Geometriae Dedicata **91:** 45–58, 2002.

Seiberg–Witten and Casson–Walker Invariants for Rational Homology 3-Spheres

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(Received: 11 January 2001)

Abstract. We consider a modified version of the Seiberg–Witten invariants for rational homology 3-spheres, obtained by adding to the original invariants a correction term which is a combination of η-invariants. We show that these modified invariants are topological invariants. We prove that an averaged version of these modified invariants equals the Casson–Walker invariant. In particular, this result proves an averaged version of a conjecture of Ozsváth and Szabó on the equivalence between their $\hat{\theta}$ invariant and the Seiberg–Witten invariant of rational homology 3-spheres.

Mathematics Subject Classifications (2000). 57R57, 57M27.

Key words. Seiberg–Witten, Casson–Walker, η-invariants.

1. Introduction

Let Y be a closed oriented 3-manifold with a Spin^c structure $\mathfrak{s}$, endowed with a Riemannian metric g. One can consider on $(Y, \mathfrak{s}, g)$ the Seiberg–Witten monopole equations, for a pair (A, ψ) consisting of a $U(1)$ connection on the determinant bundle of the Spin^c structure $\mathfrak{s}$ and a spinor ψ (cf. [8] and [4, 6, 9, 14]):

$$\partial\!\!\!/_A\psi + v.\psi = 0, \qquad *F_A = \sigma(\psi, \psi) + \eta \tag{1}$$

where η is a co-closed 1-form in $\Omega^1(Y, i\mathbb{R})$ and v is a 1-form in $\Omega^1(Y, i\mathbb{R})$ introduced to achieve smoothness of the moduli space $\mathcal{M}_Y(\mathfrak{s}, \eta, v)$ of solutions to (1) modulo gauge transformations.

For a 3-manifold Y with $b_1(Y) > 0$, for generic η, v, $\mathcal{M}_Y(\mathfrak{s}, \eta, v)$ consists of only finitely many irreducible points. The counting of points in $\mathcal{M}_Y(\mathfrak{s}, \eta, v)$ with sign given by the orientation defines the Seiberg–Witten invariant for $(Y, \mathfrak{s})$. As shown by Meng–Taubes [15] and Turaev [22], this invariant agrees with the Turaev torsion [21].

For a rational homology 3-sphere Y, assume that the generic $\eta = *dv_0$ satisfies $\mathrm{Ker}(\partial\!\!\!/_{v_0+v}) = 0$, then $\mathcal{M}_Y(\mathfrak{s}, \eta, v)$ consists of only finitely many irreducible points (where the spinor part does not vanish) and a unique, isolated, reducible point $[v_0, 0]$. The condition $\mathrm{Ker}(\partial\!\!\!/^g_{v_0+v}) \neq 0$ determines a subset of real codimension one in the space of metrics and perturbations.

In this paper, we show that a suitably modified version of the Seiberg–Witten invariant of a rational homology 3-sphere agrees with the Casson–Walker invariant. For any rational homology 3-sphere $(Y, \mathfrak{s}, g)$ with a Spin^c structure $\mathfrak{s}$ and a Riemannian metric, the counting of the irreducible Seiberg–Witten monopoles defines the Seiberg–Witten invariant

$$SW_Y(\mathfrak{s}, g) = \#\big(\mathcal{M}^*_Y(\mathfrak{s}, g)\big), \tag{2}$$

where each irreducible monopole in $\mathcal{M}^*_Y(\mathfrak{s}, g)$ has a natural orientation from the linearization of the Seiberg–Witten equations. As studied in [14], $SW_Y(\mathfrak{s}, g)$ depends on the metric and perturbation used in the definition. In order to obtain a topological invariant, we can modify $SW_Y(\mathfrak{s}, g)$ by a metric and perturbation dependent correction term as follows. Choose any four-manifold X with boundary Y, such that X is endowed with a cylindrical-end metric modeled on (Y, g_Y). Choose a Spin^c structure $\mathfrak{s}_X$ on X which agrees over the end with $\mathfrak{s}$ on Y, and choose a connection A on $(X, \mathfrak{s}_X)$ which extends the unique reducible $\theta_{\mathfrak{s}}$ on $(Y, \mathfrak{s})$. Then we set

$$\xi_Y(\mathfrak{s}, g) = \mathrm{Ind}_{\mathbb{C}}(\not{D}^X_A) - \tfrac{1}{8}\big(c_1(\mathfrak{s}_X)^2 - \sigma(X)\big), \tag{3}$$

where $\mathrm{Ind}_{\mathbb{C}}(\not{D}^X_A)$ is the complex index of the Dirac operator on $(X, \mathfrak{s}_X)$, twisted with the extending Spin^c connection A, and $\sigma(X)$ is the signature of X. By the Atiyah–Patodi–Singer index theorem, $\xi_Y(\mathfrak{s}, g)$ is independent of the choice of $(X, \mathfrak{s}_X)$ and A. Actually, $\xi_Y(\mathfrak{s}, g)$ can be expressed as a combination of the Atiyah–Patodi–Singer eta invariants for the Dirac operator and signature operator on $(Y, \mathfrak{s})$:

$$\xi_Y(\mathfrak{s}, g) = -\tfrac{1}{4}\eta_Y^{\not{\partial}_{\theta_{\mathfrak{s}}}}(0) - \tfrac{1}{8}\eta_Y^{\mathrm{sign}}(0).$$

The modified version of the Seiberg–Witten invariant is defined as

$$\hat{SW}_Y(\mathfrak{s}) = SW_Y(\mathfrak{s}, g) - \xi_Y(\mathfrak{s}, g). \tag{4}$$

We prove the following equivalence between $\hat{SW}_Y$ and the Casson–Walker invariant.

THEOREM 1.1. *Let Y be a rational homology 3-sphere. Then,*

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(Y)} \hat{SW}_Y(\mathfrak{s}) = \tfrac{1}{2}|H_1(Y, \mathbb{Z})|\lambda(Y),$$

where $\lambda(Y)$ is the Casson–Walker invariant of Y (cf. [24]).

