Solvmanifolds and noncommutative tori with real multiplication

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Based on

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Background:

- Hilbert modular surfaces

$$X = (\mathbb{H} \times \mathbb{H})/\Gamma$$

$\Gamma = \mathrm{SL}_2(O_{\mathbb{K}})$, real quadratic field $\mathbb{K} = \mathbb{Q}(\sqrt{d})$

X has isolated singularities: link 3-dim solvmanifolds

• 1970s: Hirzebruch conjecture: signature defects

$$\sigma(X) = \int_X L \; - \eta(0)$$

$$\eta(0) = L(\Lambda', V, 0)$$

Shimizu L-function (for a lattice Λ with action of units V)

$$L(\Lambda, V, s) = \sum_{\mu \in (\Lambda \smallsetminus \{0\})/V} \mathrm{sign}(N(\mu)) \, |N(\mu)|^{-s}$$

• 1983: proved Atiyah–Donnelly–Singer using Atiyah–Patodi–Singer index theorem for Dirac-type operator

$$\eta(s) = \sum_{\lambda \neq 0} \mathrm{sign}(\lambda) |\lambda|^{-s}$$

(also higher rank totally real fields)

Atiyah's question: (2007)

• Eta function $\eta(s)$ splits as $L(\Lambda, V, s)$ plus "stuff" that does not matter in the $\eta(0)$ calculation

• Does the arithmetic part of the eta function come from an underlying *noncommutative geometry*?

• Is the 3-dimensional solvmanifold a "model up to homotopy" of this noncommutative geometry?

Answer: Yes!

Notation:

• $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ real quadratic field

$\alpha_i : \mathbb{K} \hookrightarrow \mathbb{R}$, $i = 1, 2$ real embeddings

• $L \subset \mathbb{K}$ lattice $\Rightarrow$ lattice $\Lambda \subset \mathbb{R}^2$

$$L \ni \ell \mapsto (\alpha_1(\ell), \alpha_2(\ell)) \subset \mathbb{R}^2$$

$\Lambda = (\alpha_1, \alpha_2)(L)$

• $V = \epsilon^{\mathbb{Z}}$ totally positive units preserving L

$$V = \{u \in O_{\mathbb{K}}^* \,|\, uL \subset L, \;\; \alpha_i(u) \in \mathbb{R}_+^*\}$$

($L = O_{\mathbb{K}} \Rightarrow \epsilon =$ fundamental unit)

• Action of V on Λ

$$\lambda = (\alpha_1(\ell), \alpha_2(\ell)) \mapsto (\epsilon\alpha_1(\ell), \epsilon'\alpha_2(\ell)) = (\epsilon\alpha_1(\ell), \epsilon^{-1}\alpha_2(\ell))$$

The 3-dimensional solvmanifold

$$S(\Lambda, V) = \Lambda \rtimes_\epsilon V$$

action of V on Λ by

$$A_\epsilon = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon' \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$$

$$S(\mathbb{R}^2, \mathbb{R}) = \mathbb{R}^2 \rtimes \mathbb{R}$$

action of $\mathbb{R}$ on $\mathbb{R}^2$ by

$$\Theta_t(x, y) = (e^t x, e^{-t} y), \quad \Theta_t \in \mathrm{SL}_2(\mathbb{R})$$

$$X_\epsilon = S(\Lambda, V) \backslash S(\mathbb{R}^2, \mathbb{R})$$

$\pi_1(X_\epsilon) = S(\Lambda, V)$

Topology

$$H_1(X_\epsilon, \mathbb{Z}) = \Lambda/(1 - A_\epsilon)\Lambda \oplus \mathbb{Z}$$

abelianization of $\pi_1(X_\epsilon) = S(\Lambda, V)$

Surjective homomorphism $\pi(\lambda, n) = (\lambda \mod (1 - A_\epsilon)\Lambda, n)$, Kernel: commutators

Poincaré duality

$$H^0(X_\epsilon, \mathbb{Z}) = \mathbb{Z} \quad H^1(X_\epsilon, \mathbb{Z}) = \mathbb{Z}$$
$$H^3(X_\epsilon, \mathbb{Z}) = \mathbb{Z} \quad H^2(X_\epsilon, \mathbb{Z}) = \mathbb{Z} \oplus \mathrm{Coker}(1 - A_\epsilon)$$

$\mathrm{Ker}(1 - A_\epsilon) = 0$ and $\mathrm{Coker}(1 - A_\epsilon)$ torsion

Action: (equivalent formulation)

Basis $\{1, \theta\}$ of $\alpha_1(L) \subset \mathbb{R}$

$$\varphi_\epsilon = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$$

with $\epsilon = a + b\theta$, $\epsilon\theta = c + d\theta$

Fixed points $1/\theta$ and $1/\theta'$

Identifications

$$S(\Lambda, V) = \mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}$$

$$(\lambda, \epsilon^k) \mapsto (\lambda_1 = n + m\theta, k)$$

$$(\lambda, \epsilon^k) \mapsto (\lambda_2 = n + m\theta', -k)$$

Solvmanifold and universal family

$\mathbb{H}$ = hyperbolic plane, $\mathcal{U}$ = universal family

$$\mathcal{U} \xrightarrow{\pi} \mathbb{H}, \quad \pi^{-1}(\tau) = E_\tau$$

elliptic curves, up to equivalence descends to

$$\mathcal{U}/\mathrm{SL}_2(\mathbb{Z}) \to \mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$$

Galois conjugate θ, θ' for lattice Λ

Geodesic $\ell_{\theta,\theta'} \Rightarrow$ closed geodesic in $\mathbb{H}/\mathrm{SL}_2(\mathbb{Z})$

$$\mathrm{length}(\bar{\ell}_{\theta,\theta'}) = \log \epsilon$$

unit generating V acting on Λ

X_ϵ = restriction of $\mathcal{U}/\mathrm{SL}_2(\mathbb{Z})$ to geodesic $\bar{\ell}_{\theta,\theta'}$

Noncommutative tori with real multiplication

$$VU = e^{2\pi i\theta} UV$$

C^*-algebra $\mathcal{A}_\theta$

Morita equivalences (Connes, Rieffel)

$$\mathcal{A}_{\theta_1} \simeq \mathcal{A}_{\theta_2} \Leftrightarrow g \in \mathrm{SL}_2(\mathbb{Z})$$

$\theta_1 = g\theta_2$

Quadratic irrationalities: $\mathbb{Q}(\theta)$ real quadratic

$$\exists g \in \mathrm{SL}_2(\mathbb{Z}), \quad g\theta = \theta$$

Self Morita equivalences

Manin: (2005) parallel to theory of elliptic curves with complex multiplication

Twisted group algebras

- Γ finitely generated discrete group

$$\sigma : \Gamma \times \Gamma \to U(1)$$

multiplier, 2-cocycle: $\sigma(\gamma, 1) = \sigma(1, \gamma) = 1$ and

$$\sigma(\gamma_1, \gamma_2)\sigma(\gamma_1\gamma_2, \gamma_3) = \sigma(\gamma_1, \gamma_2\gamma_3)\sigma(\gamma_2, \gamma_3)$$

