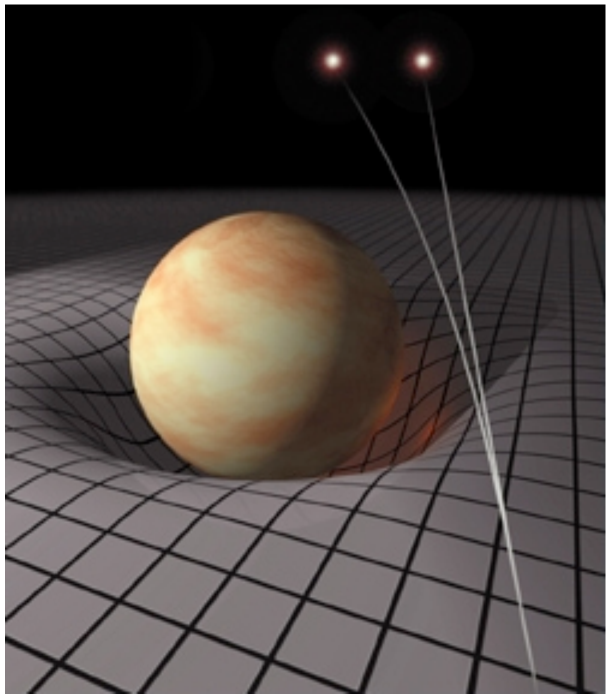
# Unit ball: distance one from a point



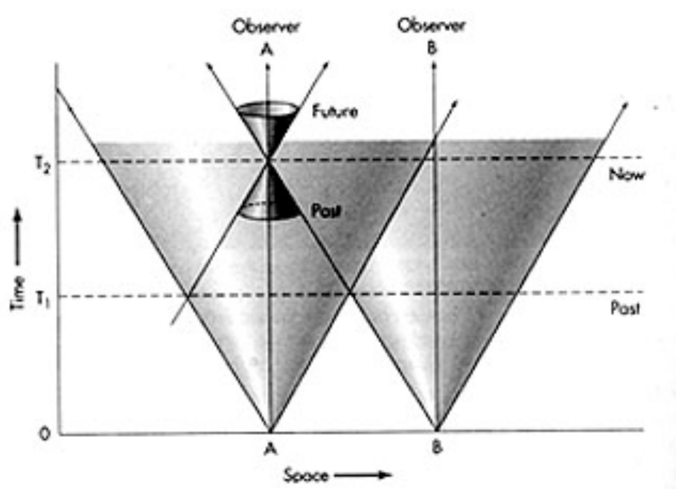

Sergels Torg Stockholm:

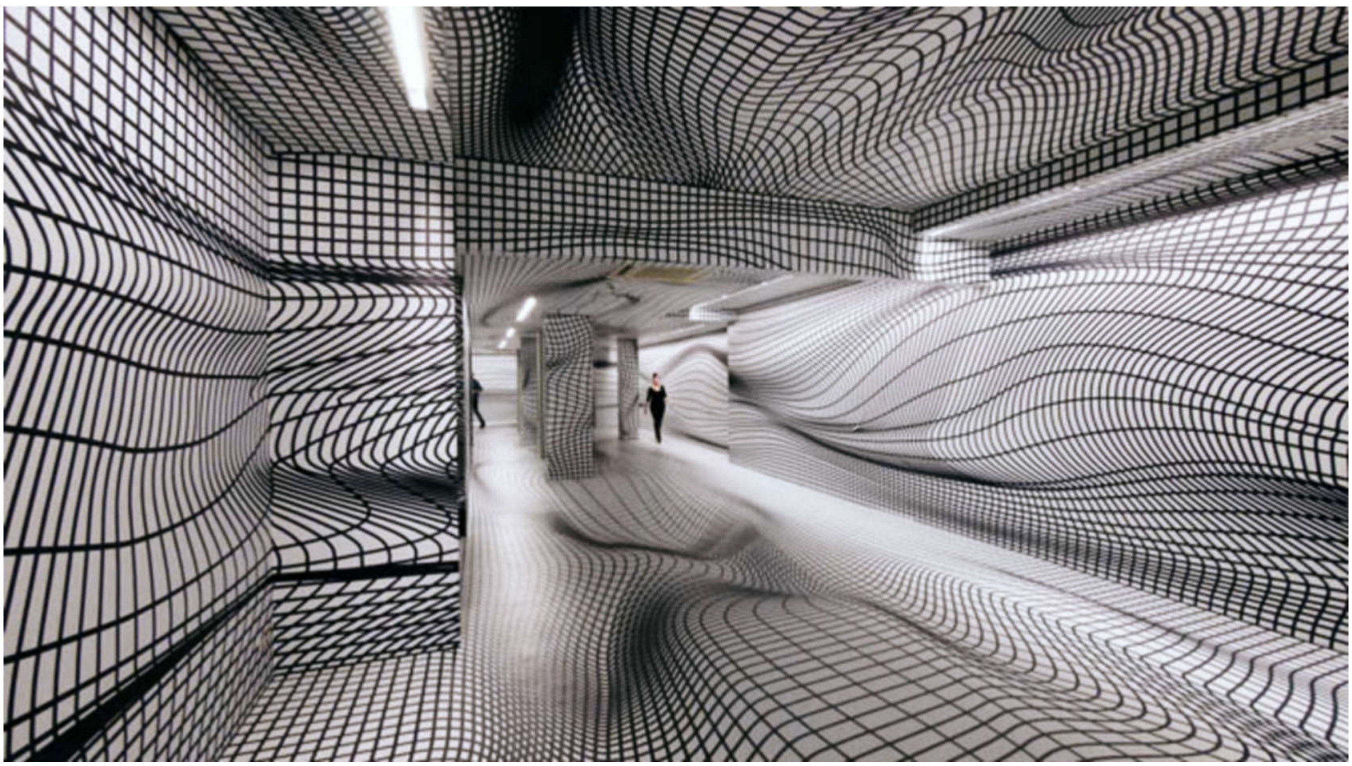
unit ball in distance $d((x, y), (0, 0)) = (x^4 + y^4)^{1/4}$

