The notion of space in mathematics

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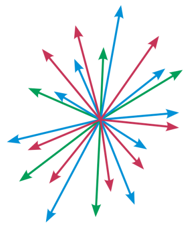
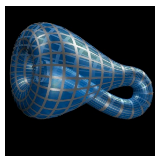
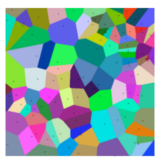
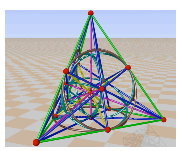
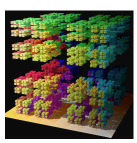
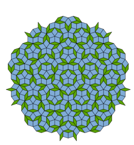
general audience lecture at Revolution Books
Berkeley 2009

“Space” is always decorated with adjectives (like numbers: integer, rational, real, complex)

- Linear space 
- Topological space 
- Metric space 
- Projective space 
- Measure space 
- Noncommutative space 

Space is a kind of structure

Often (not always) a set (points of space) with some relational properties

- Operations: adding vectors, cutting and pasting, measuring size, intersections and unions

- Proximity relation: neighborhoods, closeness, convergence, distance

- Hierarchy of structures: Smooth $\Rightarrow$ Topological $\Rightarrow$ Measure space

- Symmetries: transformations preserving all the structure (very different symmetries for different kinds of spaces)

- Telling spaces apart: invariants (numerical, algebraic structures, discrete)

Philosophical digression: *Absolute* vs. *Relational* view of space

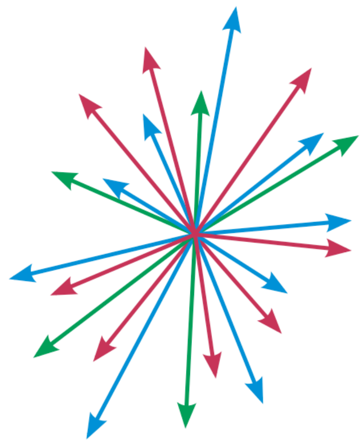
- Relational/Transformational viewpoint: Heraclitus, Democritus, Descartes, Leibniz, Bergson

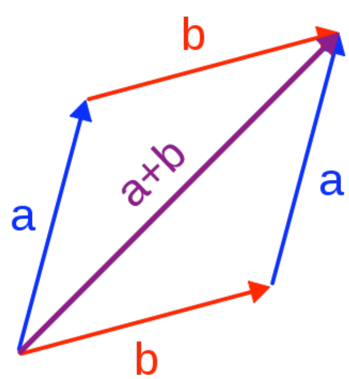
- Absolute view of space: Eleatic school (Parmenides, Zeno), Aristotle, Kant, Newton, Comte

Mathematical reconciliation of philosophical dichotomy:

- Felix Klein (Erlangen Program 1872): emphasis on transformation groups

Linear spaces (or vector spaces):

set of vectors  with dilations and composition of vectors



Examples: straight lines, planes, ...

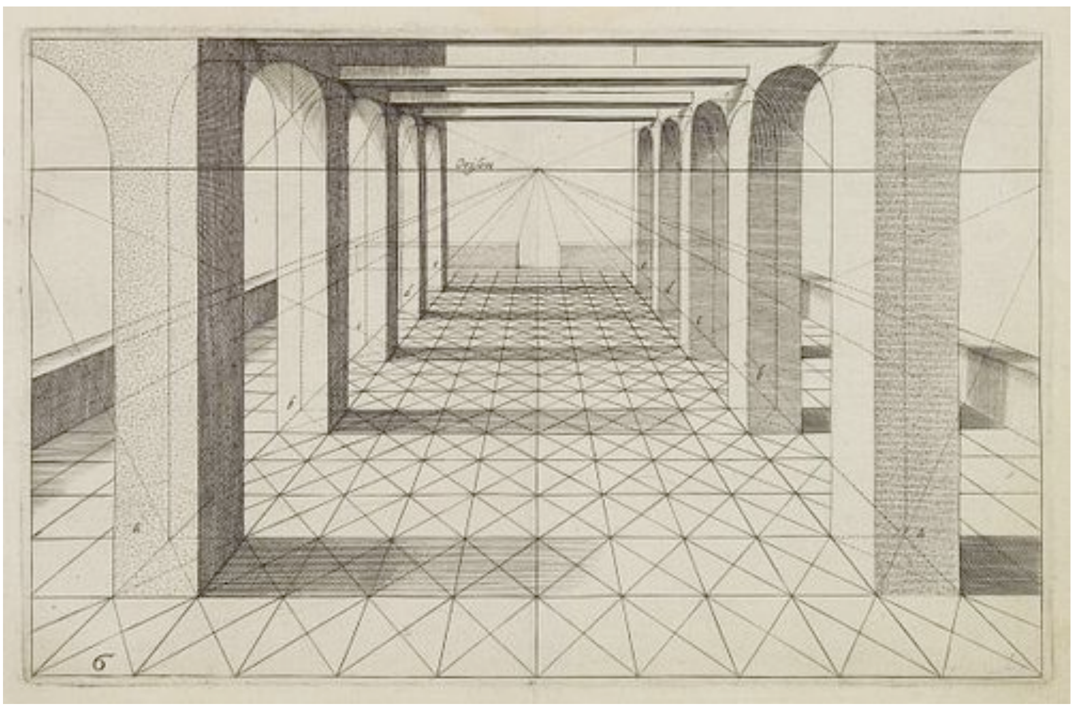
Classical mechanics: equilibrium of forces

Dimension: Maximal number of linearly independent vectors

Projective spaces:

non-zero vectors up to scaling:

(identify $v = (x_1, x_2, x_3)$ and $\lambda v = (\lambda x_1, \lambda x_2, \lambda x_3)$, scaled by nonzero λ)