- Hilbert space $\ell^2(\Gamma)$ left/right σ-regular representations

$$L^\sigma_\gamma f(\gamma') = f(\gamma^{-1}\gamma')\sigma(\gamma, \gamma^{-1}\gamma')$$

$$R^\sigma_\gamma f(\gamma') = f(\gamma'\gamma)\sigma(\gamma', \gamma)$$

$$L^\sigma_\gamma L^\sigma_{\gamma'} = \sigma(\gamma, \gamma')L^\sigma_{\gamma\gamma'}, \quad R^\sigma_\gamma R^\sigma_{\gamma'} = \sigma(\gamma, \gamma')R^\sigma_{\gamma\gamma'}$$

left σ-regular commutes with right $\bar{\sigma}$-regular

- $\mathbb{C}(\Gamma, \sigma) =$ twisted group ring, gen by R^σ_γ
$C^*_r(\Gamma, \sigma) =$ norm closure, (reduced) twisted group C^*-algebra

NC tori as twisted group algebras

$$\mathcal{A}_\theta = C^*_r(\mathbb{Z}^2, \sigma)$$

$$\sigma((n,m),(n',m')) = \exp(-2\pi i(\xi_1 nm' + \xi_2 mn'))$$

with $\theta = \xi_2 - \xi_1$

generators $U = R^\sigma_{(0,1)}$ and $V = R^\sigma_{(1,0)}$

$$Uf(n,m) = e^{-2\pi i \xi_2 n} f(n, m+1)$$

$$Vf(n,m) = e^{-2\pi i \xi_1 m} f(n+1, m)$$

$$UV = e^{2\pi i\theta} VU$$

Some freedom to choose ξ_1, ξ_2

Invariance under $\mathrm{SL}_2(\mathbb{Z})$: cocycle $\sigma((n,m),(n',m'))$ satisfies

$$\sigma((n,m),(n',m')) = \sigma((n,m)\varphi,(n',m')\varphi)$$

for all $\varphi \in \mathrm{SL}_2(\mathbb{Z})$ iff

$$\xi_2 = -\xi_1$$

$\Rightarrow$ Set $\xi_2 = \theta/2 = -\xi_1$

$$\sigma_\theta((n,m),(k,r)) = \exp(\pi i\theta(nr - mk))$$

$$= \exp(\pi i\theta(n,m)\wedge(k,r))$$

for

$$(a,b)\wedge(c,d) := \det\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

NC tori

$$\mathbb{T}_{\Lambda,1} = \quad C^*(\mathbb{Z}^2, \sigma_\theta) \quad = C^*(\Lambda, \sigma_{\theta(\theta'-\theta)^{-1}})$$
$$\mathbb{T}_{\Lambda,2} = \quad C^*(\mathbb{Z}^2, \sigma_{\theta'}) \quad = C^*(\Lambda, \sigma_{\theta'(\theta'-\theta)^{-1}})$$

For elliptic curves: natural object E/O_K^*

For NC tori: V acts on $\mathbb{T}_{\Lambda,i}$ (automorphisms)

$$\upsilon_\epsilon^k(R^\sigma_{(n,m)}) = R^\sigma_{(n,m)\varphi_\epsilon^k}$$

$$\upsilon_\epsilon^k(R^\sigma_\lambda) = R^\sigma_{A_\epsilon^k(\lambda)}$$

Crossed product $\mathbb{T}_{\Lambda,i} \rtimes_i V$

Note: in NCG crossed product replaces quotient $\mathbb{T}_\theta/\mathrm{Aut}(\mathbb{T}_\theta)$

Twisted group algebra of $S(\Lambda, V)$

Given $\sigma((n,m),(k,r)) = \exp(\pi i \theta(nr - mk))$

$$\tilde{\sigma}((n,m,k),(n',m',k')) := \sigma((n,m),(n',m')\varphi_\epsilon^k)$$

is a multiplier for $S(\Lambda, V)$

$$C^*(\Lambda, \sigma) \rtimes V = C^*(S(\Lambda, V), \tilde{\sigma})$$
$$R^{\tilde{\sigma}}_{(n,m,k)} \mapsto R^{\sigma}_{(n,m)} \upsilon_\epsilon^k$$

$\mathbb{T}_{\Lambda,V,i} = \mathbb{T}_{\Lambda,i} \rtimes_i V$

$$\begin{aligned}
\mathbb{T}_{\Lambda,V,1} = C^*(\mathbb{Z}^2, \sigma_\theta) \rtimes_{\upsilon_1} \mathbb{Z} &= C^*(\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}, \tilde{\sigma}_\theta) = \\
C^*(\Lambda, \sigma_{\theta(\theta'-\theta)^{-1}}) \rtimes_{\upsilon_1} V &= C^*(S(\Lambda, V), \tilde{\sigma}_{\theta(\theta'-\theta)^{-1}}) \\
\mathbb{T}_{\Lambda,V,2} = C^*(\mathbb{Z}^2, \sigma_{\theta'}) \rtimes_{\upsilon_2} \mathbb{Z} &= C^*(\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}, \tilde{\sigma}_{\theta'}) = \\
C^*(\Lambda, \sigma_{\theta'(\theta'-\theta)^{-1}}) \rtimes_{\upsilon_2} V &= C^*(S(\Lambda, V), \tilde{\sigma}_{\theta'(\theta'-\theta)^{-1}})
\end{aligned}$$

Note: $S(\Lambda, V)$ amenable: $C_r^*(S(\Lambda, V), \tilde{\sigma}) = C^*_{max}(S(\Lambda, V), \tilde{\sigma})$

Homotopy quotient (Baum–Connes)

• $\mathcal{A}$ noncommutative C^*-algebra (NC space)

• X topological space: is a homotopy quotient model of $\mathcal{A}$ if (analytic) K-theory of $\mathcal{A}$ computed in terms of topological K-theory of X, through a natural map (Kasparov assembly map)

Claim: $X_\epsilon =$ homotopy quotient model for $C^*(S(\Lambda, V), \tilde{\sigma})$

To check: K-theory computation for $\mathcal{A} = C^*(\Lambda, \sigma)$

$$K_0(C^*(S(\Lambda, V), \tilde{\sigma})) \cong \Lambda$$

$$K_1(C^*(S(\Lambda, V), \tilde{\sigma})) \cong \Lambda \oplus \Lambda/(1 - A_\epsilon)\Lambda$$

(Pimsner–Voiculescu six terms exact sequence)

$$\begin{tikzcd}
K_0(\mathcal{A}) \arrow[r, "1-\alpha_*"] & K_0(\mathcal{A}) \arrow[r] & K_0(\mathcal{A} \rtimes \mathbb{Z}) \arrow[d, "\partial"] \\
K_1(\mathcal{A} \rtimes \mathbb{Z}) \arrow[u, "\partial"] & K_1(\mathcal{A}) \arrow[l] & K_1(\mathcal{A}) \arrow[l, "1-\beta_*"']
\end{tikzcd}$$

Twist: no effect on K-theory

$$K_i(C^*(S(\Lambda, V), \tilde{\sigma})) \cong K_i(C^*(S(\Lambda, V)))$$

use homotopy $\exp(-t\pi i\theta(mn' - nm'))$, $t \in [0, 1]$

Kasparov assembly map isomorphism

$$\mu : K^1(X_\epsilon) \stackrel{\cong}{\to} K_0(C^*(S(\Lambda, V)))$$

$$\mu : K^0(X_\epsilon) \stackrel{\cong}{\to} K_1(C^*(S(\Lambda, V)))$$

(mapping torus, Thom isomorphism)

$$K_{i+1}(C(X_\epsilon)) = K_i(C(T^2) \rtimes_{A_\epsilon} \mathbb{Z})$$

$\mathbb{Z}^2 \rtimes_{\varphi_\epsilon} \mathbb{Z}$ satisfies BC (Chabert-Echteroff)

Conclusion: Solvmanifold X_ϵ is a model up to homotopy of NC space $\mathbb{T}_\theta/\mathrm{Aut}(\mathbb{T}_\theta)$ (analog of $E/\mathrm{Aut}(E)$ for elliptic curves)

So far topology: what about geometry?