Smooth spaces can be metric: Riemannian manifolds ⇒ General Relativity, spacetime



Lorentzian metric: light cones





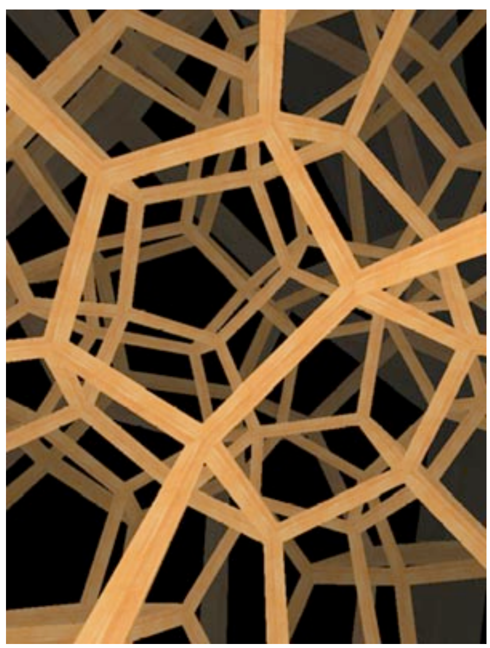
Peter Kogler, *Warped Room*, 2011

**What kind of space is space?**

(3-dimensional section of spacetime)

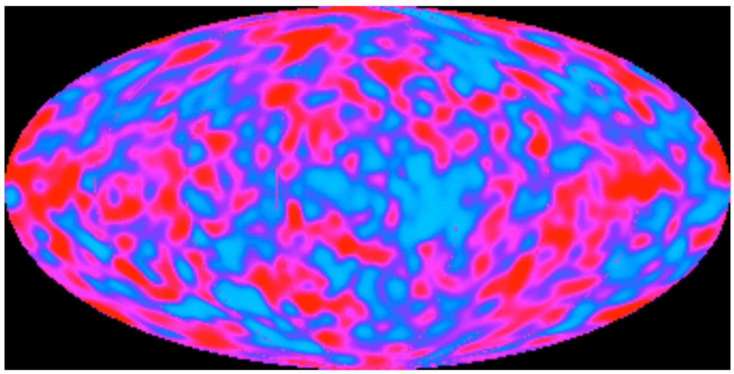
Metric properties (positive/negative curvature) related to cosmological constant