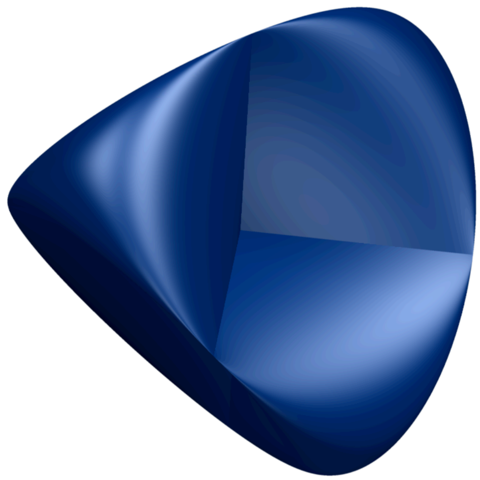
Renaissance perspective drawings

1-dimensional real projective space = circle

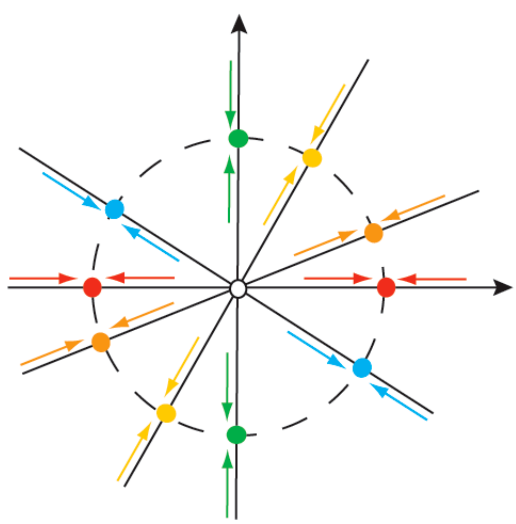
1-dimensional complex projective space = sphere

More interesting shapes:

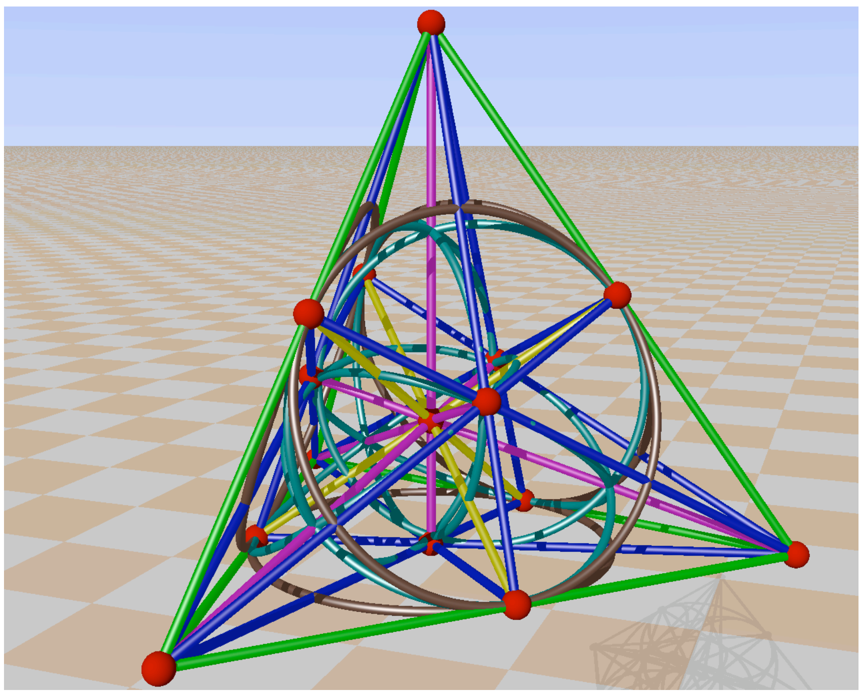
2-dimensional real projective space:



Identifying diametrically opposite points on the boundary of a disk



Different kinds of numbers (fields) $\Rightarrow$ different kinds of projective spaces



Finite projective spaces
(discrete versus continuum in geometry)

Relational properties: lines through given points, lines intersecting, planes containing lines, ...

Topological spaces formalize the relation of "being near" a point

(qualitative: does not quantify how near)

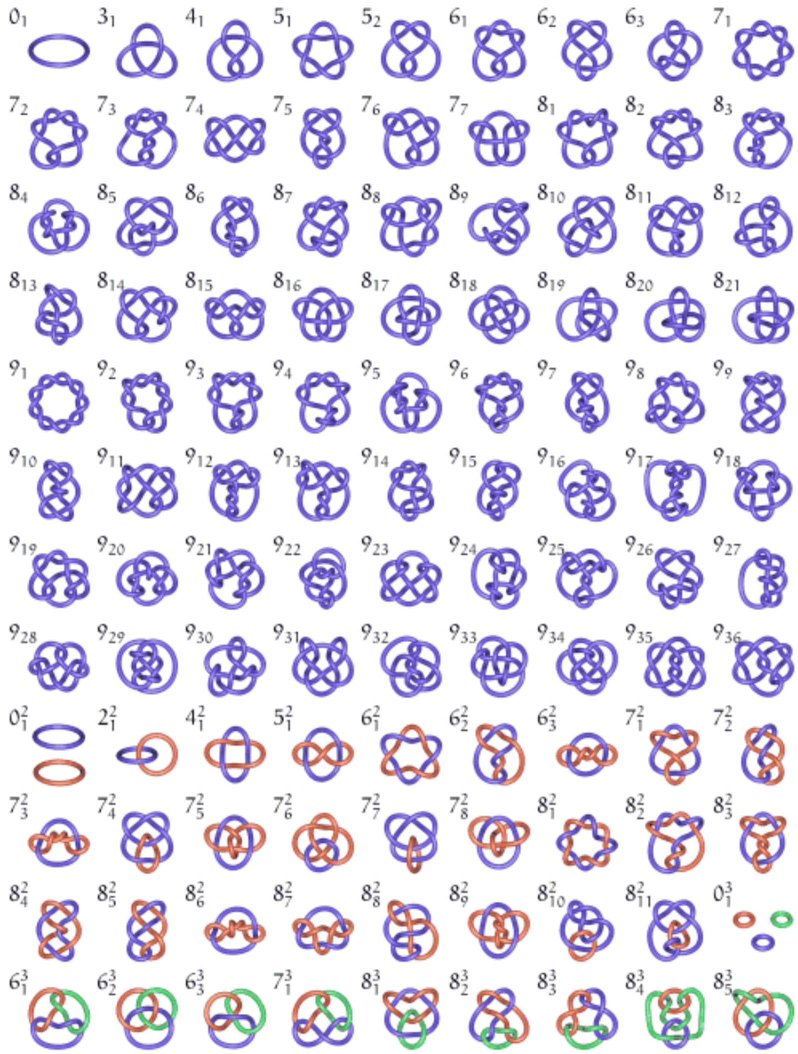
<u>Open condition</u>: stable under small variations

