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Renormalization and Motivic Galois Theory

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1 Introduction

In this paper we show that the divergences of quantum field theory are a highly structured phenomenon. More precisely, they provide data that define an action of a specific "*motivic Galois group*" U^* on the set of physical theories.

In particular, this exhibits the renormalization group as the action of a one-parameter subgroup $\mathbb{G}_a \subset U^*$ of the above Galois group.

The work of Connes and Kreimer [9, 10] provided a conceptual understanding of perturbative renormalization in terms of the Birkhoff decomposition of loops in a prounipotent Lie group G determined by the physical theory, through the Hopf algebra of Feynman graphs [9, 17].

This suggests the possibility of formulating the theory of renormalization in the context of the Riemann-Hilbert correspondence. The latter is a broad term encompassing, in various forms and levels of generalization, equivalences between geometric problems associated to differential systems with singularities and representation-theoretic data associated to the monodromy.

In this paper we construct the Riemann-Hilbert correspondence associated to perturbative renormalization, in the form of a classification of flat equisingular bundles in terms of representations of the "motivic Galois group" U^*.

More specifically, we start by considering the scattering formula

$$\gamma_-(z) = \lim_{t\to\infty} e^{-t(\beta/z+Z_0)} e^{tZ_0} \tag{1.1}$$

proved in [10], which expresses the counterterms through the residues of graphs.

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We reexpress this formula in terms of the time-ordered exponential of physicists (also known as expansional in mathematical terminology). The expression in expansional form can be recognized as solution of a differential system. This identifies a class of connections naturally associated to the differential of the regularized quantum field theory viewed as a function of the complexified dimension. The physics input that the counterterms are independent of the additional choice of a unit of mass translates, in geometric terms, into the notion of equisingularity for these connections.

Thus, the geometric problem consists of the classification of "*equisingular*" G-valued flat connections on the total space B of a principal $\mathbb{G}_m$-bundle over an infinitesimal punctured disk Δ^*. An equisingular connection is a $\mathbb{G}_m$-invariant G-valued connection, singular on the fiber over zero, and satisfying the following property: the equivalence class of the singularity of the pullback of the connection by a section of the principal $\mathbb{G}_m$-bundle only depends on the value of the section at the origin.

This classification problem stems directly from the divergences of the physical theory at the dimension D where one would like to do physics.[1] The base Δ^* is the space of complexified dimensions around D. The fibers of the principal $\mathbb{G}_m$-bundle B describe the arbitrariness in the normalization of integration in complexified dimension $z \in \Delta^*$, in the commonly used regularization procedure known as Dim-Reg (dimensional regularization). The $\mathbb{G}_m$-action corresponds to the rescaling $\hbar\,\partial/\partial\hbar$. The group G is the prounipotent Lie group whose Hopf algebra is the Hopf algebra of Feynman graphs of [9, 17].

On the other side of our Riemann-Hilbert correspondence, the representation-theoretic setting equivalent to the classification of equisingular flat connections is provided by representations $U^* \to G^*$, where U^* is a universal group, unambiguously defined independently of the physical theory. The group G^* is the semidirect product of G by the action of the grading θ_t, as in [10]. We give an explicit description of U^* as the semidirect product by its grading of the graded prounipotent Lie group U whose Lie algebra is the free graded Lie algebra

$$\mathcal{F}(1, 2, 3, \ldots)_\bullet \tag{1.2}$$

generated by elements e_{-n} of degree n, $n > 0$.

Thus, there are three different levels at which Hopf algebra structures enter the theory of perturbative renormalization. First, there is Kreimer's Hopf algebra of rooted

[1] We may assume $D = 4$ (no strings attached).

trees [17], which is adapted to the specific physical theory by decorations of the rooted trees. There is then the Connes-Kreimer Hopf algebra of Feynman graphs, which is dependent on the physical theory by construction, but which does not require decorations. There is then the algebra associated to the group $\mathbb{U}^*$, which is universal with respect to the set of physical theories.

We then construct a specific universal singular frame on principal $\mathbb{U}$-bundles over B. When using in this frame the dimensional regularization technique of quantum field theory and the minimal subtraction (MS) scheme, all divergences disappear and one obtains a finite theory, which only depends upon the choice of a local trivialization for the principal $\mathbb{G}_m$-bundle B.

The coefficients of the universal singular frame, written out in the expansional form, are the same as those appearing in the local index formula of Connes and Moscovici [12]. In particular, they are rational numbers. This means that we can view equisingular flat connections on finite-dimensional vector bundles as endowed with arithmetic structure. We show that they can be organized into a Tannakian category with a natural fiber functor to the category of vector spaces, over any field of characteristic zero. The Tannakian category obtained this way is equivalent to the category of finite-dimensional representations of the affine group scheme $\mathbb{U}^*$, which is uniquely determined by this property.

Closely related group schemes appear in motivic Galois theory and $\mathbb{U}^*$ is, for instance, abstractly (but noncanonically) isomorphic to the motivic Galois group $G_{\mathcal{M}_T}(\mathcal{O})$ (see [13, 15]) of the scheme $S_4 = \mathrm{Spec}(\mathcal{O})$ of 4-cyclotomic integers, $\mathcal{O} = \mathbb{Z}[i][1/2]$.