The problem of **Cosmic topology**

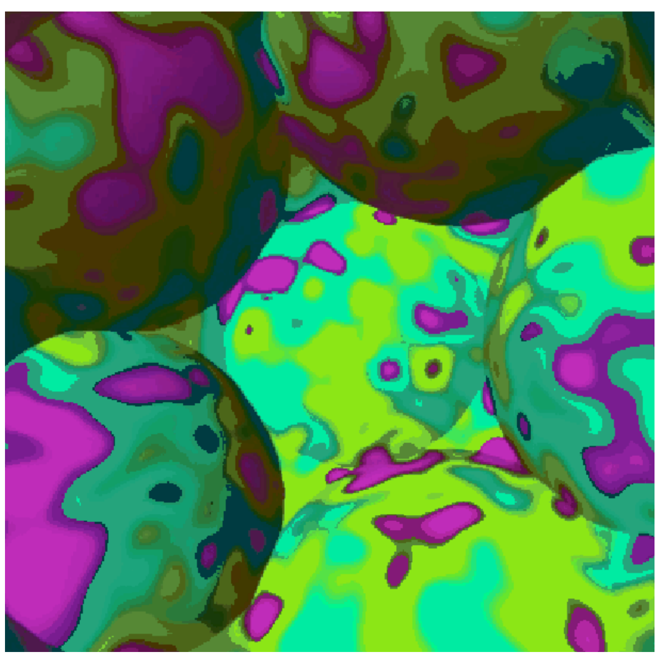


Dodecahedral universe: Poincaré sphere

Searching for dodecahedral topology in the cosmic microwave background radiation



Trying to match sides of polyhedron

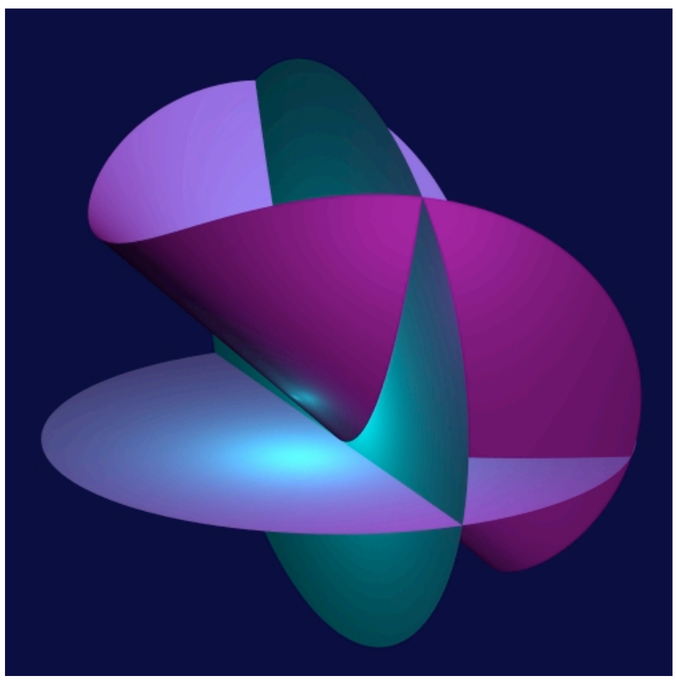


Attilio Pierelli, *Theory of the Universes: Octahedral Universe*, 1979

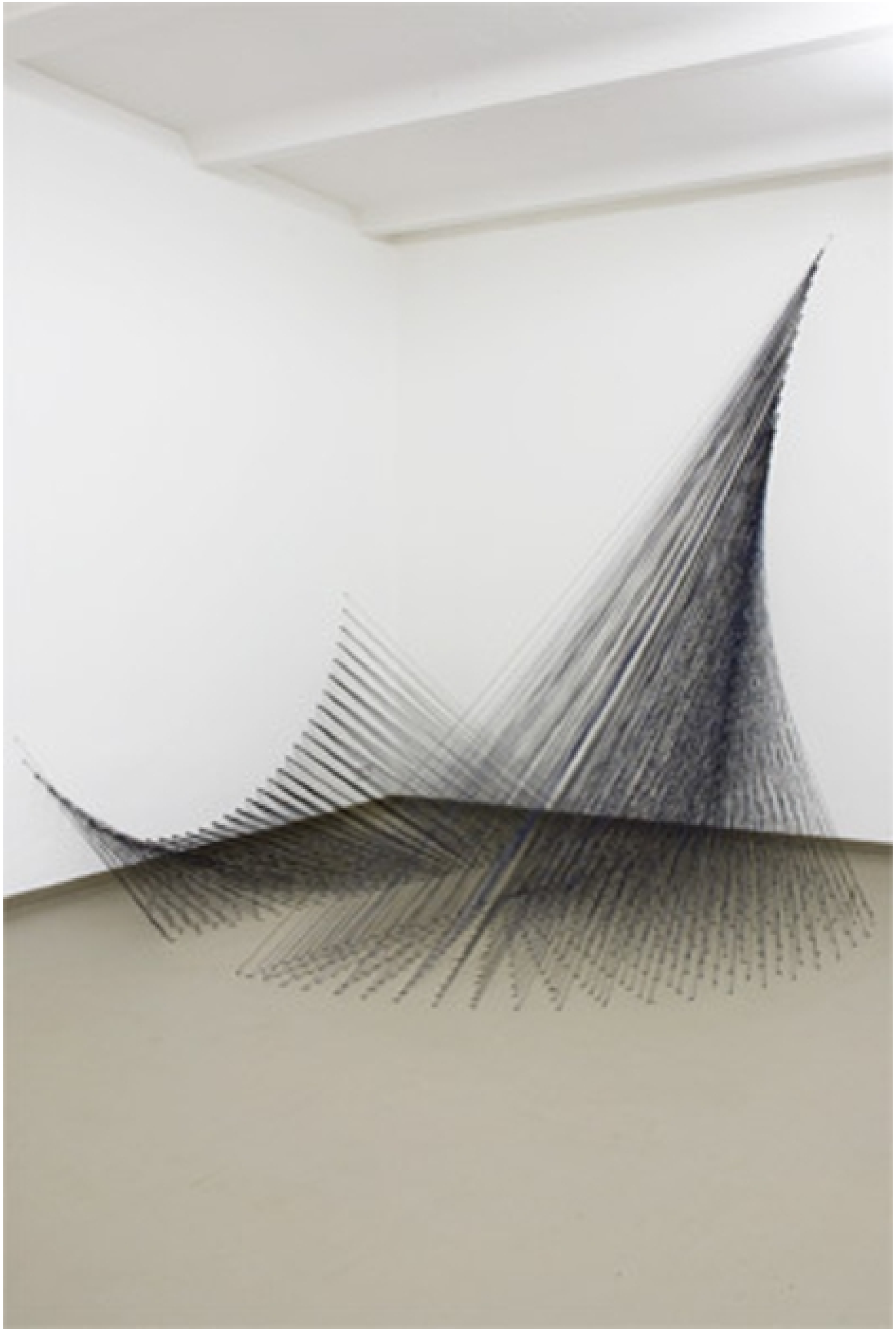
**Singular spaces**

Algebraic varieties: polynomial equations

$$yx(x^2 + y - z) = 0$$

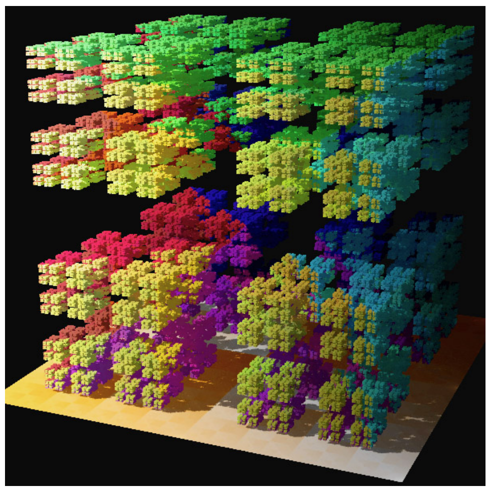


(If polynomial homogeneous: inside projective spaces)