(close condition: being on the border of two regions)

<u>Transformations</u>: continuous deformations

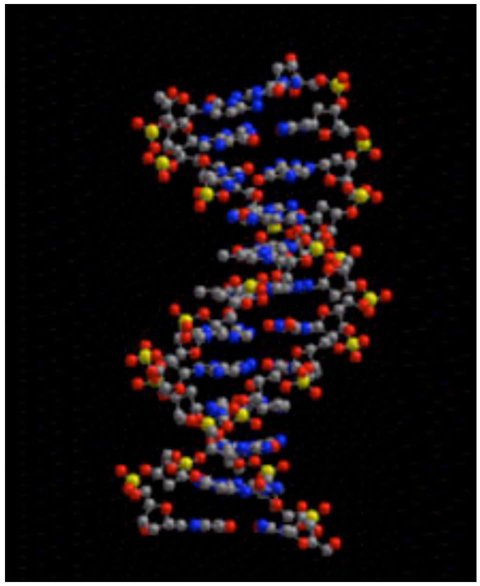
a donut is topologically the same as a cup of coffee

Knots and links

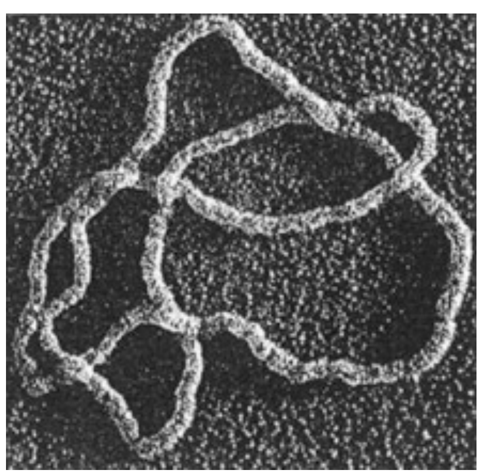


Topologically different: cannot be deformed one into the other without cutting
- *Invariants of knots*

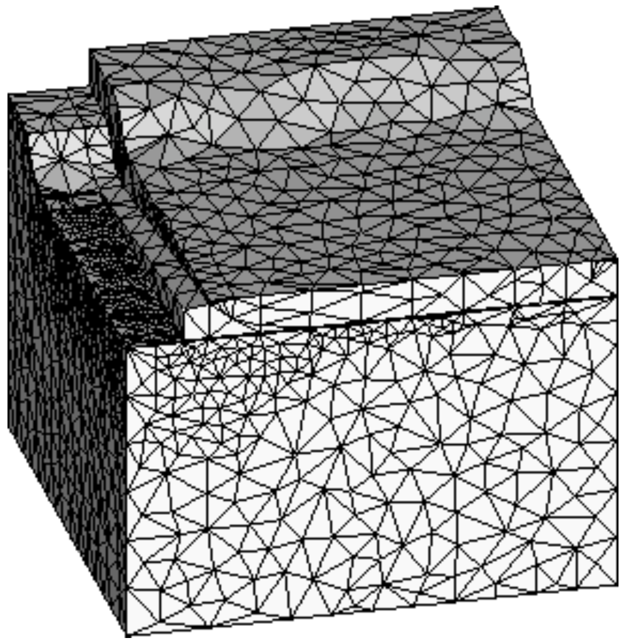
Topology of knots and DNA



Topoisomerases enzymes act on the topology: unknotting DNA prior to replication

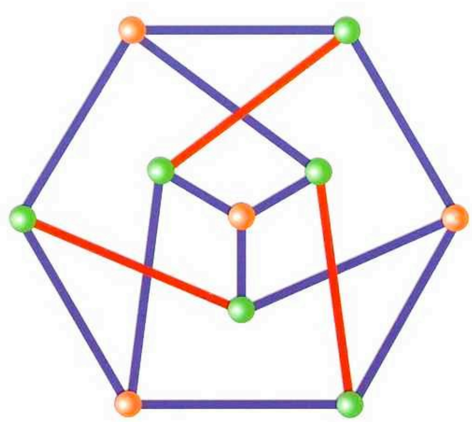


Nice topological spaces: triangulations



Essential to computer graphics

Graphs: simplest class of "piecewise linear" spaces

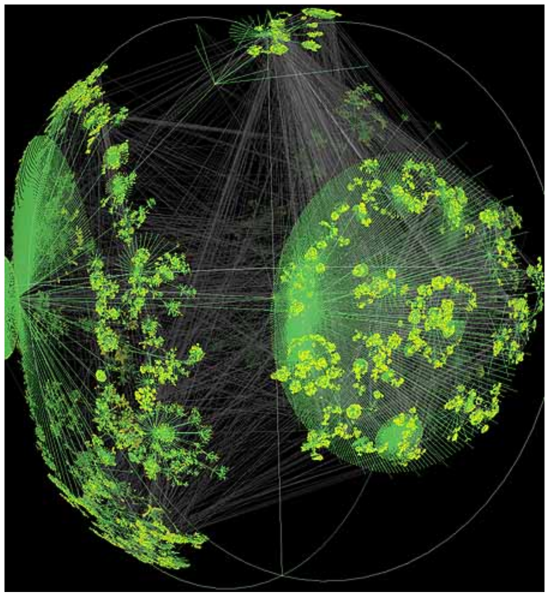


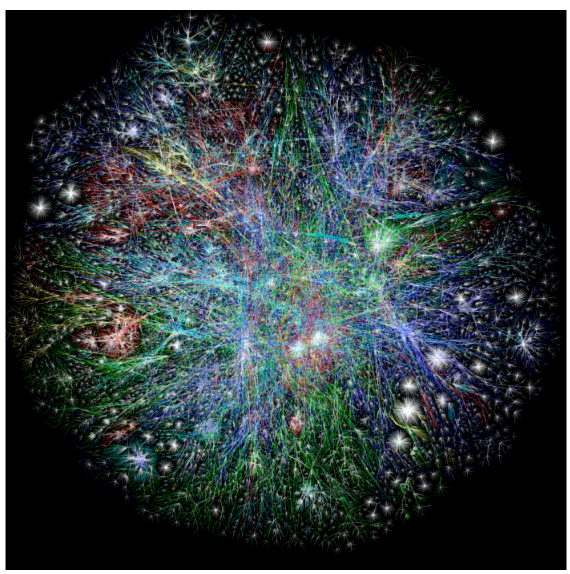
Examples of graphs:

