Introduction: What is Noncommutative Geometry?

Matilde Marcolli

Ma148b: Winter 2016
Noncommutative Geometry:

- *Geometry adapted to quantum world:* physical observables are operators in Hilbert space, these do not commute (e.g. canonical commutation relation of position and momentum: \([x, p] = i\hbar\))

- A method to describe “bad quotients” of equivalence relations as if they were nice spaces (cf. other such methods, e.g. stacks)

- Generally a method for extending smooth geometries to objects that are not smooth manifolds (fractals, quantum groups, bad quotients, deformations, …)
Simplest example of a noncommutative geometry: matrices $M_2(\mathbb{C})$

- Product is not commutative $AB \neq BA$
  \[
  \begin{pmatrix}
  a & b \\
  c & d
  \end{pmatrix}
  \begin{pmatrix}
  u & v \\
  x & y
  \end{pmatrix}
  =
  \begin{pmatrix}
  au + bx & av + by \\
  cu + dx & cv + dy
  \end{pmatrix}
  \neq
  \begin{pmatrix}
  au + cx & bu + dy \\
  ax + cy & bx + dy
  \end{pmatrix}
  \begin{pmatrix}
  a & b \\
  c & d
  \end{pmatrix}
  \]

- View product as a convolution product

  $X = \{x_1, x_2\}$ space with two points

  Equivalence relation $x_1 \sim x_2$ that identifies the two points: quotient (in classical sense) one point; graph of equivalence relation $R = \{(a, b) \in X \times X : a \sim b\} = X \times X$

  \[
  (AB)_{ij} = \sum_k A_{ik} B_{kj}
  \]

  $A_{ij} = f(x_i, x_j) : R \to \mathbb{C}$ functions on $X \times X$

  $(f_1 \ast f_2)(x_i, x_j) = \sum_{x_i \sim x_k \sim x_j} f_1(x_i, x_k) f_2(x_k, x_j)$
• The algebra $M_2(\mathbb{C})$ is the algebra of functions on $X \times X$ with convolution product

• Different description of the quotient $X/\sim$

• NCG space $M_2(\mathbb{C})$ is a point with internal degrees of freedom

• Intuition: useful to describe physical models with internal degrees of freedom

*Morita equivalence* (algebraic): rings $R$, $S$ that have equivalent categories $R - \text{Mod} \cong S - \text{Mod}$ of (left)-modules

$R$ and $M_N(R)$ are Morita equivalent
The algebra $M_2(\mathbb{C})$ represents a two point space with an identification between points. Unlike the classical quotient with algebra $\mathbb{C}$, the non-commutative space $M_2(\mathbb{C})$ “remembers” how the quotient is obtained.
Noncommutative Geometry of Quotients

Equivalence relation $\mathcal{R}$ on $X$: quotient $Y = X/\mathcal{R}$.

Even for very good $X \Rightarrow X/\mathcal{R}$ pathological!

Classical: functions on the quotient

$\mathcal{A}(Y) := \{ f \in \mathcal{A}(X) \mid f \text{ is } \mathcal{R} - \text{invariant} \}$

$\Rightarrow$ often too few functions

$\mathcal{A}(Y) = \mathbb{C}$ only constants

NCG: $\mathcal{A}(Y)$ noncommutative algebra

$\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$

functions on the graph $\Gamma_{\mathcal{R}} \subset X \times X$ of the equivalence relation

(compact support or rapid decay)

Convolution product

$$(f_1 \ast f_2)(x, y) = \sum_{x \sim u \sim y} f_1(x, u)f_2(u, y)$$

involution $f^*(x, y) = f(y, x)$. 
\( \mathcal{A}(\Gamma_R) \) noncommutative algebra \( \Rightarrow \ Y = X/R \)

noncommutative space

Recall: \( C_0(X) \Leftrightarrow X \) Gelfand–Naimark equiv of categories
abelian \( C^* \)-algebras, loc comp Hausdorff spaces

Result of NCG:

\( Y = X/R \) noncommutative space with

NC algebra of functions \( \mathcal{A}(Y) := \mathcal{A}(\Gamma_R) \) is

- as good as \( X \) to do geometry
  (deRham forms, cohomology, vector bundles, connections, curvatures, integration, points and subvarieties)