Kazuko Miyamoto *Black Poppy*, 1978

## Measure spaces and fractals

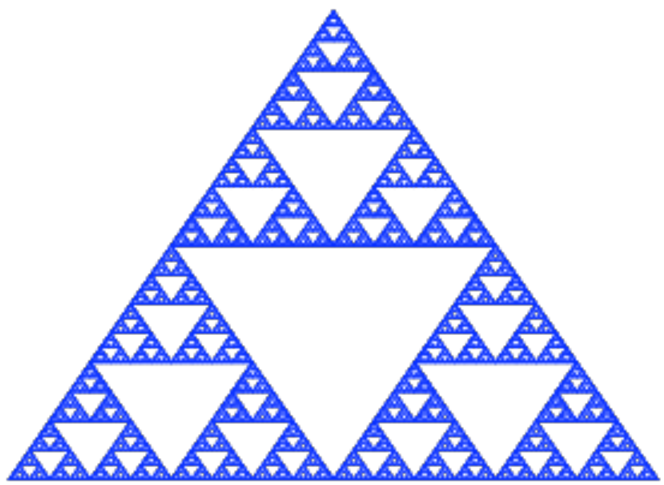


Measure the size of regions of space:
area, volume, length

Also measuring *probability* of an event
$\Rightarrow$ Quantum mechanics, observables
(theory of von Neumann algebras)

**Dimension:** Hausdorff dimension

(real number)

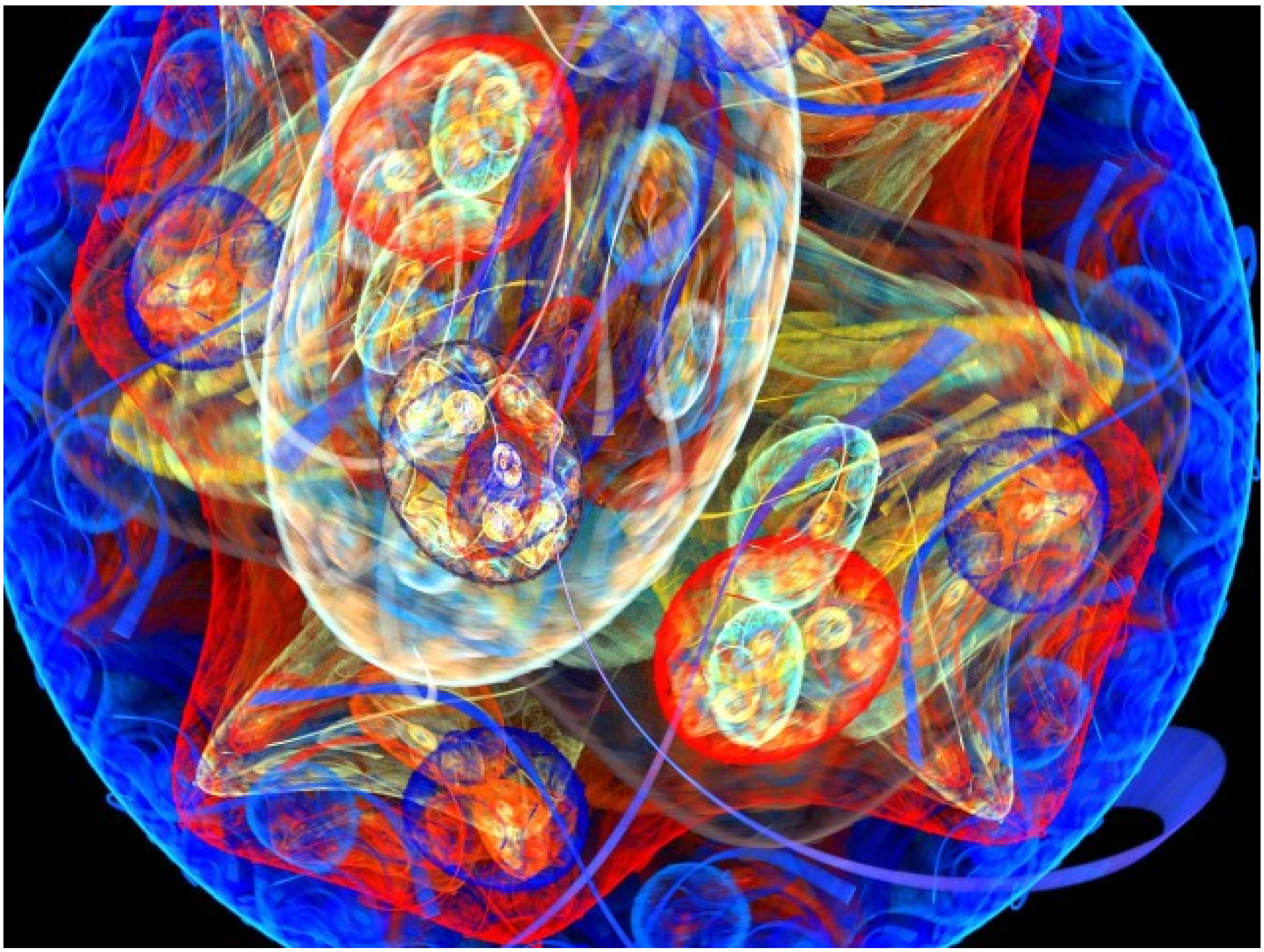


Sierpinski carpet: dimension $\frac{\log 3}{\log 2} \sim 1.585$

(union of three copies scaled down by a factor of two)