San Francisco subway system

Moscow subway system

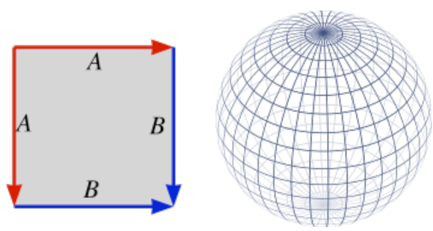
The most interesting graph of today:
the world wide web

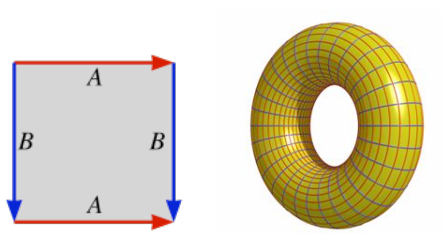


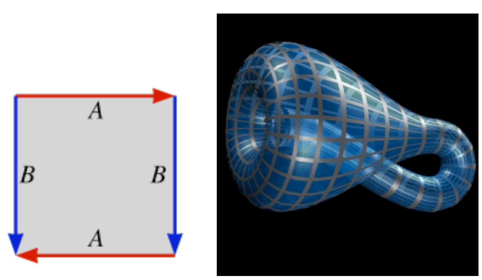


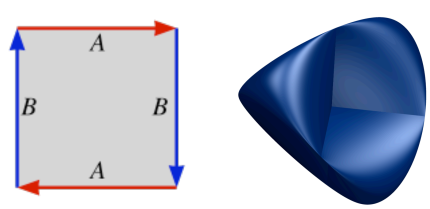
Methods of topology for internet connectivity

More examples of topological spaces:

- Sphere: 

- Torus: 

- Klein bottle: 

- Real projective plane: 

How to distinguish topological spaces?

• Euler characteristic

$$\chi = \#\text{Faces} - \#\text{Edges} + \#\text{Vertices}$$

is a topological invariant

- Sphere: $\chi = 2$, orientable

- Real projective space: $\chi = 1$, non-orientable

- Klein bottle: $\chi = 0$, non-orientable

- Torus: $\chi = 0$, orientable

- Genus g surface: $\chi = 2 - 2g$, orientable

• Orientability

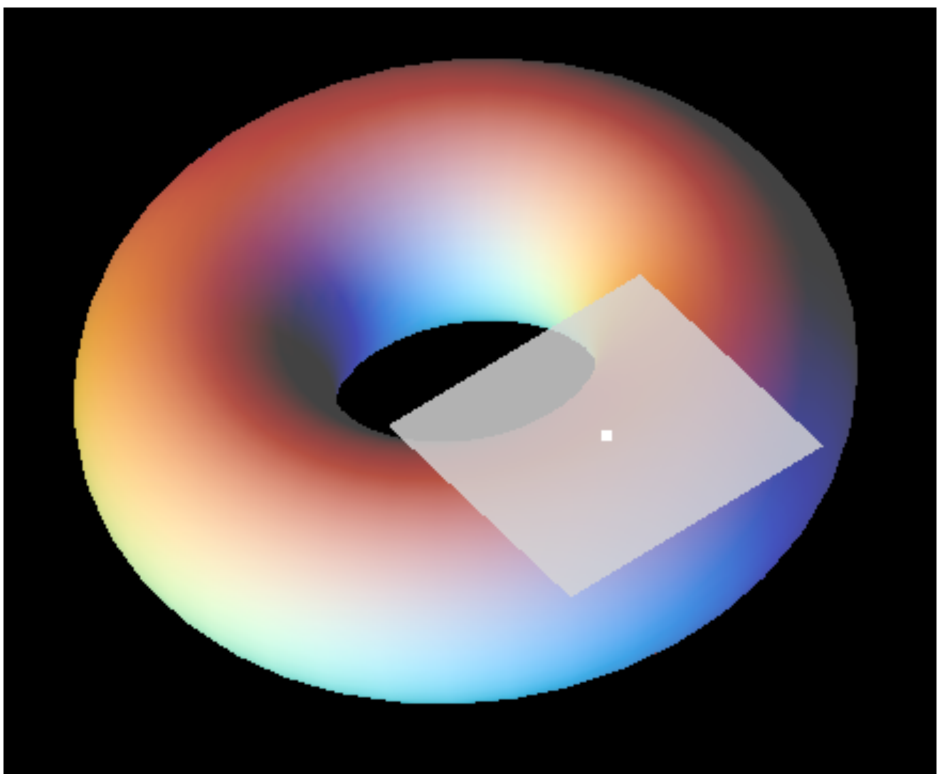
Max Bill: Möbius band sculpture

Maurits Cornelis Escher: Möbius band

Smooth spaces (or smooth manifolds):
Topological spaces locally indistinguishable from a vector space