The proof of this result relies on surgery formulae for the counting of monopoles (2), which we prove in Proposition 3.1, and for the correction term (3), which we prove in Proposition 3.2.2. The result then follows from the surgery formula of Theorem 3.3, which also proves an averaged version of a conjecture of Ozsváth and Szabó [19] on the equivalence of the SW invariant and their $\hat{\theta}$ invariant for rational homology 3-spheres.

In the case that Y is an integer homology 3-sphere, the equivalence of Seiberg–Witten and Casson invariants was established by Lim [11], and was also proved in [3].

2. Monopoles on Knot Complement and Gluing Theorem

In this section, we briefly review results in [5], concerning the Seiberg–Witten monopoles on a knot complement in any rational homology 3-sphere and the corresponding gluing theorem to obtain the Seiberg–Witten monopoles on any 3-manifold from Dehn surgery along the knot. Some of these results were also obtained by Lim in [12, 13].

Let Y be a rational homology sphere, K is a smoothly embedded knot in Y, such that the map $H_1(\partial(Y - \nu(K)), \mathbb{Z}) \to H_1(Y - \nu(K), \mathbb{Z})$ has a one-dimensional kernel whose generator has divisibility n in $H_1(T^2, \mathbb{Z})$ ($T^2 = \partial(Y - \nu(K))$), then

$$\frac{|H_1(Y, \mathbb{Z})|}{|\mathrm{Torsion}(H_1(Y - \nu(K), \mathbb{Z}))|} = n.$$

Denote by V the knot complement $Y - K$ with a cylindrical end metric modelled on $T^2 \times [0, \infty)$. Let $\mathfrak{s}$ be a Spin^c structure on V with trivial determinant over the end. We use the notation $\chi_0(T^2, V)$ for the moduli space of flat connections on $\det(\mathfrak{s})$ over T^2 modulo the gauge transformations which can be extended to V. Notice that $\chi_0(T^2, V)$ is a $\mathbb{Z} \times \mathbb{Z}_n$-covering of the moduli space $\chi(T^2)$ of flat connections modulo the full gauge group $\mathrm{Map}(T^2, U(1))$. In $\chi(T^2)$ there is a unique point θ such that the Dirac operator on T^2 coupled with θ has non-trivial kernel. We have the following structure theorem for the moduli space $\mathcal{M}_V(\mathfrak{s})$ of finite energy monopoles.

THEOREM 2.1 (cf. Theorem 1.2 in [5] and Theorem 1.3 in [13]). *For generic metrics and perturbations, the moduli space of Seiberg–Witten monopoles on V, denoted by $\mathcal{M}_V(\mathfrak{s})$ consists of the union of a circle of reducibles $\chi(V)$ and an irreducible piece $\mathcal{M}^*_V(\mathfrak{s})$ which is a smooth oriented one-dimensional manifold, compact except for finitely many ends limiting to $\chi(V)$. Moreover, there is a continuous boundary value map*

$$\mathcal{M}_V(\mathfrak{s}) \xrightarrow{\partial_\infty} \chi_0(T^2, V) \xrightarrow{\pi} \chi(T^2), \tag{5}$$

defined by taking the asymptotic limit of the Seiberg–Witten monopoles on V over the end. Under ∂_∞, $\chi(V)$ is mapped to a circle in $\chi_0(T^2, V)$, and the compactification $\bar{\mathcal{M}}^*_V(\mathfrak{s})$ is mapped to a collection of compact immersed curves in $\chi_0(T^2, V)$ whose boundary points consist of a finite set of points in $\pi^{-1}(\theta) \cup \partial_\infty(\chi(V))$. For generic perturbations the interior of the curve $\partial_\infty(\mathcal{M}^*_V(\mathfrak{s}))$ is transverse to any given finite set of curves in $\chi_0(T^2, V)$.

We also establish a gluing theorem for the moduli spaces of monopoles on 3-manifold glued along a torus where one piece is a solid torus $\nu(K)$. That is, if T^2 is a splitting torus in a closed 3-manifold $Z = V \cup_{T^2} \nu(K)$, we may cut Z along T^2 and glue in a long cylinder $[-r, r] \times T^2$, resulting in a new manifold denoted by $Z(r)$. Let $\mathfrak{s}$ be a Spin^c structure on V with trivial determinant along the boundary T^2. Let Spin$^c(Z, \mathfrak{s})$ be the set of Spin^c structures on Z whose restriction to V is $\mathfrak{s}$. Then in [5], we established the following gluing theorem.

THEOREM 2.2 (cf. Theorem 1.3 in [5] and §1.4 [12]). *For a sufficiently large r, under suitable perturbations and metrics, there exist the following diffeomorphism given by the gluing maps on the fibered products*

$$\mathcal{M}^*_{V,Z}(\mathfrak{s}) \times_{\chi_0(T^2,Z)} \chi(\nu(K) \subset Z) \longrightarrow \bigcup_{\mathfrak{s}\in \mathrm{Spin}^c(Z,\mathfrak{s})} \mathcal{M}^*_Z(\mathfrak{s}).$$

Here $\chi_0(T^2, Z)$ for the character variety (or moduli space) of flat connections on a trivial line bundle over T^2 modulo the gauge transformations on T^2 which can be extended to Z. For any tubular neighbourhood $\nu(K)$ in Z, we denote by $\chi(\nu(K) \subset Z)$ the moduli space of flat connections on $\nu(K)$ modulo the gauge transformations on $\nu(K)$ which can be extended to Z. There is a natural map $\chi(\nu(K) \subset Z) \to \chi_0(T^2, Z)$. $\mathcal{M}^*_{V,Z}(\mathfrak{s})$ is the irreducible Seiberg–Witten monopoles on $(V, \mathfrak{s})$ modulo the gauge transformations on V which can be extended to Z.