- but with new phenomena
  (time evolution, thermodynamics, quantum phenomena)
Tools needed for Physics Models

- Vector bundles and connections (gauge fields)
- Riemannian metrics (Euclidean gravity)
- Spinors (Fermions)
- Action Functional

*General idea:* reformulate usual geometry in algebraic terms (using the algebra of functions rather than the geometric space) and extend to case where algebra no longer commutative
Remark: Different forms of noncommutativity in physics

- Quantum mechanics: non-commuting operators

- Gauge theories: non-abelian gauge groups

- Gravity: hypothetical presence of noncommutativity in spacetime coordinates at high energy (some string compactifications with NC tori)

In the models we consider here the non-abelian nature of gauge groups is seen as an effect of an underlying non-commutativity of coordinates of “internal degrees of freedom” space (a kind of extra-dimensions model)
Vector bundles in the noncommutative world

- $M$ compact smooth manifold, $E$ vector bundle: space of smooth sections $C^\infty(M, E)$ is a module over $C^\infty(M)$

- The module $C^\infty(M, E)$ over $C^\infty(M)$ is finitely generated and projective (i.e. a vector bundle $E$ is a direct summand of some trivial bundle)

- Example: $TS^2 \oplus NS^2$ tangent and normal bundle give a trivial rk 3 bundle

- Serre–Swan theorem: any finitely generated projective module over $C^\infty(M)$ is $C^\infty(M, E)$ for some vector bundle $E$ over $M$
Tangent and normal bundle of $S^2$ add to trivial rank 3 bundle: more generally by Serre–Swan’s theorem all vector bundles are summands of some trivial bundle
Conclusion: algebraic description of vector bundles as finite projective modules over the algebra of functions


Vector bundles over a noncommutative space:

- Only have the algebra $\mathcal{A}$ noncommutative, not the geometric space (usually not enough two-sided ideals to even have points of space in usual sense)

- Define vector bundles purely in terms of the algebra: $\mathcal{E} =$ finitely generated projective (left)-module over $\mathcal{A}$
Connections on vector bundles

- $\mathcal{E}$ finitely generated projective module over (noncommutative) algebra $\mathcal{A}$

- Suppose have differential graded algebra $(\Omega^\bullet, d)$, $d^2 = 0$ and

  \[ d(\alpha_1 \alpha_2) = d(\alpha_1)\alpha_2 + (-1)^{\deg(\alpha_1)}\alpha_1 d(\alpha_2) \]

  with homomorphism $\mathcal{A} \to \Omega^0$ (hence bimodule)

- connection $\nabla : \mathcal{E} \to \mathcal{E} \otimes_\mathcal{A} \Omega^1$ Leibniz rule

  \[ \nabla(\eta a) = \nabla(\eta)a + \eta \otimes da \]

  for $a \in \mathcal{A}$ and $\eta \in \mathcal{E}$
Spin Geometry
(approach to Riemannian geometry in NCG)

Spin manifold

• Smooth $n$-dim manifold $M$ has tangent bundle $TM$

• Riemannian manifold (orientable): orthonormal frame bundle $FM$ on each fiber $E_x$ inner product space with oriented orthonormal basis

• $FM$ is a principal $SO(n)$-bundle

• Principal $G$-bundle: $\pi : P \to M$ with $G$-action $P \times G \to P$ preserving fibers $\pi^{-1}(x)$ on which free transitive (so each fiber $\pi^{-1}(x) \simeq G$ and base $M \simeq P/G$)
• Fundamental group $\pi_1(SO(n)) = \mathbb{Z}/2\mathbb{Z}$ so double cover universal cover:

$$Spin(n) \to SO(n)$$

• Manifold $M$ is spin if orthonormal frame bundle $FM$ lifts to a principal $Spin(n)$-bundle $PM$

• **Warning**: not all compact Riemannian manifolds are spin: there are topological obstruction

• In dimension $n = 4$ not all spin, but all at least $spin^C$

• $spin^C$ weaker form than spin: lift exists after tensoring $TM$ with a line bundle (or square root of a line bundle)

$$1 \to \mathbb{Z}/2\mathbb{Z} \to Spin^C(n) \to SO(n) \times U(1) \to 1$$
Spinor bundle