Example: the Earth from ground level looks flat

Tangent space

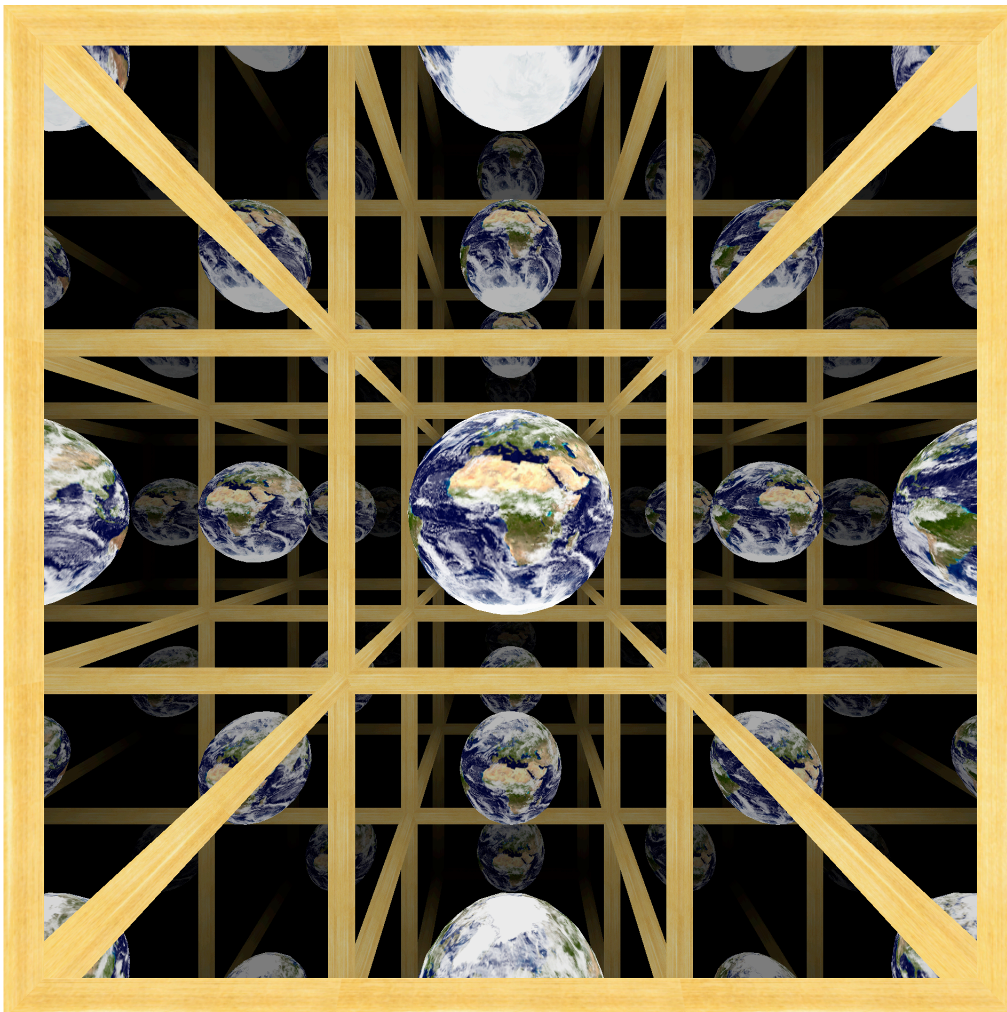


Local coordinates: number of independent parameters describing a physical system

- Dimension from tangent space (linear space)

Local versus global properties:

locally like flat space (linear space)
but globally: nontrivial topology



View from inside a 3-torus

(Jeff Weeks "The shape of space")

Smooth space ⇒ Topological space
but beware ...

<u>Exotic smoothness</u>:
4-dimensional flat space has infinitely many different smooth structures (Donaldson)

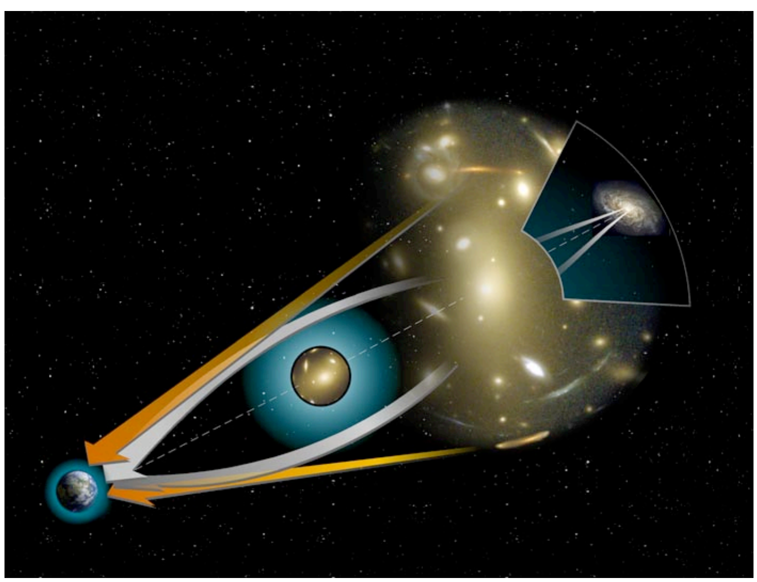
- small: contained inside ordinary flat space
- large: do not fit in ordinary space

Dimension 3 and 4 are the most complicated!!

Poincaré conjecture (Perelman):
there is only one type of 3-dimensional sphere

Smooth 4-dimensional sphere?? Unknown

Exotic smoothness can affect our understanding of the distant universe (gravitational lensing)

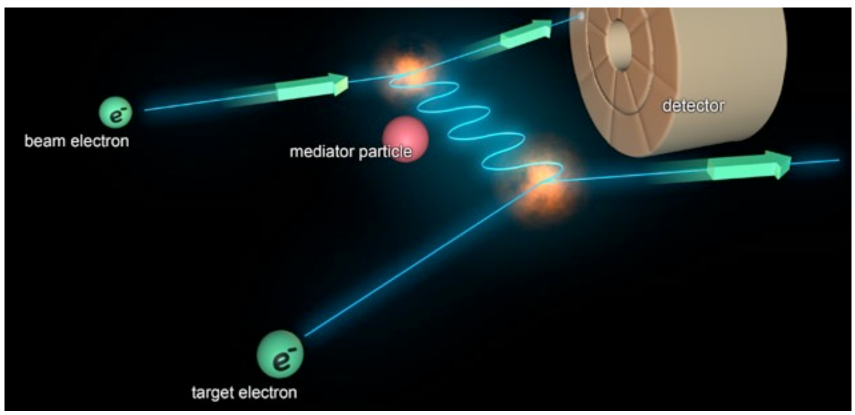


passing through a small exotic space changes lensing

What detects exotic smoothness?

Not topological invariants (Euler characteristic etc)

Different properties of particle physics!