An arithmetic quantum Hall effect

- discrete group Γ acting on contractible space $\tilde{X}$ w/ compact quotient $X = \tilde{X}/\Gamma$
- base point $x_0 \in \tilde{X}$, crystal charged ions Γx_0

$\Rightarrow$ model electron-ion interactions

- Hamiltonian $\Delta + \mathcal{V}$ on $L^2(\tilde{X})$ with $T_\gamma \Delta = \Delta T_\gamma$ and $\mathcal{V}$ invariant
- Magnetic field: closed 2-form ω w/ $\gamma^* \omega = \omega$
- global magnetic potential $\omega = d\chi$, hermitian connection $\nabla = d - i\chi$ with $\nabla^2 = i\omega$

$$\chi - \gamma^* \chi = d\phi_\gamma, \quad \phi_\gamma(x) = \int_{x_0}^{x} \chi - \gamma^* \chi$$

$$\phi_\gamma(x) - \phi_{\gamma'}(\gamma x) - \phi_{\gamma\gamma'}(x)$$

independent of $x \in \tilde{X}$ and $\phi_\gamma(x_0) = 0$

$$\sigma(\gamma, \gamma') = \exp(-i\phi_\gamma(\gamma' x_0))$$

multiplier $\sigma : \Gamma \times \Gamma \to U(1)$

After turning on magnetic field, ordinary Laplacian not invariant under magnetic translations: magnetic potential modifies connection $\nabla = d - i\chi$

Magnetic Laplacian:

$$\Delta^{\chi} = \nabla^* \nabla = (d - i\chi)^*(d - i\chi)$$

now invariant under magnetic translations

$$T^{\phi}_{\gamma} \Delta^{\chi} = \Delta^{\chi} T^{\phi}_{\gamma}$$

$T^{\phi}_{\gamma} f(x) = \exp(i\phi_{\gamma}(x)) f(\gamma^{-1}x)$

Independent electron approximation: $\mathcal{V}$ effective potential invariant under T^{ϕ}_{γ}

Discrete model: Hilbert space $\ell^2(\Gamma)$

$$\mathcal{R} = \sum_{i=1}^{r} R_{\gamma_i} \quad \text{with} \quad R_{\gamma_i} f(\gamma) = f(\gamma\gamma_i)$$

Laplacian $\Rightarrow \Delta_{discr} = r - \mathcal{R}$ random walk operator
($r = \#$ symm set if generators)

Harper operator

$$\mathcal{H}_\sigma = \sum_{i=1}^{r} R^\sigma_{\gamma_i}, \quad R^\sigma_{\gamma_i} \in \mathbb{C}(\Gamma, \sigma)$$

$$\Delta^\chi_{discr} = r - \mathcal{H}_\sigma$$

magnetic Laplacian; potential $\mathcal{V} \in \mathbb{C}(\Gamma, \sigma)$

Harper operators and noncommutative tori

$$\mathcal{H}_\sigma = U + U^* + V + V^*$$

(spectral theory: Hofstadter butterfly, rational/irrational θ)

For X_ϵ Harper operator: $\mathcal{H}_{\tilde{\sigma}} = U + U^* + V + V^* + W + W^*$ with $U = R^{\tilde{\sigma}}_{(0,1,0)}$, $V = R^{\tilde{\sigma}}_{(1,0,0)}$ and $W = R^{\tilde{\sigma}}_{(0,0,1)}$

Problem: band structure, gaps in the spectrum (detected by range of the trace on K-theory)

$\mathcal{U}(\Gamma,\sigma)$ = von Neumann alg: weak closure $\mathbb{C}(\Gamma,\sigma)$

$$P_E = \mathbf{1}_{(-\infty,E]}(\mathcal{H}_{\sigma,V}) \in \mathcal{U}(\Gamma,\sigma)$$

spectral projections of $\mathcal{H}_{\sigma,V}$ satisfy $P_E \in C^*_r(\Gamma,\sigma)$ when E in a gap in the spectrum

$\Rightarrow$ Counting of projections in $C^*_r(\Gamma,\sigma)$

$\Leftrightarrow$ count proj's mod equiv

$$[\mathrm{tr}] : K_0(C_r^*(\Gamma,\sigma)) \to \mathbb{R}$$

where $\tau(a) = \langle a\delta_1, \delta_1\rangle_{\ell^2(\Gamma)}$ on $C_r^*(\Gamma,\sigma)$ and

$$\mathrm{tr} = \tau \otimes \mathrm{Tr} : \{P \in C_r^*(\Gamma,\sigma) \otimes \mathcal{K}) \,|\, P^* = P,\ \ P^2 = P\} \to \mathbb{R}$$

Estimate by: $[\mathrm{tr}](K_0(C_r^*(\Gamma,\sigma))) \cap [0,1]$

Compute via a (twisted) index theorem:

$$[\mathrm{tr}](K_0(C^*(S(\Lambda, V), \tilde{\sigma}))) = \mathbb{Z} + \mathbb{Z}\theta(\theta' - \theta)^{-1}$$

Spectral flow: $K_1(C(X_\epsilon)) \ni [g]$, $g : X_\epsilon \to \mathrm{GL}_N(\mathbb{C})$

$$\beta(g) = g^{-1} dg \in \Omega^1(X_\epsilon, gl_N(\mathbb{C}))$$

connections $\nabla_u = d + u\beta(g)$ on $X_\epsilon \times \mathbb{C}^N \Rightarrow$ Chern–Simons form (odd Chern character)

$$Ch(g) := cs(d, d + \beta(g)) = \int_0^1 \mathrm{Tr}(\frac{d}{du}(\nabla_u) e^{\nabla_u^2}) du$$

$$Ch(g) = \sum_{k=0}^{\infty} (-1)^k \frac{k!}{(2k+1)!} \mathrm{Tr}(\beta(g)^{2k+1})$$

Odd Fredholm module pairing w/ $[g] \in K_1$

$$\langle D, [g] \rangle = SF(D, g^{-1} D g)$$

Dirac operator $\Rightarrow$ APS (mapping torus)

$$SF(\partial\!\!\!/, g^{-1} \partial\!\!\!/ g) = -\frac{1}{(2\pi i)^2} \int_{X_\epsilon} \hat{A}(X_\epsilon) Ch(g)$$

Twisted index theorem:

range of trace on $K_0(C^*(S(\Lambda, V), \tilde{\sigma}))$

$$[\mathrm{tr}](\mu_{\tilde{\sigma}}[g]) = \frac{-1}{(2\pi i)^2} \int_{X_\epsilon} \hat{A}\, e^{\omega_\epsilon}\, Ch(g)$$

$$\mu_{\tilde{\sigma}} : K^1(X_\epsilon) \to K_0(C^*(S(\Lambda, V), \tilde{\sigma}))$$

twisted Kasparov isomorphism $[g] \in K^1(X_\epsilon)$
ω_ϵ closed 2-form on X_ϵ of cocycle $\tilde{\sigma}$:
2 cocycle: $\tilde{\sigma} = \exp(2\pi i \zeta)$

$$\zeta((\lambda, k), (\eta, r)) = \frac{1}{4\pi i} \int_{\mathcal{R}} \omega$$

ω closed 2-form on $T^2 = \mathbb{R}^2/\Lambda$ flux

$$\int_{T^2} \omega = 2\pi i \theta(\theta' - \theta)^{-1}$$

parallelogram $\mathcal{R} = \{0, A^k_\epsilon(\eta), \lambda, \lambda + A^k_\epsilon(\eta)\}$