Let Y be a rational homology 3-sphere with a smoothly embedded knot K as before. Endow K with the framing (m, l) in a fixed identification: $\nu(K) \cong D^2 \times S^1$ such that l represents a generator in the kernel of the map

$$H_1(\partial(Y - \nu(K)), \mathbb{Z}) \to H_1(Y - \nu(K), \mathbb{Z}).$$

Let p and q be relatively prime integers. The Dehn surgery with coefficient $p/q \in \mathbb{Q} \cup \{\infty\}$ on K gives rise to another closed manifold $Y_{p/q}$ as follows.

Under the identification of the framing, let m be the right-handed meridian (intersecting l once), the orientation determined by $m \wedge l$ coincides with the orientation induced from Y. Similarly, let m' and l' be the meridian and longitude in the tubular neighbourhood of the knot $\nu(K)$. The meridian m' bounds a disk D^2 in $\nu(K)$, and l' generates $H_1(\nu(K), \mathbb{Z})$ and parallels to K. The Dehn surgery with coefficient $p/q \in \mathbb{Q} \cup \{\infty\}$ on K is the operation of removing $\nu(K)$ and gluing in $D^2 \times S^1$ by an orientation reversing diffeomorphism $f_{p/q}$ of T^2 that satisfies $f_{p/q}(m') = pm - ql$. Note that $Y_0 = Y_{0/1}$ is a rational homology $S^1 \times S^2$ and the other $Y_{p/q}$ is a rational homology 3-sphere.

Denote by $\mathrm{Spin}^c(V)$ the set of equivalence classes of Spin^c structures on $V = Y \backslash \nu(K)$ with trivial restriction to the boundary T^2. Then, for any $Y_{p/q}$, there is a surjective map $\imath_{Y_{p/q}}$: $\mathrm{Spin}^c(Y_{p/q}) \to \mathrm{Spin}^c(V)$, where, for any $\mathfrak{s} \in \mathrm{Spin}^c(Y_{p/q})$, $\imath_{Y_{p/q}}(\mathfrak{s})$ is given by the restriction to $V \subset Y_{p/q}$. The fiber of $\imath_{Y_{p/q}}$ is given by a cyclic group generated by the Poincaré dual of the core of $Y_{p/q} \backslash V$. Formally, for $\mathfrak{s} \in \mathrm{Spin}^c(V)$, we identify the fiber of $\imath_Y$ with the following set of Spin^c structures

$$\mathrm{Spin}^c(Y, \mathfrak{s}) = \bigcup_{m=0,\ldots,n-1} \{\mathfrak{s} \otimes L_m | c_1(L_m) = mPD([K]) \in H^2(Y, \mathbb{Z})\}.$$

Similarly the fiber of $\imath_{Y_{p/q}}$ is given by

$$\mathrm{Spin}^c(Y_{p/q}, \mathfrak{s}) = \bigcup_{m=0,\ldots,np-1} \{\mathfrak{s} \otimes L_m | c_1(L_m) = mPD([K]) \in H^2(Y_{p/q}, \mathbb{Z})\},$$

and the fiber of $\imath_{Y_0}$ is given by

$$\mathrm{Spin}^c(Y_0, \mathfrak{s}) = \bigcup_{m\in\mathbb{Z}} \{\mathfrak{s} \otimes L_m | c_1(L_m) = mPD([K]) \in H^2(Y_0, \mathbb{Z})\}.$$

Here we use the same notation $\mathfrak{s}$ on $Y_{p/q}$ (Y, or Y_0) as the corresponding Spin^c structure obtained by gluing $\mathfrak{s} \in \mathrm{Spin}^c(V)$ with the trivial Spin^c structure on $\nu(K)$ by the trivial gauge transformation on T^2. We hope this notation will not cause any confusion.

Assume that V and $\nu(K)$ are equipped with a metric with a cylindrical end modeled on T^2. Let $\mathfrak{s}$ be a Spin^c structure on Y. By Theorem 2.1, we know that the irreducible part of the moduli space of finite energy monopoles on $(V, \imath_Y(\mathfrak{s}))$, still denoted $\mathcal{M}^*_V(\mathfrak{s})$, is a smooth, oriented one-dimensional manifold. The asymptotic values along the cylindrical end and the covering map (5) define a boundary value map:

$$\partial_\infty : \mathcal{M}^*_V(\mathfrak{s}) \to \chi(T^2). \tag{6}$$

Notice that the reducible part $\chi(V)$ of the moduli space on $(V, \imath_Y(\mathfrak{s}))$ is an embedded circle $\chi(V) \subset \chi_0(T^2, V)$ under the asymptotic value map. This becomes a circle of multiplicity n in $\chi(T^2)$. There is a 'bad point' in $\chi(T^2)$, given by the flat connections such that the corresponding twisted Dirac operator has a non-trivial kernel. We can endow $\chi(T^2)$ with a coordinate system (u, v) defined by the holonomy around the longitude l and the meridian m, respectively, so that the bad point corresponds to $(u, v) = (1, 1)$. Then the reducible circle $\chi(V)$, with the holonomy around the longitude l of order n, is given by $u = u(\mathfrak{s})$, with $u(\mathfrak{s}) \in \{0, 2/n, \dots, 2(n-1)/n\}$. After a suitable perturbation, and a corresponding shift of coordinates, as discussed in [5] (cf. also [13]), we can assume that the bad point does not lie on any of these n possible circles $u = u(\mathfrak{s})$ of reducibles $\chi(V)$.

From Theorem 2.1, we know that, under the map ∂_∞ in (6), the boundary points $\partial(\mathcal{M}^*_V(\mathfrak{s}))$ are either mapped to the bad point in $\chi(T^2)$ or mapped to the reducible circle $u = u(\mathfrak{s})$ on $\chi(T^2)$.