Compare solutions of equations of motion for elementary particles:

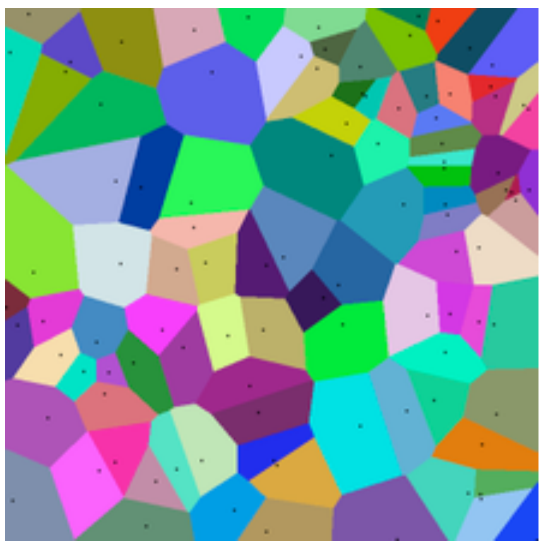
- Donaldson invariants (1980s)

from electroweak forces

- Seiberg–Witten invariants (1990s)

from string theory

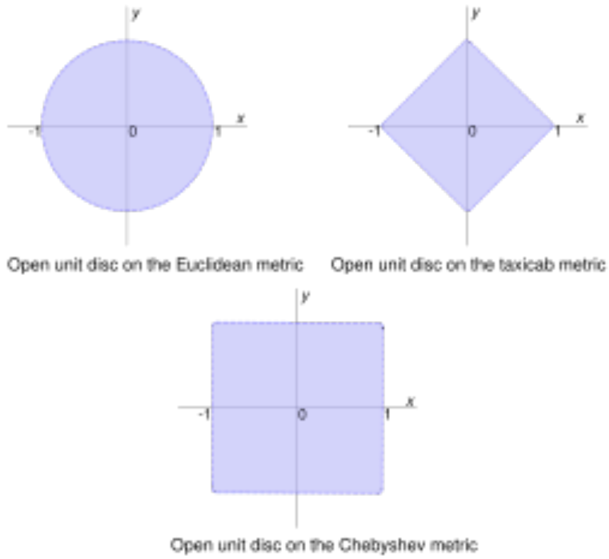
Metric spaces topological space where can measure distance between points
(Not just near but how near)



Voronoi cells: points closer to one of the "centers"

Metric space $\Rightarrow$ topological space
but not all topological spaces can be metric

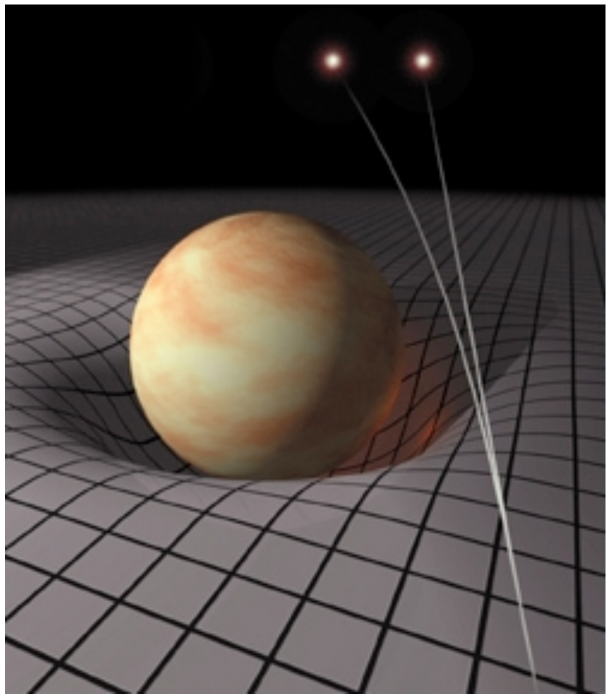
Unit ball: distance one from a point



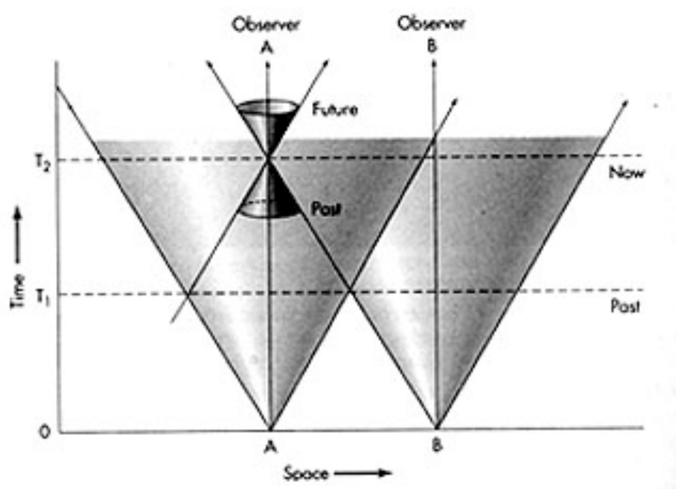
Sergels Torg Stockholm:

unit ball in distance $d((x,y),(0,0)) = (x^4 + y^4)^{1/4}$

Smooth spaces can be metric: Riemannian manifolds ⇒ General Relativity, spacetime



Lorentzian metric: light cones

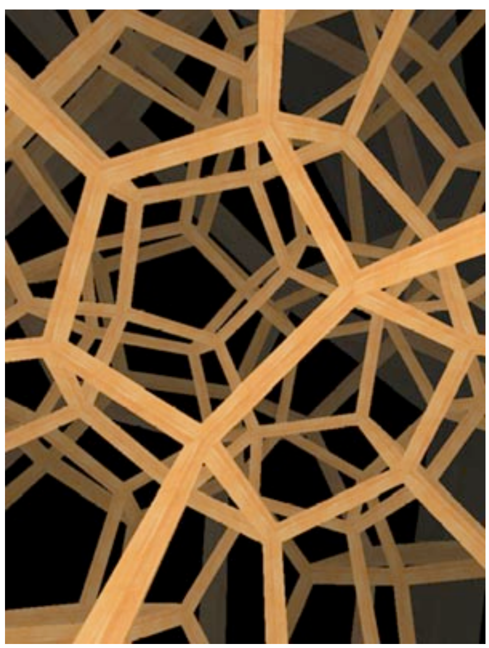


What kind of space is space?