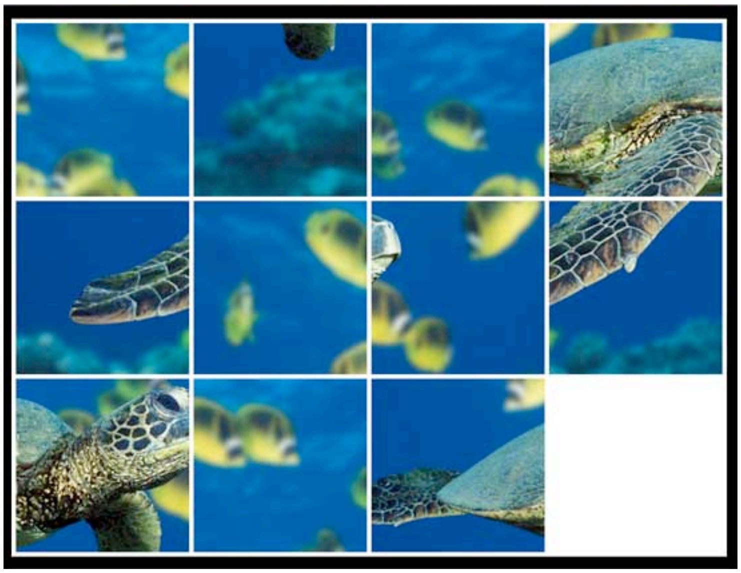
$\Rightarrow$ <u>Fractal</u>: dimension not an integer

Mandelbrot (1980s)



fractal painting generated by `Electric Sheep`

**Transformations of measure spaces**
Anything that preserves measure of sets even if it cuts and rearranges pieces



Non-measurable sets: Banach-Tarski paradox
(cut ball in finitely many pieces and reassemble them by rotating and translating into a ball twice as big)

Property of *group of transformations*

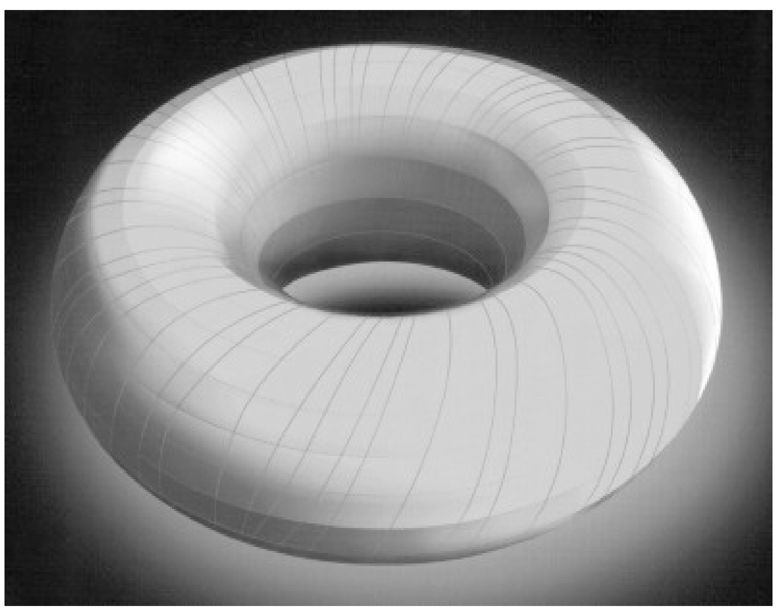
Attilio Marcolli, *Deconstruction*, 1986

**Noncommutative spaces** (Connes 1980s)

Quantum mechanics: Heisenberg uncertainty principle: positions and velocities do not commute (cannot be measured simultaneously)

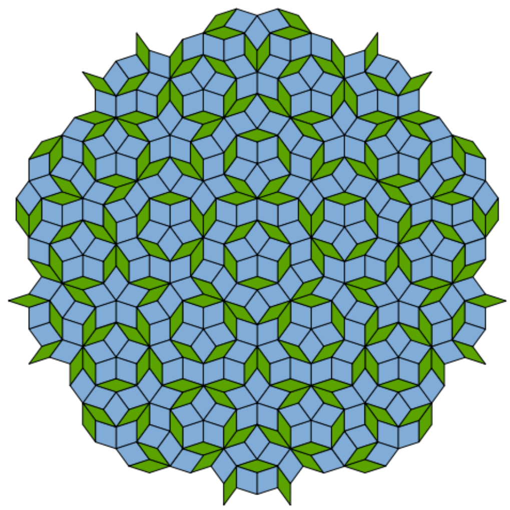
$$\Delta x \cdot \Delta v \geq \hbar$$

Quotients (gluing together points) of topological/smooth/metric/measure spaces
$\Rightarrow$ noncommutative spaces