Let $\chi(\nu(K) \subset Y_{p/q})$ be the reducible circle on $\nu(K) \subset Y_{p/q}$, which maps to a closed curve on $\chi(T^2)$ with slope p/q in the (u, v)-coordinates, parallel to $pv = qu$. Looking at the induced Spin structure on $T^2 \subset Y_{p/q}$, we know that the curve $\chi(\nu(K) \subset Y_{p/q})$ goes through $(0, 1)$ if q is odd or goes through $(0, 0)$ if q is even, cf. [5]. Again, after a suitable perturbation as in [5], and the corresponding shift of coordinates, we can assume that this p/q-curve is away from the bad point on $\chi(T^2)$ and does not meet $u = u(\mathfrak{s})$ along the coordinate line $v = 0$. Then we know that $u = u(\mathfrak{s})$ intersects $\chi(\nu(K) \subset Y_{p/q})$ inside $\chi(T^2)$ at p points, which are denoted by $\theta_1, \dots, \theta_p$, ordered according the orientation of $u = u(\mathfrak{s}) \subset \chi(T^2)$. They can be lifted to pn points in $\chi_0(T^2, V)$. We denote these points by $\theta_1^{(k)}, \dots, \theta_p^{(k)}$, $(k = 0, 1, \dots, n-1)$ according to the order.

Denote by θ_0 the intersection point of $u = u(\mathfrak{s})$ with $v = 0$ in $\chi(T^2)$. This can be lifted to n-points $\theta_0^{(0)}, \theta_0^{(1)}, \dots, \theta_0^{(n-1)}$ on $\chi_0(T^2, V)$. Moreover, we can assume that the map ∂_∞ in (6) is transverse to the curves $u = u(\mathfrak{s})$, $v = 0$ and $\chi(\nu(K) \subset Y_{p/q})$, by a suitable perturbation of the Seiberg–Witten equations on V as in [5]. We can also assume that the image $\partial_\infty \mathcal{M}^*_V(\mathfrak{s})$ does not meet the points $\theta_0, \theta_1, \dots \theta_p$ in $\chi(T^2)$, again by suitable perturbation, as discussed in [5].

Then the Seiberg–Witten monopoles on various manifolds $Y_{p/q}$ can described as follows from the gluing Theorem 2.2.

COROLLARY 2.3. *Let $\mathfrak{s}$ be a* Spin^c *structure on V with a trivial determinant over the end. Then we have*

$$\bigcup_{k=0}^{pn-1} \mathcal{M}^*_{Y_{p/q}}(\mathfrak{s}\otimes L_k) = \mathcal{M}^*_V(\mathfrak{s}) \times_{\chi(T^2)} \chi(v(K)\subset Y_{p/q}),$$

where $\chi(v(K)\subset Y_{p/q})$ is the p/q-curve in the moduli space $\chi(T^2)$. The set $\{\theta_1^{(k)},\ldots,\theta_p^{(k)} : k=0,1,\ldots,n-1\}$ consists of the unique reducible monopole for each $(Y_{p/q},\mathfrak{s}\otimes L_k)$. Similarly, we have

$$\bigcup_{k=0}^{n-1} \mathcal{M}^*_Y(\mathfrak{s}\otimes L_k) = \mathcal{M}^*_V(\mathfrak{s}) \times_{\chi(T^2)} \{v=0\},$$

and the reducible set consists of $\{\theta_0^{(0)},\theta_0^{(1)},\ldots,\theta_0^{(n-1)}\}$. After a mild perturbation of the monopole equations on Y_0, we have

$$\bigcup_{k\in\mathbb{Z}} \mathcal{M}_{Y_0}(\mathfrak{s}\otimes L_k) = \mathcal{M}^*_V(\mathfrak{s}) \times_{\chi(T^2)} \{u=u(\mathfrak{s})+\eta\}.$$

Here η is a sufficiently small positive number, introduced by effect of a small perturbation, such that there is no reducible monopole for Y_0.

3. Seiberg–Witten = Casson–Walker Invariant

In this section, we derive the relation between the topologically invariant version of the Seiberg–Witten invariant and the Casson–Walker invariant for rational homology 3-spheres. Together with the equivalence between the Casson–Walker invariant and the theta invariant introduced by Ozsváth and Szabó in [19], our result proves an averaged version of their conjecture relating the Seiberg–Witten invariant and their theta invariant.

Let Y be a rational homology 3-sphere with a smoothly embedded knot K as in the previous section. For any pair of relatively prime integers (p, q), the Dehn surgery with coefficient $p/q \in \mathbb{Q}\cup\{\infty\}$ on K gives rise to another closed manifold $Y_{p/q}$.

Assume that $V = Y - K$ and the tubular neighborhood $v(K)$ of K are equipped with a metric with a cylindrical end modeled on T^2. Let $\mathfrak{s}$ be a Spin^c structure on V. As discussed in the previous section, the moduli space $\mathcal{M}^*_V(\mathfrak{s})$ of irreducible finite energy monopoles on $(V,\mathfrak{s})$ is a smooth, oriented one-dimensional manifold, while the reducible part $\chi(V)$ of the moduli space on $(V, \iota_Y(\mathfrak{s}))$ is an embedded circle $\chi(V)\subset\chi_0(T^2,V)$. With the coordinate system introduced in the previous section, the reducible circle $\chi(V)$ is given by $u=u(\mathfrak{s})$, with $u(\mathfrak{s})\in\{0, 2/n,\ldots,2(n-1)/n\}$. We can assume that the bad point does not lie on any of these n possible circles $u=u(\mathfrak{s})$ of reducibles $\chi(V)$.