Sketch of proof:

$P^{\pm} =$ projs on L^2-kernel of $\mathcal{D}_g \mathcal{D}_g^*$ and $\mathcal{D}_g^* \mathcal{D}_g$

$$\mathcal{D}_g P^+ = 0 \qquad \mathcal{D}_g^* P^- = 0$$

$$\mathcal{D}_g = \frac{\partial}{\partial u} + D_u$$

on $\tilde{X}_\epsilon \times [0,1]$ with $D_u = \tilde{\partial\!\!\!/}_u \otimes \nabla$

$$\tilde{\partial\!\!\!/}_u = (1-u)\tilde{\partial\!\!\!/} + u g^{-1} \tilde{\partial\!\!\!/} g, \quad [g] \in K^1(X_\epsilon)$$

$P^{\pm}$ have smooth kernels $P^{\pm}(x,y)$

$$e^{-i\phi_\gamma(x)} P^{\pm}(\gamma x, \gamma y) e^{i\phi_\gamma(y)} = P^{\pm}(x,y)$$

$\Rightarrow P^{\pm}(x,x)$ is $\Gamma = S(\Lambda, V)$-invariant

$$\mathrm{tr}(P^{\pm}) = \int_{X_\epsilon \times S^1} tr P^{\pm}((x,t),(x,t))\, dx\, dt,$$

von Neumann trace, $tr\, P^{\pm}(x,x)$ ptwise trace

$$\mathrm{Ind}_{L^2}(\mathcal{D}_g) = \mathrm{tr}(P^+) - \mathrm{tr}(P^-)$$

Set $\bar{P}^{\pm}(x,y) = \int_{S^1} trP^{\pm}((x,t),(y,t))\,dt$

$$\int_{X_\epsilon \times S^1} trP^{\pm}((x,t),(x,t))\,dx\,dt = \int_{X_\epsilon} tr\,\bar{P}^{\pm}(x,x)dx$$

projections $\bar{P}^{\pm}$ in von Neumann algebra $\mathcal{U}(\Gamma,\tilde{\sigma})$
(after compact perturbation)

$$\mathrm{Ind}_{(\Gamma,\tilde{\sigma})}(\mathcal{D}_g) = [\bar{P}^+] - [\bar{P}^-] \in K_0(C_r^*(\Gamma,\tilde{\sigma}))$$

Twisted Kasparov map

$$\mu_{\tilde{\sigma}}[g] = \mathrm{Ind}_{(\Gamma,\tilde{\sigma})}(\mathcal{D}_g)$$

L^2-index:

$$\mathrm{Ind}_{L^2}(\mathcal{D}_g) = \mathrm{tr}(\bar{P}^+) - \mathrm{tr}(\bar{P}^-) = \mathrm{tr}(\mathrm{Ind}_{(\Gamma,\tilde{\sigma})}(\mathcal{D}_g))$$

Heat kernel $e^{-t\mathcal{D}^2}$

$$\mathcal{D} = \begin{pmatrix} 0 & \mathcal{D}_g^* \\ \mathcal{D}_g & 0 \end{pmatrix} \quad \text{with} \quad \mathcal{D}^2 = \begin{pmatrix} \mathcal{D}_g^* \mathcal{D}_g & 0 \\ 0 & \mathcal{D}_g \mathcal{D}_g^* \end{pmatrix}$$

$$\lim_{t\to\infty} \mathrm{tr}_s(e^{-t\mathcal{D}^2}) = \mathrm{tr}(P^+) - \mathrm{tr}(P^-)$$

$$\frac{\partial}{\partial t} \mathrm{tr}_s(e^{-t\mathcal{D}^2}) = -\mathrm{tr}_s(\mathcal{D}^2 e^{-t\mathcal{D}^2}) = \mathrm{tr}_s([\mathcal{D}e^{-t\mathcal{D}^2}, \mathcal{D}]) = 0$$

$$\mathrm{tr}(P^+) - \mathrm{tr}(P^-) = \lim_{t\to\infty} \mathrm{tr}_s(e^{-t\mathcal{D}^2}) = \lim_{t\to 0} \mathrm{tr}_s(e^{-t\mathcal{D}^2})$$

$$= \frac{-1}{(2\pi i)^2} \int_{X_\epsilon \times S^1} \hat{A}\, Ch(\nabla_u) = \frac{-1}{(2\pi i)^2} \int_{X_\epsilon} \hat{A}\, e^{\omega_\epsilon}\, Ch(g),$$

$Ch(\nabla_u) = tr(\beta e^{(d+u\beta)^2}), \quad \beta = g^{-1} dg,$
$\int_{S^1} Ch(\nabla_u) = Ch(g)$

Range computation:

3-manifold X_ϵ

$$\hat{A}(X_\epsilon) = 1 - \frac{1}{24} p_1(X_\epsilon) + \cdots \qquad e^{\omega_\epsilon} = 1 + \omega_\epsilon + \frac{1}{2}\omega_\epsilon^2 + \cdots$$

$$Ch(g) = -\frac{1}{6}\mathrm{Tr}(\beta(g)) + \frac{1}{5!}\mathrm{Tr}(\beta^3(g)) + \cdots$$

terms up to dim=3

$$\frac{1}{(2\pi)^2}\int_{X_\epsilon}\left(\frac{-1}{6}\mathrm{Tr}(\beta(g)) \wedge \omega + \frac{1}{5!}\mathrm{Tr}(\beta(g)^3)\right)$$

$$\frac{1}{(2\pi)^2}\int_{X_\epsilon}\frac{1}{5!}\mathrm{Tr}(\beta(g)^3) = \frac{1}{(2\pi)^2}\int_{X_\epsilon} Ch(g)$$

(untwisted odd Chern character)

$$\frac{1}{(2\pi)^2}\frac{-1}{6}\int_{X_\epsilon}\mathrm{Tr}(\beta(g)) \wedge \omega_\epsilon \in \mathbb{Z}R(\omega)$$

$R(\omega)$ = range of linear form

$$T_\omega : [g] \mapsto \frac{1}{(2\pi)^2}\frac{-1}{6}\int_{X_\epsilon}\mathrm{Tr}(\beta(g)) \wedge \omega_\epsilon \in \mathbb{R}$$

$$\int_C Ch_1(g) = 2\pi i \deg(g|_C) \in 2\pi i\mathbb{Z}$$

with $Ch_1(g) = \frac{-1}{6}\mathrm{Tr}(\beta(g))$ and $C \in H_1(X_\epsilon, \mathbb{Z})$

$$\frac{1}{2\pi i}\int_{X_\epsilon} Ch_1(g) \wedge PD(C) \in \mathbb{Z}$$

$PD(C) \in H^2(X_\epsilon, \mathbb{Z}) \hookrightarrow H^2(X_\epsilon, \mathbb{R})$

$$\omega_\epsilon = 2\pi i\theta(\theta' - \theta)^{-1}\bar{\omega}_\epsilon, \quad \bar{\omega}_\epsilon \in H^2(X_\epsilon, \mathbb{Z})$$

$$\bar{\omega}_\epsilon(v, w) = A^k_\epsilon(\eta) \wedge \lambda,$$

for $v = ((0,0),(\lambda,k))$ and $w = ((0,0),(\eta,r))$

$$\frac{1}{(2\pi)^2}\int_{X_\epsilon} Ch_1(g) \wedge \omega_\epsilon$$

$$= -\frac{1}{2\pi i}\theta(\theta' - \theta)^{-1} \int_{PD(\bar{\omega}_\epsilon)} Ch_1(g)$$

$$= \theta(\theta' - \theta)^{-1}\deg(g|_{PD(\bar{\omega}_\epsilon)}) \in \theta(\theta' - \theta)^{-1}\mathbb{Z}$$