- Spin group $Spin(n)$ and Clifford algebra: vector space $V$ with quadratic form $q$

$$Cl(V, q) = T(V) / I(V, q)$$
tensor algebra mod ideal gen by $uv + vu = 2\langle u, v \rangle$ with $\langle u, v \rangle = (q(u + v) - q(u) - q(v))/2$

- Spin group is subgroup of group of units

$$Spin(V, q) \hookrightarrow \text{GL}_1(Cl(V, q))$$
elements $v_1 \cdots v_{2k}$ prod of even number of $v_i \in V$ with $q(v_i) = 1$

- $Cl^C(\mathbb{R}^n)$ complexification of Clifford alg of $\mathbb{R}^n$ with standard inn prod: unique min dim representation $dim \Delta_n = 2^{[n/2]} \Rightarrow$ rep of $Spin(n)$ on $\Delta_n$ not factor through $SO(n)$

15
• Associated vector bundle of a principal $G$-bundle: $V$ linear representation
  \[ \rho : G \to GL(V) \] get vector bundle
  \[ E = P \times_G V \] (diagonal action of $G$

• \textit{Spinor bundle} $\mathcal{S} = P \times \rho \Delta_n$
  on spin manifold $M$

• \textit{Spinors} sections $\psi \in C^\infty(M, \mathcal{S})$

• Module over $C^\infty(M)$ and also action by
  forms (Clifford multiplication) $c(\omega)$

• as vector space $Cl(V, q)$ same as $\Lambda^\bullet(V)$ not
  as algebra: under this vector space identifi-
  cation Clifford multiplication by a diff form
**Dirac operator**

- first order linear differential operator (elliptic on $M$ compact): “square root of Laplacian”

- $\gamma_a = c(e_a)$ Clifford action on basis of $(V, q)$

- even dimension $n = 2m$: $\gamma = (-i)^m \gamma_1 \cdots \gamma_n$ with $\gamma^* = \gamma$ and $\gamma^2 = 1$ sign

\[
\frac{1 + \gamma}{2} \quad \text{and} \quad \frac{1 - \gamma}{2}
\]

orthogonal projections: $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$

- Spin connection $\nabla^S : \mathbb{S} \to \mathbb{S} \otimes \Omega^1(M)$

\[
\nabla^S(c(\omega)\psi) = c(\nabla \omega)\psi + c(\omega)\nabla^S \psi
\]

for $\omega \in \Omega^1(M)$ and $\psi \in C^\infty(M, \mathbb{S})$ and $\nabla = \text{Levi-Civita connection}$
• Dirac operator \( \mathcal{D} = -ic \circ \nabla^S \)

\[
\mathcal{D} : \mathbb{S} \xrightarrow{\nabla^S} \mathbb{S} \otimes_{C^\infty(M)} \Omega^1(M) \xrightarrow{-ic} \mathbb{S}
\]

• \( \mathcal{D} \psi = -ic(dx^\mu) \nabla^S_{\partial_\mu} \psi = -i\gamma^\mu \nabla^S_\mu \psi \)

• Hilbert space \( \mathcal{H} = L^2(M, \mathbb{S}) \) square integrable spinors

\[
\langle \psi, \xi \rangle = \int_M \langle \psi(x), \xi(x) \rangle_x \sqrt{g} \, d^n x
\]

• \( C^\infty(M) \) acting as bounded operators on \( \mathcal{H} \)
(Note: \( M \) compact)

• Commutator: 

\[
[\mathcal{D}, f] \psi = -ic(\nabla^S(f \psi)) + ifc(\nabla^S \psi) \\
= -ic(\nabla^s(f \psi) - f \nabla^S \psi) = -ic(df \otimes \psi) = -ic(df) \psi
\]

\( [\mathcal{D}, f] = -ic(df) \) bounded operator on \( \mathcal{H} \)
(Note: \( M \) compact)
Analytic properties of Dirac on $\mathcal{H} = L^2(M, \mathbb{S})$ on a compact Riemannian $M$

- Unbounded operator

- Self adjoint: $\mathcal{D}^* = \mathcal{D}$ with dense domain

- Compact resolvent: $(1 + \mathcal{D}^2)^{-1/2}$ is a compact operator (if no kernel $\mathcal{D}^{-1}$ compact)