We use the notations from the previous section. The reducible circle $\chi(v(K)\subset Y_{p/q})$ on $v(K)\subset Y_{p/q}$ is mapped to a closed curve on $\chi(T^2)$ with slope p/q in the

(u, v)-coordinates. $u = u(\mathfrak{s})$ intersects $\chi(v(K) \subset Y_{p/q})$ inside $\chi(T^2)$ at p points, which are denoted by $\theta_1, \ldots, \theta_p$, ordered according the orientation of $u = u(\mathfrak{s}) \subset \chi(T^2)$. We denote the lifting points in $\chi_0(T^2, V)$ by $\theta_1^{(k)}, \ldots, \theta_p^{(k)}$, $(k = 0, 1, \ldots, n-1)$ according to the order. The point θ_0 is the intersection of $u = u(\mathfrak{s})$ with $v = 0$ in $\chi(T^2)$ which can be lifted to n-points $\theta_0^{(0)}, \theta_0^{(1)}, \ldots, \theta_0^{(n-1)}$ on $\chi_0(T^2, V)$. Moreover, we can assume that the map ∂_∞ in (6) is transverse to the curves $u = u(\mathfrak{s})$, $v = 0$ and $\chi(v(K) \subset Y_{p/q})$, and the image $\partial_\infty \mathcal{M}_V^*(\mathfrak{s})$ does not meet the points $\theta_0, \theta_1, \ldots, \theta_p$ in $\chi(T^2)$.

Let I be any open interval in

$$\chi(V) = \{u = u(\mathfrak{s})\} \subset \chi_0(T^2, V).$$

We denote by $SF_{\mathbb{C}}(\partial\!\!\!/_I^V)$ the complex spectral flow of Dirac operator on V, twisted with the path of reducible connections I on V. From the analysis in [5] and [14], we know that

$$\#\left(\partial_\infty|_{\partial \mathcal{M}_V^*(\mathfrak{s})}\right)^{-1}(I) = SF_{\mathbb{C}}(\partial\!\!\!/_I^V). \tag{7}$$

For convenience, we define

$$SF_{\mathbb{C}}(\partial\!\!\!/_{[\theta_i, \theta_j]}^V) = \sum_{k=0}^{n-1} SF_{\mathbb{C}}(\partial\!\!\!/_{[\theta_i^{(k)}, \theta_j^{(k)}]}^V). \tag{8}$$

With this notation understood, we can state the following proposition relating the Seiberg–Witten invariants on $Y_{p/q}$, Y and Y_0.

PROPOSITION 3.1. *Consider generic compatible small perturbations of the Seiberg–Witten equations on $Y_{p/q}$, Y and Y_0, such that the map ∂_∞ as in (6) is transverse to the curves $u = u(\mathfrak{s})$, $v = 0$ and $\chi(v(K) \subset Y_{p/q})$ and misses the points $\theta_0, \theta_1, \ldots, \theta_p$ in $\chi(T^2)$. Then we have the following relation:*

$$\begin{aligned}
&\sum_{k=0}^{pn-1} SW_{Y_{p/q}}(\mathfrak{s} \otimes L_k, g_{Y_{p/q}}) \\
&\quad = p \sum_{k=0}^{n-1} SW_Y(\mathfrak{s} \otimes L_k, g_Y) + q \sum_{k \in \mathbb{Z}} SW_{Y_0}(\mathfrak{s} \otimes L_k) \\
&\qquad + \sum_{i=1}^{p} SF_{\mathbb{C}}(\partial\!\!\!/_{[\theta_0, \theta_i]}^V).
\end{aligned}$$

Proof. By the gluing theorem for three-dimensional monopoles as in Theorem 2.2 and Corollary 2.3, we have

$$\sum_{k=0}^{pn-1} SW_{Y_{p/q}}(\mathfrak{s} \otimes L_k, g_{Y_{p/q}}) = \#\left(\mathcal{M}_V^*(\mathfrak{s}) \times_{\chi(T^2)} \chi(v(K) \subset Y_{p/q})\right). \tag{9}$$

Notice that the set $\{\theta_1^{(k)}, \ldots, \theta_p^{(k)} : k = 0, 1, \ldots, n-1\}$ consists of the unique reducible monopole for each $(Y_{p/q}, \mathfrak{s} \otimes L_k)$.

Similarly, we have

$$\sum_{k=0}^{n-1} SW_Y(\mathfrak{s}\otimes L_k, g_Y) = \#\big(\mathcal{M}_V^*(\mathfrak{s}) \times_{\chi(T^2)} \{v=0\}\big). \tag{10}$$

Here the reducible set consists of $\{\theta_0^{(0)}, \theta_0^{(1)}, \dots, \theta_0^{(n-1)}\}$.

In order to avoid the circle of reducibles on $(Y_0, \mathfrak{s}\otimes L_0)$, we need to introduce a small perturbation such that $\chi(\nu(K)\subset Y_0)$ on $\chi(T^2)$ is a small parallel shifting of $u = u(\mathfrak{s})$ such that the bad point is not contained in the narrow strip bounded by these two parallel curves. We denote this small shift of $u = u(\mathfrak{s})$ by $u = u(\mathfrak{s}) + \eta$, where η is a sufficiently small positive number. This can be achieved by a perturbation of the equations as in [5]. Then we have

$$\sum_{k\in\mathbb{Z}} SW_{Y_0}(\mathfrak{s}\otimes L_k) = \#\big(\mathcal{M}_V^*(\mathfrak{s}) \times_{\chi(T^2)} \{u = u(\mathfrak{s})+\eta\}\big). \tag{11}$$

In order to compare the three countings in (9) – (11), we need to choose an oriented
ted
2-chain C in $\chi(T^2)$ whose boundary 1-chain is given by

$$\begin{aligned}&\chi(\nu(K)\subset Y_{p/q}) - p\chi(\nu(K)\subset Y) - q\chi(\nu(K)\subset Y_0)\\ &\quad = \chi(\nu(K)\subset Y_{p/q}) - p\{v=0\} - q\{u = u(\mathfrak{s})+\eta\},\end{aligned}$$

and such that C does not contain the bad point in $\chi(T^2)$. Then, counting the boundary points of $\partial_\infty^{-1}(C)$, as a 0-chain, we obtain

$$\begin{aligned}&\#\big(\partial_\infty^{-1}(\chi(\nu(K)\subset Y_{p/q})\big)\\ &\quad = p\#\big(\partial_\infty^{-1}(\{v=0\})\big) + q\#\big(\partial_\infty^{-1}(\{u=u(\mathfrak{s})+\eta\})\big)\\ &\qquad + \#\big(\partial_\infty\,|_{\partial(\mathcal{M}_V^*(\mathfrak{s}))}\big)^{-1}(C).\end{aligned} \tag{12}$$