(3-dimensional section of spacetime)

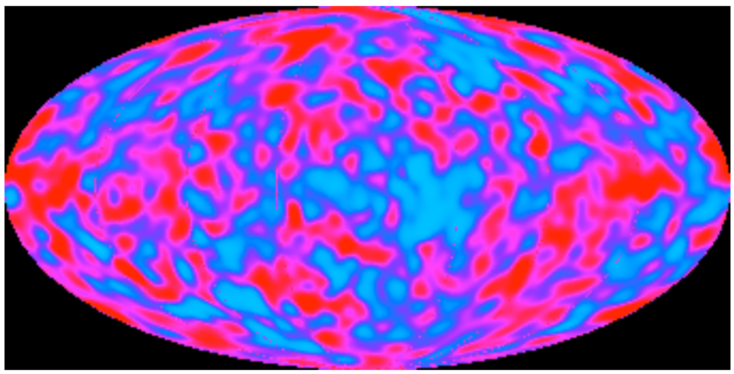
Metric properties (positive/negative curvature) related to cosmological constant

The problem of **Cosmic topology**

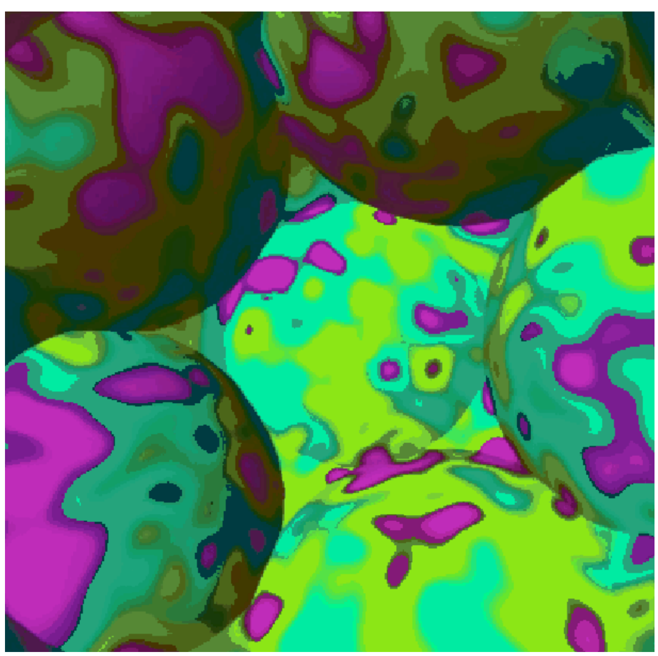


Dodecahedral universe: Poincaré sphere

Searching for dodecahedral topology in the cosmic microwave background radiation



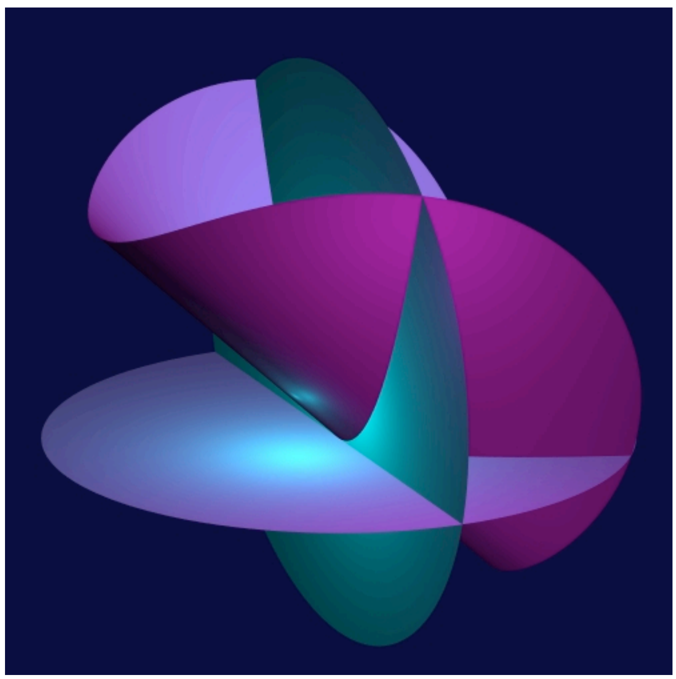
Trying to match sides of polyhedron



Singular spaces

Algebraic varieties: polynomial equations

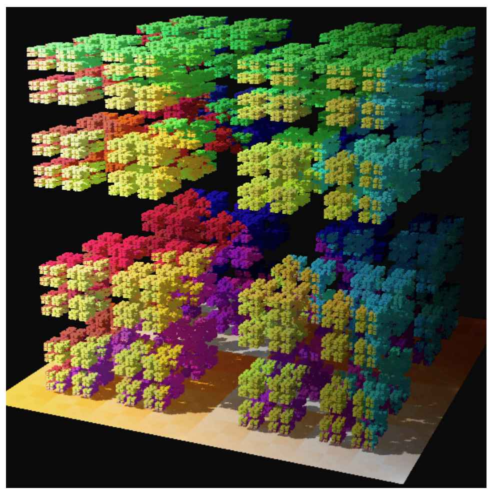
$$yx(x^2 + y - z) = 0$$



(If polynomial homogeneous: inside projective spaces)

Singularities: black holes, big bang, gravitational lensing

Measure spaces and fractals

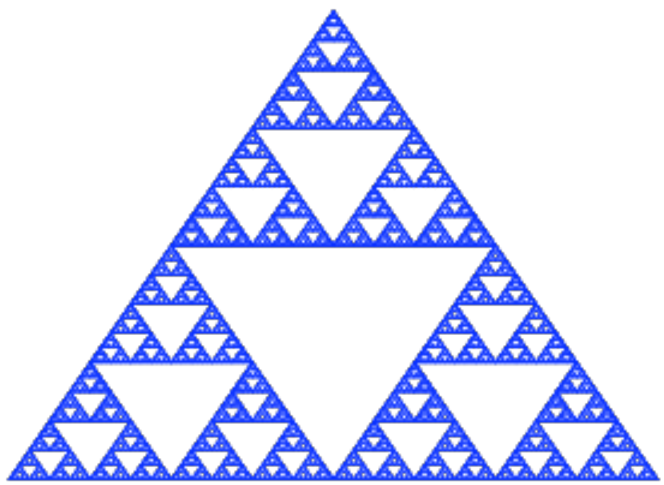


Measure the size of regions of space:
area, volume, length

Also measuring *probability* of an event
$\Rightarrow$ Quantum mechanics, observables
(theory of von Neumann algebras)

Dimension: Hausdorff dimension

(real number)