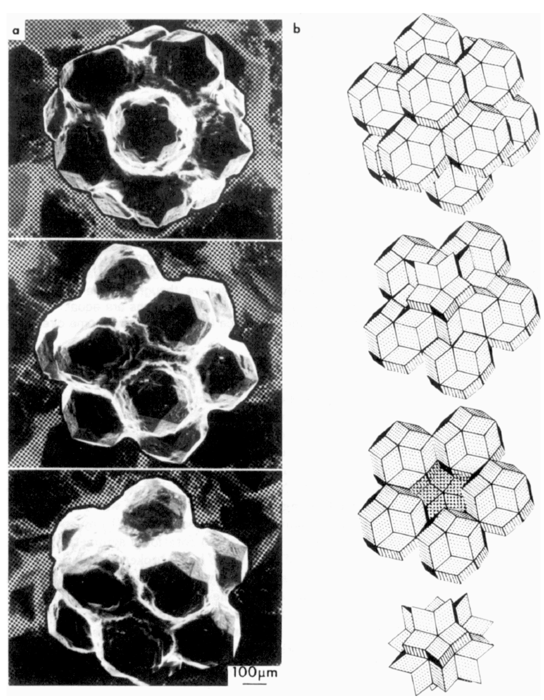


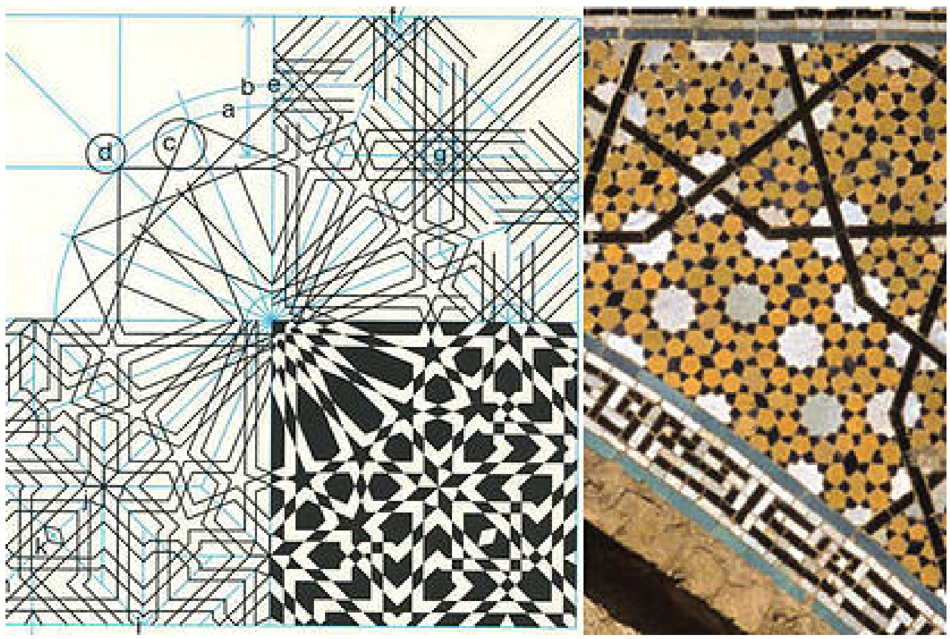
Models for particle physics

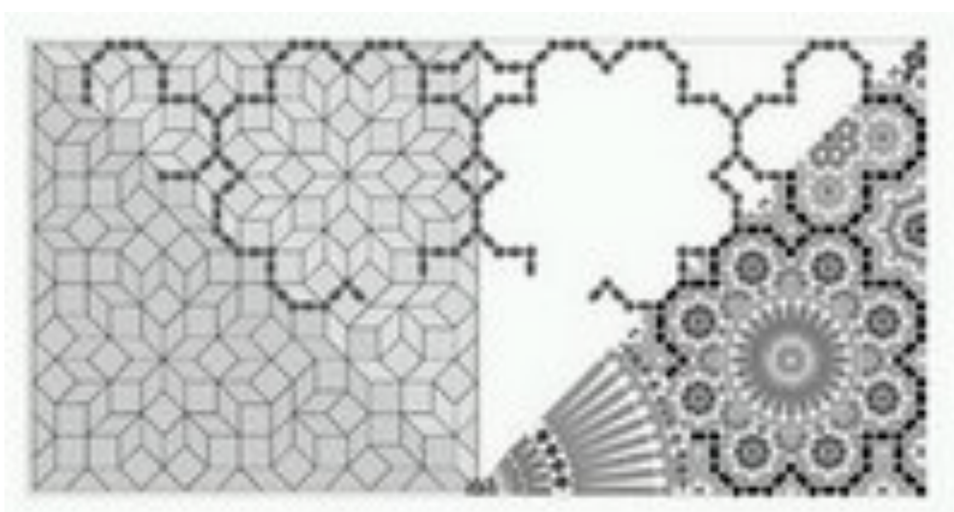
# Examples of noncommutative spaces:



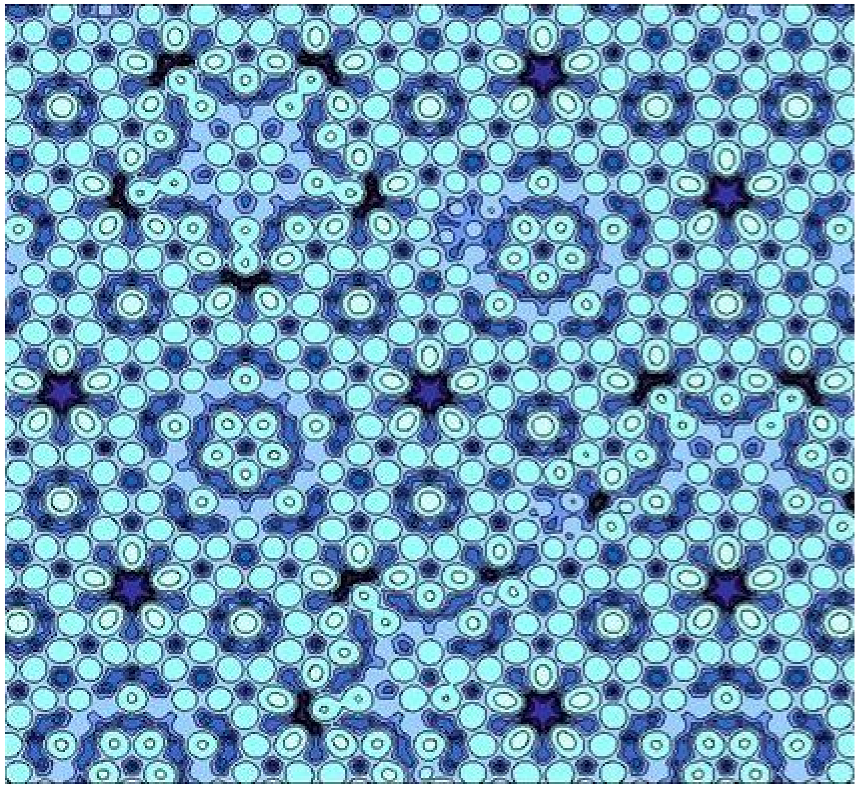
Space of Penrose tilings $\Rightarrow$ Quasicrystals





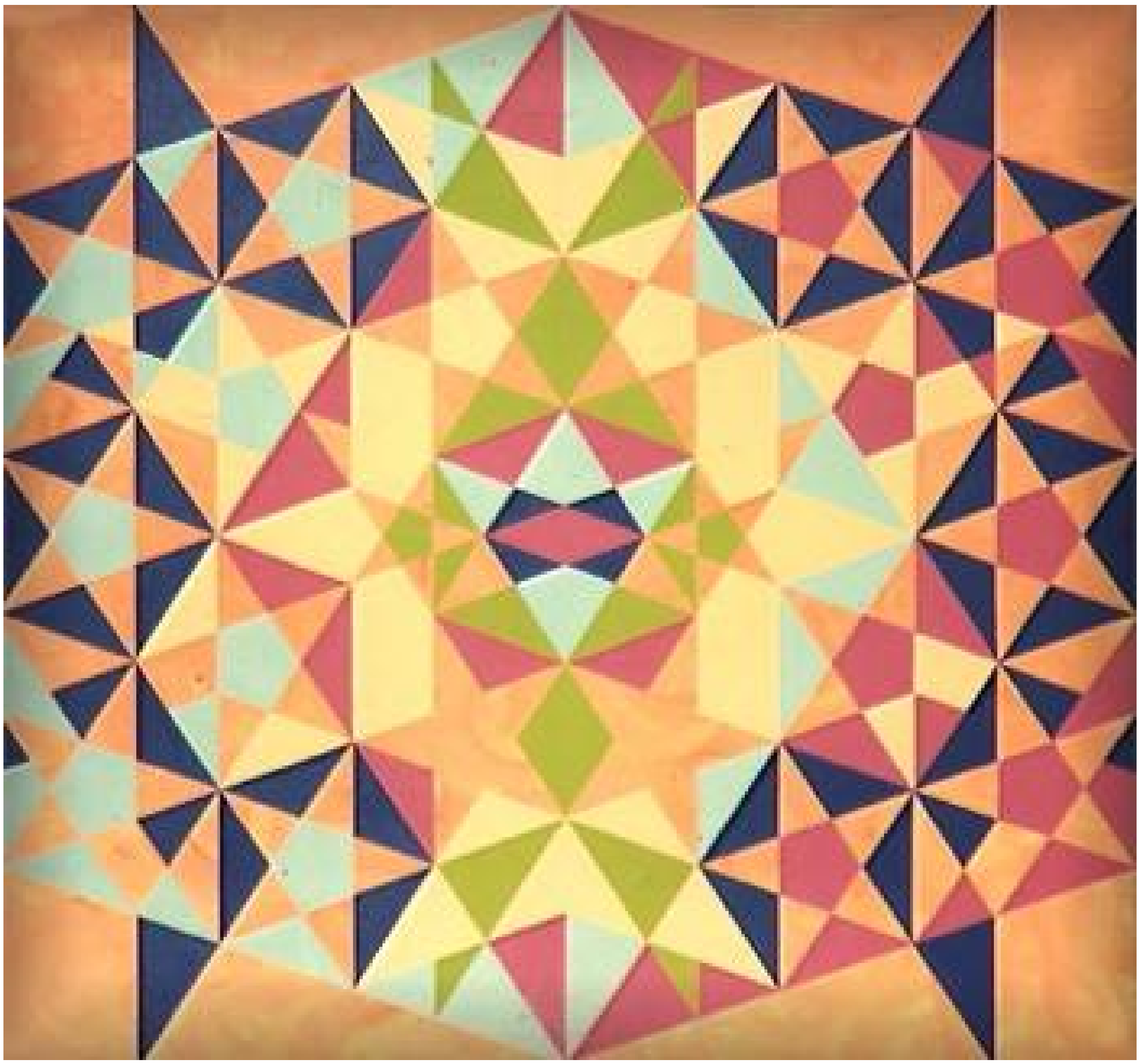
Quasiperiodic tilings were used in traditional Islamic architecture