As C does not contain the bad points, we know that the possible points of $\partial_\infty(\partial(\mathcal{M}_V^*(\mathfrak{s})))\cap C$ all lie on the curve $u = u(\mathfrak{s})$, away from the points $\theta_0, \theta_1, \dots, \theta_p$. It is easy to see that C covers the intervals of $u = u(\mathfrak{s})$ between two consecutive points θ_i with different multiplicities: the multiplicities are $p, p-1, \dots, 1, 0$, for the intervals

$$[\theta_0, \theta_1], [\theta_1, \theta_2], \dots, [\theta_{p-1}, \theta_p], [\theta_p, \theta_0],$$

respectively. By the identity (7) and the definition (8), we know that

$$\#\big(\partial_\infty|_{\partial(\mathcal{M}_V^*(\mathfrak{s}))}\big)^{-1}(C) = \sum_{i=1}^{p} SF_{\mathbb{C}}(\not{\partial}^V_{[\theta_0,\theta_i]}). \tag{13}$$

Combining all the identities in (9)–(13), we obtain the proof of the proposition. □

The Seiberg–Witten invariant for any rational homology 3-sphere depends on metric and perturbation (cf. [14]). We now consider the correction term (3) as defined in the introduction. We have the following proposition relating the correction terms for $Y_{p/q}$ and Y.

PROPOSITION 3.2. (1) *For any rational homology* 3*-sphere Y with a* Spin^c *structure* $\mathfrak{s}$ *and a Riemannian metric* g_Y,

$$\hat{SW}_Y(\mathfrak{s}) = SW_Y(\mathfrak{s}, g_Y) - \xi(\mathfrak{s}, g_Y)$$

is a well-defined topological invariant.

(2) *For any relatively prime integers p and q, a positive integer n, and $u \in \{0, 2/n, \ldots, 2(n-1)/n\}$, we have that*

$$\sum_{k=0}^{pn-1} \xi_{Y_{p/q}}(\mathfrak{s} \otimes L_k, g_{Y_{p/q}}) - p \sum_{k=0}^{n-1} \xi_Y(\mathfrak{s} \otimes L_k, g_Y) - \sum_{i=1}^{p} SF_{\mathbb{C}}(\partial\!\!\!/^{V}_{[\theta_0, \theta_i]})$$

is independent of the manifold Y and depends only on p, q, n, and $u(\mathfrak{s}) \in \{0, 2/n, \ldots, 2(n-1)/n\}$.

Proof. Claim (1) follows from the wall-crossing formulae in [14] and the Atiyah–Patodi–Singer index theorem. This was also proved in [12]. The proof of claim (2) is analogous to the proof of Proposition 7.9 in [19]. We adapt their arguments to our situation. We write the standard surgery cobordism between S^3 and the Lens space $L(p, q)$ as

$$W(S^3, L(p, q)) = \left([0, 1] \times S^1 \times D^2\right) \cup_{[0,1]\times S^1\times S^1} X_{p/q},$$

Then the surgery cobordism between Y and $Y_{p/q}$ can be identified as

$$W_{p/q} = \left([0, 1] \times V\right) \cup_{[0,1]\times S^1\times S^1} X_{p/q}.$$

We fix a metric on $W_{p/q}$ which respects the product structure $[0, 1] \times V$ and $[0, 1] \times S^1 \times S^1$, and agrees with g_Y and $g_{Y_{p/q}}$ on the boundaries Y and $Y_{p/q}$, respectively.

For a Spin^c structure $\mathfrak{s} \otimes L_i^{(m)}$ in $\{\mathfrak{s} \otimes L_k : k = 0, \ldots, pn-1\}$ on $Y_{p/q}$, whose reducible monopole corresponds to $\theta_i^{(m)}$ (with $i \in \{1, \ldots, p\}$ and $m \in \{0, \ldots, n-1\}$), we consider the Spin^c structure $\mathfrak{s} \otimes L_m$ on Y whose reducible monopole is $\theta_0^{(m)}$. Then we claim that

$$\xi_{Y_{p/q}}(\mathfrak{s} \otimes L_i^{(m)}, g_{Y_{p/q}}) - \xi_Y(\mathfrak{s} \otimes L_m, g_Y) - SF_{\mathbb{C}}(\partial\!\!\!/^{V}_{[\theta_0^{(m)}, \theta_i^{(m)}]}) \qquad (14)$$

is independent of Y and depends only on p, q, n and on $u(\mathfrak{s}) \in \{0, 2/n, \ldots, 2(n-1)/n\}$.

To prove this claim, we choose a Spin^c structure $\tilde{\mathfrak{s}}$ on $W_{p/q}$ whose restriction to Y and $Y_{p/q}$ is given by $\mathfrak{s} \otimes L_m$ and $\mathfrak{s} \otimes L_i^{(m)}$, respectively, and such that $c_1(\tilde{\mathfrak{s}})^2 = 1$. On $(W_{p/q}, \tilde{\mathfrak{s}})$, we choose a connection A, whose restriction to $V \times [0, 1]$ is the path of reducibles connecting $\theta_0^{(m)}$ to $\theta_i^{(m)}$ along the curve $\chi(V) \subset \chi_0(T^2, V)$. Then we have