The natural appearance of the "motivic Galois group" $\mathbb{U}^*$ in the context of renormalization confirms a suggestion made by Cartier in [4], that in the Connes-Kreimer theory of perturbative renormalization one should find a hidden "cosmic Galois group" closely related in structure to the Grothendieck-Teichmüller group. The question of relations between the work of Connes-Kreimer, motivic Galois theory, and deformation quantization was further emphasized by Kontsevich in [16]. At the level of the Hopf algebra of rooted trees, relations between renormalization and motivic Galois theory were also investigated by Goncharov in [14].

The "motivic Galois group" $\mathbb{U}$ acts on the set of dimensionless coupling constants of physical theories, through the map of the corresponding group G to formal diffeomorphisms constructed in [10].

This also realizes the hope formulated in [6] of relating concretely the renormalization group to a Galois group. Here, we are dealing with the Galois group dictated by renormalization and the renormalization group appears as a canonical one-parameter subgroup $\mathbb{G}_a \subset \mathbb{U}$.

These facts altogether indicate that the divergences of quantum field theory, far from just being an unwanted nuisance, are a clear sign of the presence of totally unexpected symmetries of geometric origin. This shows, in particular, that one should understand how the universal singular frame "renormalizes" the geometry of space-time using the Dim-Reg MS scheme and the universal counterterms.

2 Expansional form of the counterterms

The following discussion will be quite general. We let G be a complex graded prounipotent Lie group, $\mathfrak{g} = \operatorname{Lie} G$ its Lie algebra, and $\theta_t = e^{tY}$ the one-parameter group of automorphisms implementing the grading Y. We assume that the grading Y is integral and strictly positive.

We let G^* be the semidirect product

$$G^* = G \rtimes_\theta \mathbb{R} \tag{2.1}$$

of G by the action of the grading θ_t, hence the Lie algebra of G^* has an additional generator Z_0, such that

$$[Z_0, X] = Y(X) \quad \forall X \in \operatorname{Lie} G. \tag{2.2}$$

We let $\mathcal{H}$ be the commutative Hopf algebra of coordinates on G. For any unital algebra A over $\mathbb{C}$, we let $G(A)$ be the group of points of G over A, that is, of homomorphisms

$$\mathcal{H} \longrightarrow A, \tag{2.3}$$

with the product coming from the coproduct of $\mathcal{H}$.

We identify the elements of the Lie algebra $\mathfrak{g} = \operatorname{Lie} G$ with linear forms L on $\mathcal{H}$ such that

$$L(XY) = L(X)\varepsilon(Y) + \varepsilon(X)L(Y), \quad \forall X, Y \in \mathcal{H}, \tag{2.4}$$

where ε is the augmentation of $\mathcal{H}$, playing the role of the unit in the dual algebra. More generally, for any unital algebra A over $\mathbb{C}$, one defines $\mathfrak{g}(A)$ as the Lie algebra of linear maps $\mathcal{H} \to A$, fulfilling the above derivation rule.

In [10] a complete characterization is given of those G-valued loops $\gamma_\mu(z)$ satisfying the properties

$$\begin{aligned} &\gamma_{e^t\mu}(z) = \theta_{tz}\big(\gamma_\mu(z)\big) \quad \forall t \in \mathbb{R}, \\ &\frac{\partial}{\partial\mu}\gamma_\mu^-(z) = 0. \end{aligned} \tag{2.5}$$

Here, γ_μ^- is the negative part of the Birkhoff decomposition

$$\gamma_\mu(z) = \gamma_\mu^-(z)^{-1}\gamma_\mu^+(z), \quad z \in \partial\Delta, \tag{2.6}$$

where γ_μ^+ and γ_μ^- extend to holomorphic maps on Δ and $\mathbb{P}^1(\mathbb{C}) \setminus \{0\}$, respectively.

In this Birkhoff decomposition, γ_μ^+ provides the renormalized values at D, and γ_μ^- provides the counterterms for the renormalization procedure of quantum field theory (cf. [10]). The properties (2.5) originate from physical considerations, namely, from the fact that the counterterms are independent of the choice of the mass scale parameter μ (cf. [5, (7.1.4a)–(7.1.4c), page 170]).

We can regard the γ_μ as elements of $G(K)$, where we let K be the field $\mathbb{C}(\{z\})$ of convergent Laurent series in z.

Given a $\mathfrak{g} = \operatorname{Lie} G$-valued smooth function $\alpha(t)$, where $t \in [a, b] \subset \mathbb{R}$ is a real parameter, one defines the expansional (cf. [1]), or time-ordered exponential, by the equality

$$\mathrm{T}e^{\int_a^b \alpha(t)dt} = 1 + \sum_1^\infty \int_{a\le s_1\le\cdots\le s_n\le b} \alpha(s_1)\cdots\alpha(s_n)\prod ds_j, \tag{2.7}$$

where the product comes from the coproduct in $\mathcal{H}$.