- Lichnerowicz formula: $\mathcal{D}^2 = \Delta^S + \frac{1}{4}R$ with $R$ scalar curvature and Laplacian

\[ \Delta^S = -g^{\mu\nu}(\nabla^S_\mu \nabla^S_\nu - \Gamma^\lambda_{\mu\nu} \nabla^S_\lambda) \]

Main Idea: abstract these properties into an algebraic definition of Dirac on NC spaces
How to get metric $g_{\mu\nu}$ from Dirac $\slashed{D}$

- Geodesic distance on $M$: length of curve $\ell(\gamma)$, piecewise smooth curves
  $$d(x, y) = \inf_{\gamma: [0,1] \to M} \{ \ell(\gamma) \}$$
  for $\gamma(0) = x, \gamma(1) = y$

- *Myers–Steenrod theorem*: metric $g_{\mu\nu}$ uniquely determined from geodesic distance

- Show that geodesic distance can be computed using Dirac operator and algebra of functions

- $f \in \mathcal{C}(M)$ have
  $$|f(x) - f(y)| \leq \int_0^1 |\nabla f(\gamma(t))| |\dot{\gamma}(t)| \, dt$$
  $$\leq \|\nabla f\|_\infty \int_0^1 |\dot{\gamma}(t)| \, dt = \|\nabla f\|_\infty \ell(\gamma) = \|[\slashed{D}, f]\| \ell(\gamma)$$
• $|f(x) - f(y)| \leq \|[[\mathcal{D}, f]]\|\ell(\gamma)$ gives

$$\sup_{f : \|[\mathcal{D}, f]\| \leq 1} \left\{ |f(x) - f(y)| \right\} \leq \inf_{\gamma} \ell(\gamma) = d(x, y)$$

• Note: sup over $f \in C^\infty(M)$ or over $f \in Lip(M)$ Lipschitz functions

$$|f(x) - f(y)| \leq Cd(x, y)$$

• Take $f(x)(y) = d(x, y)$ Lipschitz with

$$|f(x)(y) - f(x)(z)| \leq d(y, z)$$

(triangle inequality)

• $[[\mathcal{D}, f]] = -ic(df_x)$ and $|\nabla f_x| = 1$, then $|f_x(y) - f_x(x)| = f_x(y) = d(x, y)$ realizes sup

• Conclusion: distance from Dirac

$$d(x, y) = \sup_{f : \|[\mathcal{D}, f]\| \leq 1} \left\{ |f(x) - f(y)| \right\}$$
Some references for Spin Geometry:


*Spin Geometry and NCG, Dirac and distance:*


**Spectral triples**: abstracting Spin Geometry

- involutive algebra $\mathcal{A}$ with representation $\pi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$
- self adjoint operator $D$ on $\mathcal{H}$, dense domain
- compact resolvent $(1 + D^2)^{-1/2} \in \mathcal{K}$
- $[a, D]$ bounded $\forall a \in \mathcal{A}$
- even if $\mathbb{Z}/2$- grading $\gamma$ on $\mathcal{H}$
  
  $[\gamma, a] = 0, \forall a \in \mathcal{A}, \quad D\gamma = -\gamma D$

**Main example**: $(\mathcal{C}^\infty(M), L^2(M, \mathcal{S}), \mathcal{D})$ with chirality $\gamma = (-i)^m \gamma_1 \cdots \gamma_n$ in even-dim $n = 2m$

Real Structures in Spin Geometry

- Clifford algebra $Cl(V, q)$ non-degenerate quadratic form of signature $(p, q)$, $p + q = n$

- $Cl^+_n = Cl(\mathbb{R}^n, g_{n,0})$ and $Cl^-_n = Cl(\mathbb{R}^n, g_{0,n})$

- Periodicity: $Cl^\pm_{n+8} = Cl^\pm_n \otimes M_{16}(\mathbb{R})$

- Complexification: $Cl^\pm_n \subset \mathbb{C}l_n = Cl^\pm_n \otimes_\mathbb{R} \mathbb{C}$