Aluminium-Palladium-Manganese quasicrystal

Quasicrystals were first discovered in the 1982 (Dan Schechtman)

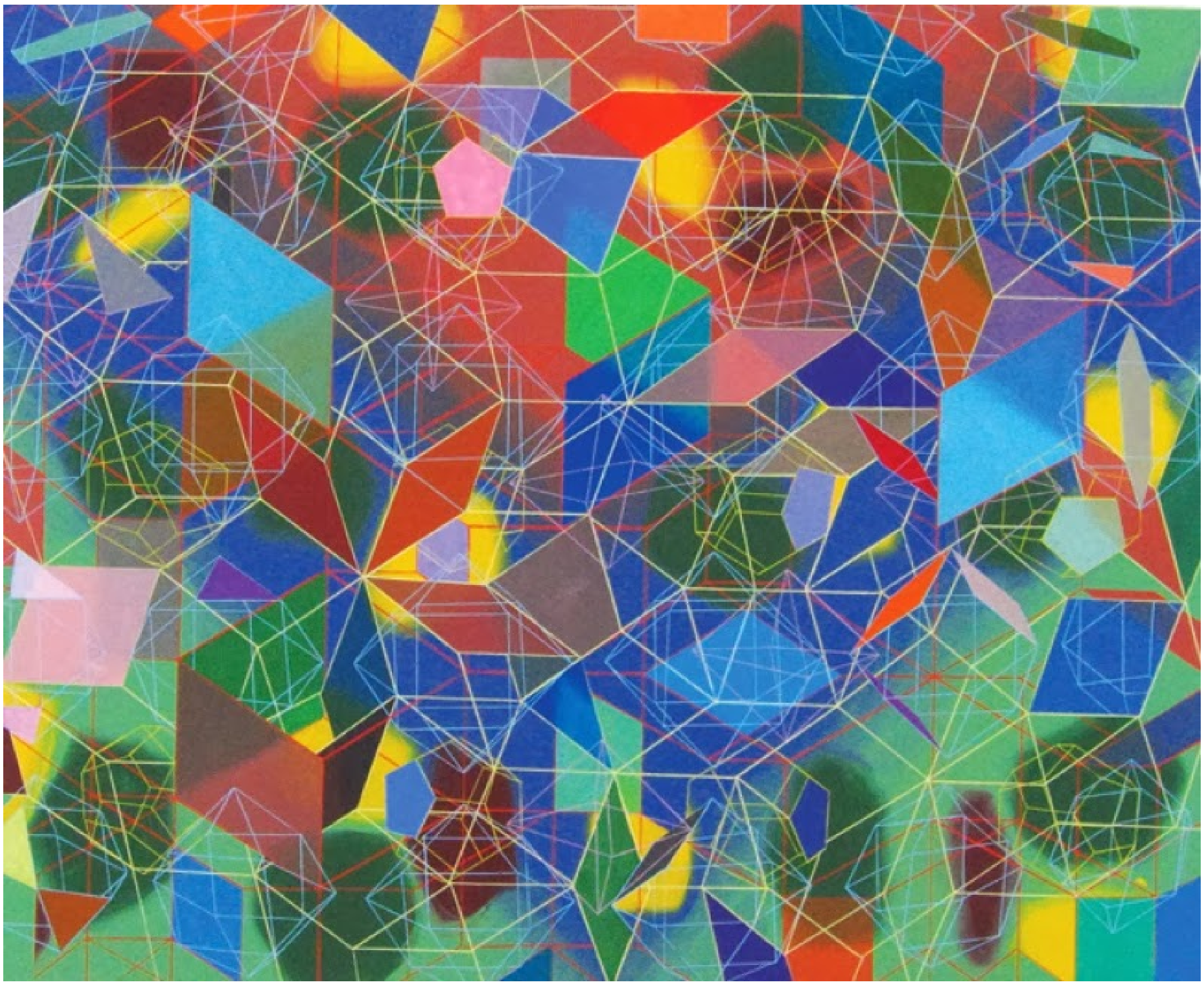
The modern mathematical theory of quasi-periodic tilings was developed in the 1960s and 1970s (Hao Wang, Roger Penrose)



Matjuska Teja Krasek, *Quasicube*, 2005

Tony Robbin, *COAST*, 1994

Modern uses of quasicrystals in architecture



Tony Robbin, *Pattern*, 2013

**Do we need all these notions of space?**

Yes: interplay of different structures

- Topological spaces can be smooth in different ways or not at all (exotic smoothness).

- Topological spaces acquire a new notion of dimension when seen as measure spaces (fractals).

- Riemannian manifolds (like spacetime) can be locally isometric but globally different due to topology (cosmic topology).

- Different physics on different spaces.