The notion of space in mathematics

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“Space” is always decorated with adjectives (like numbers: integer, rational, real, complex)

- Linear space
- Topological space
- Metric space
- Projective space
- Measure space
- Noncommutative space
Space is a kind of structure

Often (not always) a set (points of space) with some relational properties

- **Operations**: adding vectors, cutting and pasting, measuring size, intersections and unions

- **Proximity relation**: neighborhoods, closeness, convergence, distance

- **Hierarchy of structures**: Smooth $\Rightarrow$ Topological $\Rightarrow$ Measure space

- **Symmetries**: transformations preserving all the structure (very different symmetries for different kinds of spaces)

- **Telling spaces apart**: invariants (numerical, algebraic structures, discrete)
Philosophical digression: Absolute vs. Relational view of space

- Relational/Transformational viewpoint: Heraclitus, Democritus, Descartes, Leibniz, Bergson

- Absolute view of space: Eleatic school (Parmenides, Zeno), Aristotle, Kant, Newton, Comte

Mathematical reconciliation of philosophical dichotomy:

- Felix Klein (Erlangen Program 1872): emphasis on transformation groups
**Linear spaces** (or vector spaces):

set of vectors with dilations and composition of vectors

Examples: straight lines, planes, ...

Classical mechanics: equilibrium of forces

**Dimension:** Maximal number of linearly independent vectors
**Projective spaces:**
non-zero vectors up to scaling:
(identify $v = (x_1, x_2, x_3)$ and $\lambda v = (\lambda x_1, \lambda x_2, \lambda x_3)$, scaled by nonzero $\lambda$)

Renaissance perspective drawings

1-dimensional real projective space = circle

1-dimensional complex projective space = sphere
More interesting shapes:

2-dimensional real projective space:

Identifying diametrically opposite points on the boundary of a disk
Different kinds of numbers (fields) $\Rightarrow$ different kinds of projective spaces

Finite projective spaces
(discrete versus continuum in geometry)

Relational properties: lines through given points, lines intersecting, planes containing lines, ...
Topological spaces formalize the relation of “being near” a point
(qualitative: does not quantify how near)

Open condition: stable under small variations
(close condition: being on the border of two regions)

Transformations: continuous deformations

a donut is topologically the same as a cup of coffee
Knots and links

Topologically different: cannot be deformed one into the other without cutting
- Invariants of knots
Topology of knots and DNA

Topoisomerases enzymes act on the topology: unknotting DNA prior to replication
Nice topological spaces: triangulations

Essential to computer graphics

Graphs: simplest class of “piecewise linear” spaces
Examples of graphs:

San Francisco subway system

Moscow subway system
The most interesting graph of today: the world wide web

Methods of topology for internet connectivity
More examples of topological spaces:

- Sphere:

- Torus:

- Klein bottle:

- Real projective plane:
How to distinguish topological spaces?

- Euler characteristic

\[ \chi = \#\text{Faces} - \#\text{Edges} + \#\text{Vertices} \]

is a topological invariant

- Sphere: \( \chi = 2 \), orientable

- Real projective space: \( \chi = 1 \), non-orientable

- Klein bottle: \( \chi = 0 \), non-orientable

- Torus: \( \chi = 0 \), orientable

- Genus \( g \) surface: \( \chi = 2 - 2g \), orientable
• Orientability

Max Bill: Möbius band sculpture

Maurits Cornelis Escher: Möbius band
Smooth spaces (or smooth manifolds): Topological spaces locally indistinguishable from a vector space
Example: the Earth from ground level looks flat

Tangent space

Local coordinates: number of independent parameters describing a physical system

- Dimension from tangent space (linear space)
Local versus global properties:
locally like flat space (linear space)
but globally: nontrivial topology

View from inside a 3-torus
(Jeff Weeks “The shape of space”)
Smooth space ⇒ Topological space
but beware …

**Exotic smoothness:**
4-dimensional flat space has infinitely many different smooth structures (Donaldson)

- small: contained inside ordinary flat space
- large: do not fit in ordinary space

Dimension 3 and 4 are the most complicated!!

Poincaré conjecture (Perelman):
there is only one type of 3-dimensional sphere

Smooth 4-dimensional sphere?? Unknown
Exotic smoothness can affect our understanding of the distant universe (gravitational lensing)

passing through a small exotic space changes lensing
What detects exotic smoothness?
Not topological invariants (Euler characteristic etc)

Different properties of particle physics!

Compare solutions of equations of motion for elementary particles:
- Donaldson invariants (1980s) from electroweak forces
- Seiberg–Witten invariants (1990s) from string theory
Metric spaces topological space where can measure distance between points
(Not just near but how near)

Voronoi cells: points closer to one of the “centers”

Metric space $\Rightarrow$ topological space
but not all topological spaces can be metric
Unit ball: distance one from a point

Sergels Torg Stockholm:

unit ball in distance $d((x, y), (0, 0)) = (x^4 + y^4)^{1/4}$
Smooth spaces can be metric: Riemannian manifolds $\Rightarrow$ General Relativity, spacetime

Lorentzian metric: light cones
What kind of space is space?
(3-dimensional section of spacetime)

Metric properties (positive/negative curvature) related to cosmological constant

The problem of Cosmic topology

Dodecahedral universe: Poincaré sphere
Searching for dodecahedral topology in the cosmic microwave background radiation

Trying to match sides of polyhedron
Singular spaces

Algebraic varieties: polynomial equations

\[ yx(x^2 + y - z) = 0 \]

(If polynomial homogeneous: inside projective spaces)

Singularities: black holes, big bang, gravitational lensing
Measure spaces and fractals

Measure the size of regions of space: area, volume, length

Also measuring *probability* of an event
⇒ Quantum mechanics, observables
   (theory of von Neumann algebras)
**Dimension:** Hausdorff dimension
(real number)

Sierpinski carpet: dimension $\frac{\log 3}{\log 2} \sim 1.585$
(union of three copies scaled down by a factor of two)

$\Rightarrow$ **Fractal:** dimension not an integer

Mandelbrot (1980s)
Transformations of measure spaces
Anything that preserves measure of sets even if it cuts and rearranges pieces

Non-measurable sets: Banach-Tarski paradox
(cut ball in finitely many pieces and reassemble them by rotating and translating into a ball twice as big)

Property of group of transformations
**Noncommutative spaces** (Connes 1980s)

Quantum mechanics: Heisenberg uncertainty principle: positions and velocities do not commute (cannot be measured simultaneously)

\[ \Delta x \cdot \Delta v \geq \hbar \]

Quotients (gluing together points) of topological/smooth/metric/measure spaces \( \Rightarrow \) noncommutative spaces

Models for particle physics
Examples of noncommutative spaces:

Space of Penrose tilings $\Rightarrow$ Quasicrystals
Do we need all these notions of space?

Yes: interplay of different structures

- Topological spaces can be smooth in different ways or not at all (exotic smoothness).

- Topological spaces acquire a new notion of dimension when seen as measure spaces (fractals).

- Riemannian manifolds (like spacetime) can be locally isometric but globally different due to topology (cosmic topology).

- Different physics on different spaces.