Dirac operator on X_ϵ
$\{dt, e^t dx, e^{-t} dy\}$ basis of cotangent bundle
of $S(\mathbb{R}^2, \mathbb{R}, \epsilon) = \mathbb{R}^2 \rtimes_\epsilon \mathbb{R}$

$$\partial\!\!\!/_{X_\epsilon} = c(dt)\frac{\partial}{\partial t} + c(e^t dx)\frac{\partial}{\partial x} + c(e^{-t} dy)\frac{\partial}{\partial y}$$

$$\partial\!\!\!/_{X_\epsilon} = \frac{\partial}{\partial t}\sigma_0 + e^t \frac{\partial}{\partial x}\sigma_1 + e^{-t}\frac{\partial}{\partial y}\sigma_2$$

$$\partial\!\!\!/_{X_\epsilon} = \begin{pmatrix} \frac{\partial}{\partial t} & e^{-t}\frac{\partial}{\partial y} - ie^t \frac{\partial}{\partial x} \\ e^{-t}\frac{\partial}{\partial y} + ie^t \frac{\partial}{\partial x} & -\frac{\partial}{\partial t} \end{pmatrix}$$

σ_i, for $i = 0, 1, 2$ Pauli matrices

Commutative spectral triple

$$(C^\infty(X_\epsilon), L^2(X_\epsilon, S), \partial\!\!\!/_{X_\epsilon})$$

Isospectral deformations (Connes–Landi)

Given $(C^\infty(X), L^2(X,S), \partial\!\!\!/_X)$ with $T^2 \subset \mathrm{Isom}(X)$

$$\pi(f) = \sum_{n,m\in\mathbb{Z}} \pi(f_{n,m})$$

$$\alpha_\tau(\pi(f_{n,m})) = e^{2\pi i(n\tau_1+m\tau_2)}\,\pi(f_{n,m}), \quad \forall \tau = (\tau_1,\tau_2) \in T^2$$

$\alpha_\tau(T) = U(\tau)TU(\tau)^*$, for $T \in \mathcal{B}(\mathcal{H})$, $\tau \in T^2$,
$U(\tau)\psi(x) = \psi(\tau^{-1}(x))$ on $\mathcal{H} = L^2(X,S)$

$$U(\tau) = \exp(2\pi i \tau L) = \exp(2\pi i(\tau_1 L_1 + \tau_2 L_2))$$

w/ L_1 and L_2 the infinitesimal generators

Algebra in $\mathcal{B}(\mathcal{H})$ gen by

$$\pi_{\xi_1,\xi_2}(f) = \sum_{n,m} \pi(f_{n,m}) e^{-2\pi i(\xi_1 n L_2 + \xi_2 m L_1)}$$

$$\pi_{\xi_1,\xi_2}(f_{n,m})\pi_{\xi_1,\xi_2}(h_{k,r}) = e^{-2\pi i(\xi_1 nr + \xi_2 mk)}\pi(f_{n,m})\pi(h_{k,r})$$

$\Rightarrow$ cocycle $\sigma((n,m),(k,r)) = \exp(-2\pi i(\xi_1 nr + \xi_2 mk))$
NC Spectral triple $(C^\infty(X)_{\xi_1,\xi_2}, L^2(X,S), \partial\!\!\!/_X)$

Deforming X_ϵ isospectrally

Fibration $T^2 \to X_\epsilon \to S^1$ torus action

$$U(\tau)\psi((x,y),t) = \psi((x + e^t\tau_1, y + e^{-t}\tau_2), t)$$

preserves the metric $dt^2 + e^t dx^2 + e^{-t}dy^2 \Rightarrow U(\tau)\partial_{X_\epsilon}U(\tau)^* = \partial_{X_\epsilon}$

Infinitesimal generators

$$2\pi L_1 = e^t \frac{\partial}{\partial x}, \quad 2\pi L_2 = e^{-t}\frac{\partial}{\partial y}$$

$U(\tau) = \exp(2\pi i(\tau_1 L_1 + \tau_2 L_2))$

$$E_\lambda((x,y),t) := e^{2\pi i \langle \Theta_{-t}(x,y),\lambda\rangle}$$

$\Theta_{-t}(x,y) = (e^{-t}x, e^t y)$ and $\langle (a,b),\lambda\rangle = a\lambda_1 + b\lambda_2$

$$\Xi_u(\lambda, L_1, L_2) := \exp\left(i\pi \frac{u}{(\theta' - \theta)} \lambda \wedge (L_1, L_2)\right)$$

Connes-Landi deformation $C^\infty(X_\epsilon)_u$
subalg of $\mathcal{B}(L^2(X_\epsilon, S)$ gen by

$$\pi_u(f) = E_\lambda \Xi_u(\lambda, L_1, L_2)$$

$\xi_2 = u/2 = -\xi_1$ twisted translations

$$\pi(R^\sigma_\lambda) = E_\lambda \, \Xi_u(\lambda, L_1, L_2)$$

$$\pi(R^\sigma_\lambda)\pi(R^\sigma_\eta) = \sigma(\lambda, \eta)\pi(R^\sigma_{\lambda+\eta})$$

$$\sigma(\lambda, \eta) = \exp(2\pi i u_\theta \, \lambda \wedge \eta)$$

for $u = \theta$ and $u = \theta'$ get $\mathbb{T}_{\Lambda,i}$

$$\Rightarrow (C^*(\Lambda, \sigma), L^2(X_\epsilon, S), \partial\!\!\!/_{X_\epsilon})$$

Conclusion: metric (Dirac) on solvmanifold induces metric (spectral triple) on NC space $C^*(\Lambda, \sigma)$

Modify Dirac by unitary equivalences (after ADS)

- Fourier modes on the fibers

$$\partial\!\!\!/_{X_\epsilon} \psi_\lambda = (\frac{\partial}{\partial t}\sigma_0 + 2\pi i \lambda_1 \sigma_1 + 2\pi i \lambda_2)\psi_\lambda$$

$$\pi(R^\sigma_\eta)\psi_\lambda = e^{2\pi i u\, \eta\wedge\lambda}\psi_{\eta+\lambda}$$

$$[\partial\!\!\!/_{X_\epsilon}, \pi(R^\sigma_\eta)] = (\eta_1\sigma_1 + \eta_2\sigma_2)\, R^\sigma_\eta$$