Sierpinski carpet: dimension $\frac{\log 3}{\log 2} \sim 1.585$

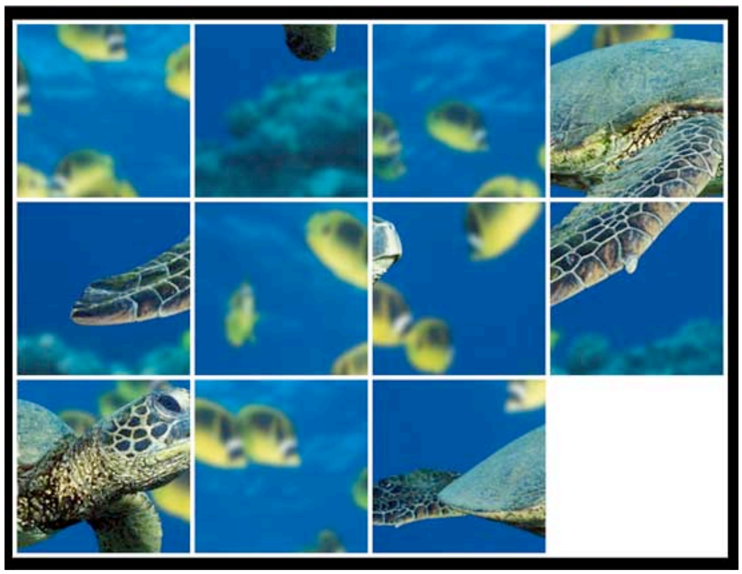
(union of three copies scaled down by a factor of two)

$\Rightarrow$ Fractal: dimension not an integer

Mandelbrot (1980s)

Transformations of measure spaces

Anything that preserves measure of sets even if it cuts and rearranges pieces



Non-measurable sets: Banach-Tarski paradox

(cut ball in finitely many pieces and reassemble them by rotating and translating into a ball twice as big)

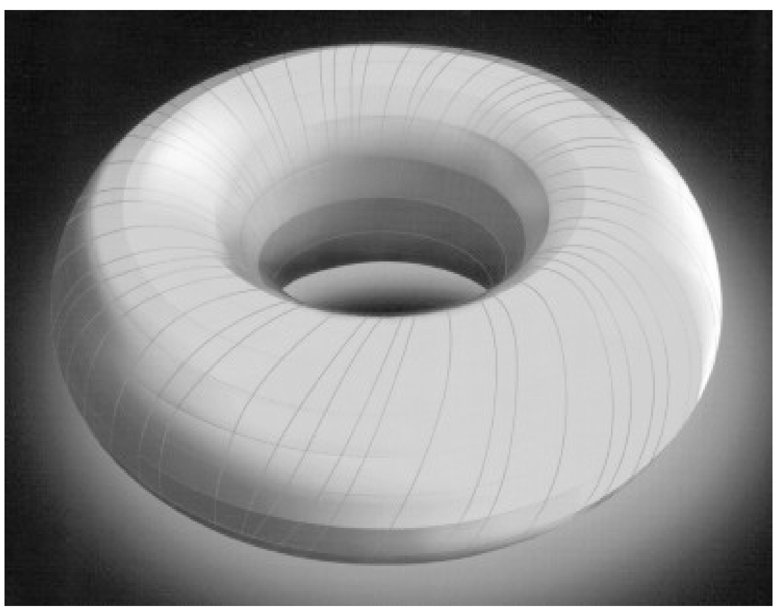
Property of *group of transformations*

Noncommutative spaces (Connes 1980s)

Quantum mechanics: Heisenberg uncertainty principle: positions and velocities do not commute (cannot be measured simultaneously)

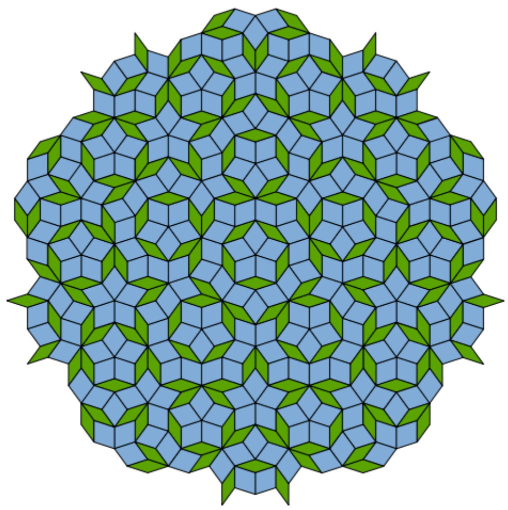
$$\Delta x \cdot \Delta v \geq \hbar$$

Quotients (gluing together points) of topological/smooth/metric/measure spaces
$\Rightarrow$ noncommutative spaces

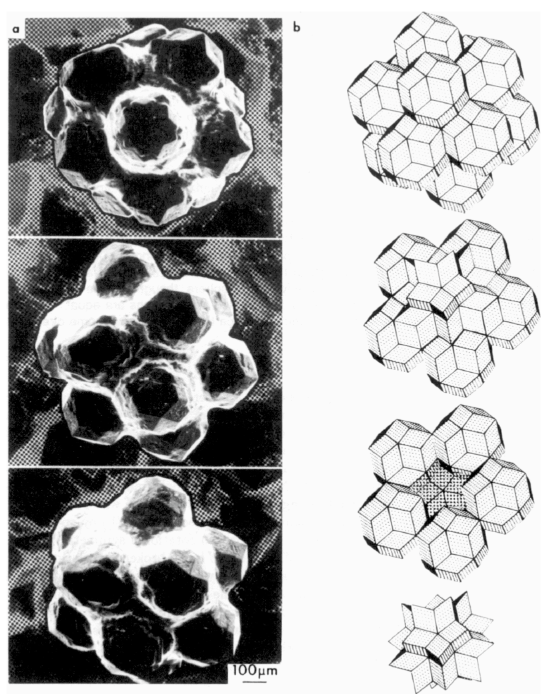


Models for particle physics

Examples of noncommutative spaces:



Space of Penrose tilings $\Rightarrow$ Quasicrystals



Do we need all these notions of space?

Yes: interplay of different structures

- Topological spaces can be smooth in different ways or not at all (exotic smoothness).

- Topological spaces acquire a new notion of dimension when seen as measure spaces (fractals).

- Riemannian manifolds (like spacetime) can be locally isometric but globally different due to topology (cosmic topology).

- Different physics on different spaces.