$$\begin{aligned}
&\xi_{Y_{p/q}}(\mathfrak{s}\otimes L_i^{(m)}, g_{Y_{p/q}}) - \xi_Y(\mathfrak{s}\otimes L_m, g_Y)\\
&\quad = \mathrm{Ind}_{\mathbb{C}}(\not{\partial}_A^{W_{p/q}}) - \left(\frac{c_1(\tilde{\mathfrak{s}})^2 - \sigma(W_{p/q})}{8}\right)\\
&\quad = \mathrm{Ind}_{\mathbb{C}}(\not{\partial}_A^{W_{p/q}})\\
&\quad = \mathrm{Ind}_{\mathbb{C}}(\not{\partial}_A^{[0,1]\times V}) + \mathrm{Ind}_{\mathbb{C}}(\not{\partial}_A^{X_{p/q}})
\end{aligned} \tag{15}$$

where the third equality follows from the splitting principle for the index, as the Dirac operator has no kernel on the various boundaries and corners ([2, 16]). Notice that we have

$$\mathrm{Ind}_{\mathbb{C}}(\not{\partial}_A^{[0,1]\times V}) = SF_{\mathbb{C}}(\not{\partial}^V_{[\theta_0^{(m)}, \theta_i^{(m)}]}),$$

and the connection $A|_{X_{p/q}}$ extends to connection A_0 on $W(S^3, L(p,q))$ by a flat connection, whose index on $[0,1]\times S^1\times D^2$ satisfies

$$\mathrm{Ind}_{\mathbb{C}}(\not{\partial}_{A_0}^{[0,1]\times S^1\times D^2}) = 0.$$

In fact, we can choose the metric on $W(S^3, L(p,q))$ with positive scalar curvature on $[0,1]\times S^1\times D^2$. Therefore, we have

$$\mathrm{Ind}_{\mathbb{C}}(\not{\partial}_A^{X_{p/q}}) = \mathrm{Ind}_{\mathbb{C}}(\not{\partial}_{A_0}^{W(S^3, L(p,q))}),$$

which depends only on p, q, n and $u(\mathfrak{s})$, and so does the quantity

$$\xi_{Y_{p/q}}(\mathfrak{s}\otimes L_i^{(m)}, g_{Y_{p/q}}) - \xi_Y(\mathfrak{s}\otimes L_m, g_Y) - SF_{\mathbb{C}}(\not{\partial}^V_{[\theta_0^{(m)}, \theta_i^{(m)}]}). \tag{16}$$

When summing the identity (16) over $i\in\{1,\ldots,p\}$ and $m\in\{0,\ldots,n-1\}$, notice that the term $\xi_Y(\mathfrak{s}\otimes L_m, g_Y)$ is independent of $i\in\{1,\ldots,p\}$, hence we obtain the proof of the claim (2) by using the definition (8). □

With these two propositions in place, we now have the following surgery formula for the modified version of the Seiberg–Witten invariant.

THEOREM 3.3. *Given any two relatively prime integers p and q, a positive integer n and* $u\in\{0, 2/n, 2(n-1)/n\}$, *there is a rational valued function* $s(p,q,n,u)$, *depending only on p, q, n and u, satisfying the following property. Let Y be a rational homology* 3-*sphere with a smoothly embedded knot and a canonical framing* (m,l) *such that* $\nu(K)\cong D^2\times S^1$. *Assume that K represents a torsion element of order n in* $H_1(Y,\mathbb{Z})$. *Let* $\mathfrak{s}$ *be a* Spin^c *structure on Y. Then we have*

$$\begin{aligned}
&\sum_{k=0}^{pn-1} S\hat{W}_{Y_{p/q}}(\mathfrak{s}\otimes L_k)\\
&\quad = p\sum_{k=0}^{n-1} S\hat{W}_Y(\mathfrak{s}\otimes L_k) + q\sum_{k\in\mathbb{Z}} SW_{Y_0}(\mathfrak{s}\otimes L_k) + s(p,q,n,u).
\end{aligned}$$

Proof. Following from Proposition 3.2, we know that

$$\sum_{k=0}^{pn-1} \xi_{Y_{p/q}}(\mathfrak{s} \otimes L_k, g_{Y_{p/q}}) - p \sum_{k=0}^{n-1} \xi_Y(\mathfrak{s} \otimes L_k, g_Y) - \sum_{i=1}^{p} SF_{\mathbb{C}}(\not{\partial}^V_{[\theta_0, \theta_i]}) \tag{17}$$

depends only on p, q, n and $u = u(\mathfrak{s}) \in \{0, 2/n, \ldots, 2(n-1)/n\}$. We denote this term by $s(p, q, n, u)$. By subtracting (17) from the surgery formula for the Seiberg–Witten invariants in Proposition 3.1, we obtain the proof of this theorem. □

Now we can establish the equivalence between the modified version of the Seiberg–Witten invariant $S\hat{W}$ and the Casson–Walker invariant for rational homology 3-spheres.

THEOREM 3.4. *For any rational homology* 3-*sphere* Y, *we have*

$$\sum_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} S\hat{W}_Y(\mathfrak{s}) = \tfrac{1}{2}|H_1(Y, \mathbb{Z})|\lambda(Y)$$

where $\lambda(Y)$ *is the Casson–Walker invariant*.

Proof. We first derive the surgery formula for the invariant $\sum_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} S\hat{W}_Y(\mathfrak{s})$ from Theorem 3.3 and the Seiberg-Witten invariant for Y_0 (a rational homology $S^1 \times S^2$, i.e., $b_1(Y_0) = 1$) (see [15, 7]):

$$\begin{aligned}
&\sum_{\mathfrak{s} \in \mathrm{Spin}^c(Y_{p/q})} S\hat{W}_{Y_{p/q}}(\mathfrak{s}) \\
&\quad = p \sum_{\mathfrak{s} \in \mathrm{Spin}^c(Y)} S\hat{W}_Y(\mathfrak{s}) + q \sum_{j=0}^{\infty} a_j j^2 + |H_1(Y, \mathbb{Z})| s(p, q, n),
\end{aligned} \tag{18}$$