- $\mathcal{U}\,\psi_\lambda = \sigma_\lambda\,\psi_\lambda$: $\sigma_\lambda = \text{prod } \sigma_i$, $i = 1, 2$, $\lambda_i < 0$

$$\mathcal{U}\hat{\partial\!\!\!/}_{X_\epsilon}\mathcal{U}^* = \mathrm{sign}(N(\lambda))(\frac{\partial}{\partial t}\sigma_0 + 2\pi i|\lambda_1|\sigma_1 + 2\pi i|\lambda_2|\sigma_2)$$

$$\mathcal{U}\hat{\pi}(R^\sigma_\eta)\mathcal{U}^* : \sigma_\lambda\psi_\lambda \mapsto \sigma_{\lambda+\eta}\psi_{\lambda+\eta}$$

- For $\lambda = A^k_\epsilon(\mu) \neq 0$, $\mu \in \mathcal{F}_V$

$$\tilde{\mathcal{U}}(\sigma_\lambda\psi_\lambda)(t) = \sigma_\lambda\psi_\lambda(t - \log\frac{|\mu_1|}{|N(\mu)|^{1/2}})$$

$$\tilde{\partial\!\!\!/} = \tilde{\partial\!\!\!/}^{(0)} + \sum_{\mu \in (\Lambda \smallsetminus \{0\})/V} \tilde{\partial\!\!\!/}^{(\mu)}$$

$$\tilde{\psi}_\lambda := \tilde{\mathcal{U}}(\sigma_\lambda \psi_\lambda) = \sigma_\lambda \psi_{|N(\lambda)|^{1/2}(\mathrm{sign}(\lambda_1)\epsilon^k, \mathrm{sign}(\lambda_2)\epsilon^{-k})}$$

Unitarily equivalent: $\tilde{\pi}(R^\sigma_\eta)\, \tilde{\psi}_\lambda = \tilde{\psi}_{\lambda+\eta}$

$$\tilde{\partial\!\!\!/}^{(\mu)}\, \tilde{\psi}_{A^k_\epsilon(\mu)} = \mathrm{sign}(N(\mu))|N(\mu)|^{1/2}.$$

$$\left(|N(\mu)|^{-1/2} \frac{\partial}{\partial t}\sigma_0 + 2\pi i \epsilon^k \sigma_1 + 2\pi i \epsilon^{-k}\sigma_2 \right) \tilde{\psi}_{A^k_\epsilon(\mu)}$$

$$\tilde{\partial\!\!\!/}^{(\mu)} = D_\mu B_\mu$$

$$D_\mu\, \tilde{\psi}_{A^k_\epsilon(\mu)} = \mathrm{sign}(N(\mu))|N(\mu)|^{1/2}\, \tilde{\psi}_{A^k_\epsilon(\mu)}$$

$$B_\mu\, \tilde{\psi}_{A^k_\epsilon(\mu)} = \left(|N(\mu)|^{-1/2} \frac{\partial}{\partial t}\sigma_0 + 2\pi i \epsilon^k \sigma_1 + 2\pi i \epsilon^{-k}\sigma_2 \right) \tilde{\psi}_{A^k_\epsilon(\mu)}$$

Foliations, transverse measures, differential operators

$$\Theta_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

Fixed point $(0,0)$, stable manifold axis $(0,y)$, unstable manifold axis $(x,0)$

$$(x,y) = (s_1 + s_2\theta, s_1 + s_2\theta'), \quad (s_1,s_2) \in \mathbb{R}^2/\mathbb{Z}^2$$

Kronecker foliations

$$L_\theta = \{s_1 + s_2\theta\}, \quad L_{\theta'} = \{s_1 + s_2\theta'\}$$

operators $e^t \frac{\partial}{\partial x}$ and $e^{-t}\frac{\partial}{\partial y}$ leaf-wise derivations
factors e^t and e^{-t} scaling of transverse measure

$$\delta_\theta : \psi_{n,m} \mapsto (n + m\theta)\,\psi_{n,m}$$

$$\delta_{\theta'} : \psi_{n,m} \mapsto (n + m\theta')\,\psi_{n,m}$$

$$D\!\!\!/_{\theta,\theta'} = \begin{pmatrix} 0 & \delta_{\theta'} - i\delta_\theta \\ \delta_{\theta'} + i\delta_\theta & 0 \end{pmatrix}$$

$\not{D}_{\theta,\theta',0}$ on complement of zero modes

$$\not{D}_{\theta,\theta',0} = \sum_{\mu \in (\Lambda \smallsetminus \{0\})/V} \not{D}^{\mu}_{\theta,\theta'}$$

$$\not{D}^{\mu}_{\theta,\theta'} \, \psi_{A^k_\epsilon(\mu)} = (\lambda_1 \sigma_1 + \lambda_2 \sigma_2) \, \psi_{A^k_\epsilon(\mu)}$$

adiabatic limit of

$$\hat{\not{\partial}}_{X_\epsilon} : \psi_\lambda \mapsto (\frac{\partial}{\partial t} \sigma_0 + 2\pi i \lambda_1 \sigma_1 + 2\pi i \lambda_2 \sigma_2) \, \psi_\lambda$$

Unitary equivalence

$$\tilde{\not{D}}^{\mu}_{\theta,\theta'} \, \tilde{\psi}_{A^k_\epsilon(\mu)} = \mathrm{sign}(N(\mu)) \, |N(\mu)|^{1/2} \, (\epsilon^k \sigma_1 + \epsilon^{-k} \sigma_2) \, \tilde{\psi}_{A^k_\epsilon(\mu)}$$

$$\tilde{\not{D}}^{\mu}_{\theta,\theta'} = D^{\mu}_{\theta} B_{\theta}$$

$$D^{\mu}_{\theta} \, \tilde{\psi}_{A^k_\epsilon(\mu)} = \mathrm{sign}(N(\mu)) \, |N(\mu)|^{1/2} \, \tilde{\psi}_{A^k_\epsilon(\mu)}$$

$$B_{\theta} \, \tilde{\psi}_{A^k_\epsilon(\mu)} = (\epsilon^k \sigma_1 + \epsilon^{-k} \sigma_2) \, \tilde{\psi}_{A^k_\epsilon(\mu)}$$

Lorentzian geometry

$$N(\lambda) = \lambda_1 \lambda_2 = (n + m\theta)(n + m\theta') \Leftrightarrow \Box = p_0^2 - p_1^2$$

wave operator $\mathcal{D}_\lambda^2 = \Box_\lambda$

$$\mathcal{D}_\lambda = \begin{pmatrix} 0 & \mathcal{D}_\lambda^+ \\ \mathcal{D}_\lambda^- & 0 \end{pmatrix} := \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix}$$

$$\Box_\lambda = \begin{pmatrix} N(\lambda) & 0 \\ 0 & N(\lambda) \end{pmatrix}$$

On $\mathcal{H} = \ell^2(\Lambda) \oplus \ell^2(\Lambda)$ with $C^*(\Lambda, \sigma)$ acting diagonally, operator

$$\mathcal{D} e_{\lambda,\pm} = \mathcal{D}_\lambda e_{\lambda,\pm}$$

$\mathbb{Z}/2\mathbb{Z}$-grading

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Bounded commutators:

$$[\mathcal{D}, R_\eta^\sigma] e_{\lambda,\pm} = \sigma(\lambda, \eta)\, \eta_\mp \; e_{\eta+\lambda,\pm}$$

$\eta_+ = \eta_1$ and $\eta_- = \eta_2$

Problems with spectral triples in Lorentzian geometry

• $\mathcal{D}$ not self-adjoint

• $\mathcal{D}$ has infinite multiplicities in the spectrum: noncompact symmetry group $V = \epsilon^{\mathbb{Z}}$