<table>
<thead>
<tr>
<th>n</th>
<th>$Cl^+_n$</th>
<th>$Cl^-_n$</th>
<th>$\mathbb{C}l_n$</th>
<th>$\Delta_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\mathbb{R} \oplus \mathbb{R}$</td>
<td>$\mathbb{C}$</td>
<td>$\mathbb{C} \oplus \mathbb{C}$</td>
<td>$\mathbb{C}$</td>
</tr>
<tr>
<td>2</td>
<td>$M_2(\mathbb{R})$</td>
<td>$\mathbb{H}$</td>
<td>$M_2(\mathbb{C})$</td>
<td>$\mathbb{C}^2$</td>
</tr>
<tr>
<td>3</td>
<td>$M_2(\mathbb{C})$</td>
<td>$\mathbb{H} \oplus \mathbb{H}$</td>
<td>$M_2(\mathbb{C}) \oplus M_2(\mathbb{C})$</td>
<td>$\mathbb{C}^2$</td>
</tr>
<tr>
<td>4</td>
<td>$M_2(\mathbb{H})$</td>
<td>$M_2(\mathbb{H})$</td>
<td>$M_4(\mathbb{C})$</td>
<td>$\mathbb{C}^4$</td>
</tr>
<tr>
<td>5</td>
<td>$M_2(\mathbb{H}) \oplus M_2(\mathbb{H})$</td>
<td>$M_4(\mathbb{C})$</td>
<td>$M_4(\mathbb{C}) \oplus M_4(\mathbb{C})$</td>
<td>$\mathbb{C}^4$</td>
</tr>
<tr>
<td>6</td>
<td>$M_4(\mathbb{H})$</td>
<td>$M_8(\mathbb{R})$</td>
<td>$M_8(\mathbb{C})$</td>
<td>$\mathbb{C}^8$</td>
</tr>
<tr>
<td>7</td>
<td>$M_8(\mathbb{C})$</td>
<td>$M_8(\mathbb{R}) \oplus M_8(\mathbb{R})$</td>
<td>$M_8(\mathbb{C}) \oplus M_8(\mathbb{C})$</td>
<td>$\mathbb{C}^8$</td>
</tr>
<tr>
<td>8</td>
<td>$M_{16}(\mathbb{R})$</td>
<td>$M_{16}(\mathbb{R})$</td>
<td>$M_{16}(\mathbb{C})$</td>
<td>$\mathbb{C}^{16}$</td>
</tr>
</tbody>
</table>
• Both real Clifford algebra and complexification act on spinor representation $\Delta_n$.

• $\exists$ antilinear $J : \Delta_n \to \Delta_n$ with $J^2 = 1$ and $[J, a] = 0$ for all $a$ in real algebra $\Rightarrow$ real subbundle $Jv = v$

• antilinear $J$ with $J^2 = -1$ and $[J, a] = 0$ $\Rightarrow$ quaternion structure

• real algebra: elements $a$ of complex algebra with $[J, a] = 0$, $JaJ^* = a$. 
Real Structures on Spectral Triples

$KO$-dimension $n \in \mathbb{Z}/8\mathbb{Z}$

antilinear isometry $J : \mathcal{H} \to \mathcal{H}$

$$J^2 = \varepsilon, \quad JD = \varepsilon' DJ, \quad \text{and} \quad J\gamma = \varepsilon''\gamma J$$

<table>
<thead>
<tr>
<th>$n$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\varepsilon$</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\varepsilon'$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\varepsilon''$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td></td>
</tr>
</tbody>
</table>

Commutation: $[a, b^0] = 0 \quad \forall a, b \in \mathcal{A}$

where $b^0 = Jb^* J^{-1} \quad \forall b \in \mathcal{A}$

Order one condition:

$$[[D, a], b^0] = 0 \quad \forall a, b \in \mathcal{A}$$
Finite Spectral Triples $F = (\mathcal{A}_F, \mathcal{H}_F, D_F)$

- A finite dimensional (real) $C^*$-algebra
  $$\mathcal{A} = \bigoplus_{i=1}^{N} M_{n_i}(K_i)$$
  $K_i = \mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$ quaternions (Wedderburn)

- Representation on finite dimensional Hilbert space $\mathcal{H}$, with bimodule structure given by $J$ (condition $[a, b^0] = 0$)

- $D_F^* = D_F$ with order one condition
  $$[[D_F, a], b^0] = 0$$

- No analytic conditions: $D_F$ just a matrix

$\Rightarrow$ Moduli spaces (under unitary equivalence)

Branimir Čačić, Moduli spaces of Dirac operators for finite spectral triples, arXiv:0902.2068