where $s(p, q, n) = \sum_u s(p, q, n, u)/n$ and a_j is the coefficient of the symmetrized Alexander polynomial of Y_0,

$$A(t) = a_0 + \sum_{j=1}^{\infty} a_j (t^j + t^{-j})$$

normalized so that $A(1) = |\mathrm{Torsion}(H_1(Y_0, \mathbb{Z}))|$. Set $\bar{\lambda}(Y) = \frac{1}{2}|H_1(Y, \mathbb{Z})|\lambda(Y)$ as the normalized Casson–Walker invariant. Then the surgery formula in [24] for $\bar{\lambda}(Y)$ can be expressed as (cf. [19]):

$$\begin{aligned}
\bar{\lambda}(Y_{p/q}) &= p\bar{\lambda}(Y) + q \sum_{j=0}^{\infty} a_j j^2 \\
&\quad + |H_1(Y, \mathbb{Z})| \left(\frac{q(n^2 - 1)}{12n^2} - \frac{ps(p, q)}{2} \right).
\end{aligned} \tag{19}$$

Here $s(p, q)$ is the Dedekind sum of relatively prime integers p and q (cf. [24]). Comparing (18) and (19), we only need to show that

$$s(p, q, n) = \frac{q(n^2 - 1)}{12n^2} - \frac{ps(p, q)}{2}. \tag{20}$$

Since $s(p, q, n)$ is independent of the manifold Y, we can choose some examples that can be computed explicitly, and use them to identify the coefficient $s(p, q, n)$. The Lens space $L(p, q)$ can be obtained by a p/q-surgery on an unknot in S^3. The calculation of Nicolaescu [17] for $L(p, q)$ gives us that

$$\sum_{\mathfrak{s}\in \mathrm{Spin}^c(L(p,q))} S\hat{W}_{L(p,q)}(\mathfrak{s}) = -\frac{ps(p, q)}{2}.$$

This implies that (20) holds for $n = 1$. Now we can prove (20) by induction on n. This is exactly the same argument as in the proof of Theorem 7.5 in [19] on the equivalence of their theta invariant and the Casson–Walker invariant. The example is the Seifert manifold $M(n, 1; -n, 1; q, -p)$, obtained by p/q surgery on a knot of order n in $L(n, 1)\#\overline{L(n, 1)}$. By Kirby calculus it is possible to show that $M(n, 1; -n, 1; q, -p)$ can be obtained as $(-n)$-surgery on a knot in the Lens space $L(pn - q, q)$, and can be obtained as a sequence of surgeries on knots of order less than n, see the proof of Theorem 7.5 in [19] for details. □

4. Additional Remarks

In the classical theory of topological invariants of 3-manifolds, there is an interesting dichotomy between two types of invariants: the Casson invariant and torsion. Turaev torsion is defined for manifolds with $b_1(Y) > 0$, as an invariant with values in $\mathbb{Z}[H]$, $H = H_1(Y, \mathbb{Z})$ being the first homology, and can be extended to a $\mathbb{Q}[H]$-valued invariant for rational homology 3-spheres, [21]. On the other hand, the Casson invariant was originally defined only for integral homology spheres [1], and was later extended to rational homology 3-spheres as the Casson–Walker invariant [24]. A further generalization to all 3-manifolds due to Lescop [10] revealed that, for $b_1(Y) > 0$, this invariant gives no more information than Turaev torsion, and in fact can be described as an averaged version of torsion. Thus, we have on the one hand a first invariant that lives naturally on 3-manifolds with $b_1(Y) > 0$ and is trivial on integral homology spheres, and on the other hand a second classical invariant which lives naturally on integral homology spheres and reduces to an averaged version of the first for manifolds with $b_1(Y) > 0$. It was also apparent that these are independent invariants in the intermediate range of rational homology spheres, where both are defined.

The picture that recently emerged, from the results of [11, 15, 18, 22], and the present paper, identifies the Seiberg–Witten invariant as a natural unifying theory behind both Casson invariant and torsion. A recent paper of Nicolaescu [18] shows how the Seiberg–Witten invariants, with the correction terms we describe here in Section 2 for rational homology spheres, can be organized as an invariant $SW^0_Y \in \mathbb{Q}[H]$, which satisfies the relation $SW^0_Y = \mathcal{T}^0_Y$. The invariant $\mathcal{T}^0_Y$ is a modified version of Turaev torsion, which agrees with Turaev torsion $\mathcal{T}_Y$ when $b_1(Y) > 0$, and is corrected so as to satisfy $\mathcal{T}^0_Y(1) = \frac{1}{2}|H|\lambda(Y)$ when $b_1(Y) = 0$. The proof given by Nicolaescu combines our surgery formula of Proposition 3.1 with surgery formulae for torsion and Casson–Walker invariant, and shows that the class of 3-manifolds for which the difference of these invariants $D_Y := SW_Y^0 - \mathcal{T}_Y^0$ (or rather its Fourier transform) is trivial can be enlarged by admissible surgeries to eventually encompass all rational homology 3-spheres.

A unified theory of Casson–Walker invariant and Turaev torsion was also derived by Ozsváth and Szabó [19, 20], via their theta invariant.

Given the results of [11, 22], and the present paper, the result $SW_Y^0 = \mathcal{T}_Y^0$ for all 3-manifolds amounts to showing that Seiberg-Witten invariants determine Turaev torsion for rational homology 3-spheres. There is an axiomatic characterization of Turaev torsion for rational homology 3-spheres [23], which seems suitable for comparison with Seiberg–Witten invariants, but this seems to require a more refined version of the surgery formula for Seiberg–Witten invariants, which is stronger than our Proposition 3.1. This approach may lead to a simplified proof of the main result of [18].

Acknowledgements

Some of our arguments are inspired by the paper of Ozsváth and Szabó on the theta divisor and the Casson–Walker invariant. The first author is partially supported by the Humboldt Foundation (Sofja Kovalevskaya Award). The second author is supported by the Australian Research Council.

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