Two issues:

• work over $\mathbb{K} = \mathbb{Q}(\sqrt{d})$ instead of $\mathbb{C}$

• use indefinite inner product spaces (Krein)

Arithmetic Krein spaces

$c : \mathbb{K} \to \mathbb{K}$ Galois involution $c : x \mapsto x'$

$\mathcal{V}$ a $\mathbb{K}$-vector space, $T : \mathcal{V} \to \mathcal{V}$ is c-linear if

$$T(av + bw) = c(a)T(v) + c(b)T(w)$$

• Lorentzian pairing:

$$(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{K}$$

non-degenerate, c-linear in first variable, linear in the second

• $\mathbb{K}$-Krein space: $(\mathcal{V}, (\cdot,\cdot))$ with c-linear involution $\kappa : \mathcal{V} \to \mathcal{V}$

– $(\kappa\cdot,\cdot) = c(\cdot,\kappa\cdot)$

– For all $v \neq 0$ in $\mathcal{V}$, the elements $(\kappa v, v) \in \mathbb{K}$ are totally positive

• Krein adjoint $T^\dagger$ of $\mathbb{K}$-linear T

$$(v, Tw) = (T^\dagger v, w)$$

• Associated real Hilbert spaces

$$\mathcal{V}_{\mathbb{R},i} := \mathcal{V} \otimes_{\iota_i(\mathbb{K})} \mathbb{R}$$

two embeddings $\iota_i : \mathbb{K} \hookrightarrow \mathbb{R}$

$$\begin{aligned} \langle v, w \rangle = & \ \tfrac{1}{2}\iota_1 \left((\kappa v, w) + (v, \kappa w) \right) \\ = & \ \tfrac{1}{2}\iota_2 \left((\kappa v, w) + (v, \kappa w) \right) \end{aligned}$$

• For $\mathbb{K}$-linear operator T on $\mathbb{K}$-Krein space $\mathcal{V}$

$$\mathbb{M}_i(T) := \inf_{(v,v)=1} \iota_i(Tv, Tv), \quad i = 1, 2$$

Krein-bounded: $\mathbb{M}_i(T) > -\infty$
(Note: need not be bounded in associated Hilbert space)

Lorentzian $\mathbb{K}$-spectral triples: $(\mathcal{A}, \mathcal{V}, \mathcal{D})$
– $\mathcal{A}$ involutive $\mathbb{K}$-algebra
– $\mathcal{V}$ a $\mathbb{K}$-Krein space with action

$$\pi : \mathcal{A} \to \mathrm{End}_{\mathbb{K}}(\mathcal{V})$$

by Krein-bounded

$$\mathbb{M}_i(\pi(a)) > -\infty$$

$\pi(a^*) = \pi(a)^\dagger$
– Densely defined $\mathbb{K}$-linear $\mathcal{D}$ with $\mathcal{D}^\dagger = \mathcal{D}$
– Krein-bounded commutators $C_a := [\mathcal{D}, \pi(a)]$

$$\mathbb{M}_i(C_a) > -\infty, \quad \forall a \in \mathcal{A}$$

Continue...

...continue

– $\exists$ densely def $\mathbb{K}$-linear $U : \mathcal{V} \to \mathcal{V}$

$$(Uv, Uv) = (v, v), \quad v \in Dom(U)$$

$U^\dagger = U^{-1}$ and $U^\dagger \mathcal{D} U = \mathcal{D}$

– Commutators $C_{a,U} := [\mathcal{D}, \pi_U(a)]$ Krein-bounded

$$\mathbb{M}_i(C_{a,U}) > -\infty, \quad \forall a \in \mathcal{A}$$

$\pi_U(a) = U^\dagger \pi(a) U$

– U unbounded $U = U^*$ in assoc Hilbert space

– p-summability

$$\sum_n \left| \langle U e_n, |\mathcal{D}^2| U e_n \rangle \right|^{-s/2} < \infty, \qquad \forall s \geq p$$

$e_n =$ o.n.basis of complement of zero modes of $|\mathcal{D}^2|$ on Hilbert space

Lorentzian $\mathbb{K}$-spectral triple for real multiplication NC tori

- $\mathbb{K}$-Krein space $\mathcal{V}_\Lambda$ spanned by e_λ, $\lambda \in \Lambda$

$$(v, w) := \sum_\lambda c(a_\lambda) b_\lambda$$

$v = \sum_\lambda a_\lambda e_\lambda$ and $w = \sum_\lambda b_\lambda e_\lambda$

- Action of group ring $\mathbb{K}[\Lambda]$

$$R_\lambda e_\eta = e_{\lambda+\eta} \quad \text{with} \quad \mathbb{M}_i(R_\lambda) > -\infty$$

In this case: bounded in assoc Hilbert space

- Twisted group ring $\omega \in \mathbb{K}^*$ with $N(\omega) = \omega\omega' = 1$

$$\varpi(\lambda, \eta) = \omega^{(n,m)\wedge(r,k)}$$

$\lambda = (n + m\theta, n + m\theta')$ and $\eta = (r + k\theta, r + k\theta')$
$\mathbb{K}^*$-valued 2-cocycle ϖ on Λ

$$R^\varpi_\lambda R^\varpi_\eta = \varpi(\lambda, \eta) R^\varpi_{\lambda+\eta}$$

twisted $\mathbb{K}(\Lambda, \varpi)$

• Action of $\mathbb{K}(\Lambda, \varpi)$ on $\mathcal{V}_\Lambda$

$$R^{\varpi}_\lambda e_\eta = \varpi(\eta, \lambda) e_{\lambda+\eta}$$

Krein-bounded $\mathbb{M}_i(R^{\varpi}_\lambda) > -\infty$ not-bounded in assoc Hilbert

• Preserves Lorentzian pairing

$$(R^{\varpi}_\lambda e_\eta, R^{\varpi}_\lambda e_\zeta) = c(\omega(\eta, \lambda))\omega(\zeta, \lambda)\delta_{\eta,\zeta} =$$

$$N(\omega(\eta, \lambda))\delta_{\eta,\zeta} = (e_\eta, e_\zeta)$$

• Lorentzian Dirac operator $\mathcal{D}_{\mathbb{K}}$

$$\mathcal{D}_{\mathbb{K},\lambda} e_{\lambda,\pm} = \begin{pmatrix} 0 & \mathcal{D}^+_\lambda \\ \mathcal{D}^-_\lambda & 0 \end{pmatrix} e_{\lambda,\pm} := \begin{pmatrix} 0 & \ell \\ c(\ell) & 0 \end{pmatrix} e_{\lambda,\pm}$$

Krein self-adjoint $\mathcal{D}^\dagger_{\mathbb{K}} = \mathcal{D}_{\mathbb{K}}$

- On associated real Hilbert spaces

$$\mathcal{D}_\lambda = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix} \quad \text{and} \quad c(\mathcal{D}_\lambda) = \begin{pmatrix} 0 & \lambda_2 \\ \lambda_1 & 0 \end{pmatrix}$$

- Commutators Krein-bounded

$$[\mathcal{D}_{\mathbb{K}}, R_\lambda^{\varpi}] e_{\eta,\pm} = \varpi(\eta,\lambda) \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix} e_{\lambda+\eta,\pm}$$

$$([\mathcal{D}_{\mathbb{K}}, R_\lambda^{\varpi}]v, [\mathcal{D}_{\mathbb{K}}, R_\lambda^{\varpi}]v) = N(\lambda)(v,v)$$

- Fundamental domain $\mathcal{F}_V$ in Λ

$\rho(\lambda) \in \mathbb{Z}$ unique s.t. $\lambda = A_\epsilon^{\rho(\lambda)}(\mu)$, with $\mu \in \mathcal{F}_V$

$$T_\epsilon e_{\lambda,\pm} := \begin{pmatrix} \epsilon^{\rho(\lambda)} & 0 \\ 0 & \epsilon^{-\rho(\lambda)} \end{pmatrix} e_{\lambda,\pm}$$

$$J e_{\lambda,\pm} = e_{J(\lambda),\pm}, \quad J(A_\epsilon^k(\mu)) = A_\epsilon^{-k}(\mu), \quad U_\epsilon = T_\epsilon J$$

$$U_\epsilon e_{\lambda,\pm} = \begin{pmatrix} \epsilon^{-\rho(\lambda)} & 0 \\ 0 & \epsilon^{\rho(\lambda)} \end{pmatrix} e_{J(\lambda),\pm}$$

- $U_\epsilon^\dagger = U_\epsilon^{-1}$ and $(U_\epsilon v, U_\epsilon v) = (v, v)$. Symmetry

$$U_\epsilon^\dagger \mathcal{D}_{\mathbb{K}} U_\epsilon = \mathcal{D}_{\mathbb{K}}$$

- Krein-bounded commutators

$$\mathbb{M}_i([\mathcal{D}_{\mathbb{K}}, U_\epsilon^\dagger R_\lambda^\varpi U_\epsilon]) > -\infty$$

$$(U_\epsilon^\dagger [\mathcal{D}_{\mathbb{K}}, R_\lambda^\varpi] U_\epsilon v, U_\epsilon^\dagger [\mathcal{D}_{\mathbb{K}}, R_\lambda^\varpi] U_\epsilon v) = N(\lambda)(v, v)$$

- Hilbert space adjoint $U_\epsilon^* = U_\epsilon$

$$U_\epsilon^* e_{\lambda,\pm} = (\kappa U_\epsilon^\dagger \kappa) e_{\lambda,\pm} = c(U_\epsilon^\dagger) e_{\lambda,\pm}$$

$$= \begin{pmatrix} \epsilon^{-\rho(\lambda)} & 0 \\ 0 & \epsilon^{\rho(\lambda)} \end{pmatrix} e_{J(\lambda),\pm} = U_\epsilon e_{\lambda,\pm}$$

unbounded

- Finite summability:

$$|\mathcal{D}_{\mathbb{K}}^2| e_{\lambda,\pm} = \begin{pmatrix} |N(\lambda)| & 0 \\ 0 & |N(\lambda)| \end{pmatrix} e_{\lambda,\pm}$$

- On complement of zero modes

$$\sum_{\lambda \neq 0} \left| \langle U_\epsilon e_{\lambda,\pm}, |\mathcal{D}_{\mathbb{K}}^2| U_\epsilon e_{\lambda,\pm} \rangle \right|^{-s/2} =$$

$$\sum_{\lambda \neq 0} (\epsilon^{2\rho(\lambda)} + \epsilon^{-2\rho(\lambda)})^{-s/2} |N(\lambda)|^{-s/2} =$$

$$\sum_{k \in \mathbb{Z}} (\epsilon^{2k} + \epsilon^{-2k})^{-s/2} \sum_{\mu \in (\Lambda \smallsetminus \{0\})/V} |N(\mu)|^{-s/2}$$

Eta function

$$\eta_{\mathcal{D}}(s) := \sum_n \mathrm{sign}(\langle Ue_n, \mathcal{D}^2 Ue_n\rangle) \left|\langle Ue_n, |\mathcal{D}^2| Ue_n\rangle\right|^{-s/2}$$

Shimizu L-function

$$\eta_{\mathcal{D}_{\mathbb{K}}}(s) = L\left(\Lambda, V, \frac{s}{2}\right) Z_\epsilon\left(\frac{s}{2}\right)$$

$L(\Lambda, V, s)$ Shimizu L-function and

$$Z_\epsilon(s/2) = \sum_{k\in\mathbb{Z}} (\epsilon^{2k} + \epsilon^{-2k})^{-s/2}$$

3-dim geometry adiabatic $\tilde{\not{D}}_{\theta,\theta'}$

$$\zeta_{\tilde{\not{D}}_{\theta,\theta'}}(s) = 2 Z_\epsilon(s/2) \sum_{\mu\in(\Lambda\smallsetminus\{0\})/V} |N(\mu)|^{-s/2}$$

Eta vanishes: symmetry
Restriction $\tilde{\not{D}}^+_{\theta,\theta'}$ to positive modes of B_θ

$$\eta_{\tilde{\not{D}}^+_{\theta,\theta'}}(s) = L(\Lambda, V, s/2) Z_\epsilon(s/2) = \eta_{\mathcal{D}_{\mathbb{K}}}(s)$$

Residue:

$$\mathrm{Res}_{s=0}\ \eta_{\tilde{D}^+_{\theta,\theta'}}(s) = \frac{L(\Lambda, V, 0)}{\log \epsilon}$$

$$Z_\epsilon(s) := \sum_{k\in\mathbb{Z}} (\epsilon^{2k} + \epsilon^{-2k})^{-s} \quad \text{with} \quad \Gamma(s) Z_\epsilon(s) = \int_0^\infty g_\epsilon(t)\, t^{s-1} dt$$

$$g_\epsilon(t) = \left(\sum_{k\in\mathbb{Z}} e^{-(\epsilon^{2k}+\epsilon^{-2k})t} \right)$$

$$g_\epsilon(t) = -e^{-2t} + 2h_\epsilon(t) - 2\sum_{k=0}^\infty e^{-\epsilon^{2k} t}(1 - e^{-\epsilon^{-2k} t})$$

$$h_\epsilon(t) = \sum_{k=0}^\infty e^{-\epsilon^{2k} t} \quad \text{and} \quad h_\epsilon(t) - h_\epsilon(\epsilon^2 t) = e^{-t} = \sum_{r=0}^\infty \frac{(-1)^r}{r!} t^r$$

$$h_\epsilon(t) = \frac{1}{2\log\epsilon} \log(1/t) + C - \sum_{r=0}^\infty \frac{(-1)^r}{r!(\epsilon^{2r}-1)} t^r$$

$\Gamma(s) Z_\epsilon(s)$ double pole $s=0$, simple poles $s \in \mathbb{Z}_{<0}$

$$\mathrm{Res}_{s=0} Z_\epsilon(s) = \frac{1}{\log \epsilon}$$

Conclusions

• The Shimuzu L-function $L(\Lambda, V, s)$ arises as the η-function of a spectral triple on $C^*(\Lambda, V, \tilde{\sigma}) = C^*(\Lambda, \sigma) \rtimes V$ obtained by deforming the commutative spectral triple of the solvmanifold X_ϵ turning the fiber T^2 into NC tori $\mathbb{T}_\theta$ with real multiplication

• The induced spectral triple on $\mathbb{T}_\theta$ is the Wick rotation of a Lorentzian geometry, where the modes of the wave operator are norms in the real quadratic field

• the 3-dimensional solvmanifold is the homotopy quotient (in the sense of Baum–Connes) of the NC space
$\mathbb{T}_\theta / \mathrm{Aut}(\mathbb{T}_\theta) = C^*(\Lambda, V, \tilde{\sigma})$