This defines an element of $G(\mathbb{C})$, which is the value $A(b)$ at b of the unique solution $A(t)$ with $A(a) = 1$ at $t = a$ of the differential equation

$$dA(t) = A(t)\alpha(t)dt. \tag{2.8}$$

The basic property of the expansional is the identity

$$\mathrm{T}e^{\int_a^c \alpha(t)dt} = \mathrm{T}e^{\int_a^b \alpha(t)dt}\mathrm{T}e^{\int_b^c \alpha(t)dt}. \tag{2.9}$$

With this notation, we can rewrite the scattering formula (1.1) as follows.

Theorem 2.1. Let $\gamma_\mu(z)$ be a family of G-valued loops fulfilling (2.5). Then there exist uniquely $\beta \in \mathfrak{g}$ and a loop $\gamma_{\text{reg}}(z)$ regular at $z = 0$ such that

$$\gamma_\mu(z) = \mathrm{T}e^{-(1/z)\int_\infty^{-z\log\mu} \theta_{-t}(\beta)dt}\theta_{z\log\mu}\big(\gamma_{\text{reg}}(z)\big). \tag{2.10}$$

Conversely, given any β and any regular loop $\gamma_{\text{reg}}(z)$, the expression (2.10) gives a solution to equations (2.5). □

The Birkhoff decomposition of the loop $\gamma_\mu(z)$ of (2.10) is given by

$$\begin{aligned}\gamma_\mu^+(z) &= \mathrm{T}e^{-(1/z)\int_0^{-z\log\mu} \theta_{-t}(\beta)dt}\theta_{z\log\mu}\big(\gamma_{\text{reg}}(z)\big),\\ \gamma_\mu^-(z) &= \mathrm{T}e^{-(1/z)\int_0^{\infty} \theta_{-t}(\beta)dt}.\end{aligned} \tag{2.11}$$

3 Local equivalence of meromorphic connections

We consider the local behavior, on an infinitesimal punctured disk Δ^* centered at $z = 0$, of solutions of G-differential systems.

As above, we work with convergent Laurent series. Namely, we let K be the field $\mathbb{C}(\{z\})$ of convergent Laurent series in z and let $\mathcal{O} \subset K$ be the subring of series without a pole at 0. The field K is a differential field and we let Ω^1 be the 1-forms on K with

$$d : K \longrightarrow \Omega^1 \tag{3.1}$$

the differential, $df = (df/dz)dz$.

A connection on the trivial principal G-bundle $P = \Delta^* \times G$ is specified by the restriction of the connection form to $\Delta^* \times 1$, that is, by a $\mathfrak{g}$-valued 1-form ω on Δ^*. We let $\Omega^1(\mathfrak{g})$ denote $\mathfrak{g}$-valued 1-forms on Δ^*, so that every element of $\Omega^1(\mathfrak{g})$ is of the form $A dz$ with $A \in \mathfrak{g}(K)$.

The operator

$$D : G(K) \longrightarrow \Omega^1(\mathfrak{g}), \quad Df = f^{-1}df, \tag{3.2}$$

satisfies

$$D(fh) = Dh + h^{-1}Df h. \tag{3.3}$$

We consider differential equations of the form

$$Df = \omega, \tag{3.4}$$

where $\omega \in \Omega^1(\mathfrak{g})$ specifies the connection on the trivial principal G-bundle.

Definition 3.1. Two connections ω and ω' are equivalent if and only if

$$\omega' = Dh + h^{-1}\omega h, \tag{3.5}$$

for some $h \in G(\mathcal{O})$.

This simply identifies connections that differ by a change of local frame, given by a G-valued map regular in Δ.

By construction, the group G is a projective limit of linear algebraic groups G_i whose Hopf algebras are finitely generated graded Hopf subalgebras $\mathcal{H}_i \subset \mathcal{H}$. Given $\omega \in \Omega^1(\mathfrak{g})$, its projections $p_i(\omega) \in \Omega^1(\mathfrak{g}_i)$ have a positive radius of convergence $\rho_i > 0$. Thus, for a choice of a base point $z_0 \neq 0$ with $|z_0| < \rho_i$, we obtain the monodromy in the form

$$M = \mathbf{T}e^{\int_0^1 c^*(\omega)}, \tag{3.6}$$

where $c(t)$ is a simple closed path of winding number one in the punctured disk of radius ρ_i, with endpoints $c(0) = z_0 = c(1)$.

When passing to the projective limit, one has to take care of the change of base point, but the triviality of the monodromy, $M = 1$, is a well-defined condition. It ensures the existence of solutions $f \in G(K)$ for equation (3.4).

A solution f of (3.4) defines a G-valued loop. By our assumptions on G, any $f \in G(K)$ has a unique Birkhoff decomposition of the form

$$f = \left(f^-\right)^{-1} f^+, \tag{3.7}$$

with

$$f^+ \in G(\mathcal{O}), \qquad f^- \in G(\mathcal{Q}), \tag{3.8}$$

where $\mathcal{O} \subset K$ is the subalgebra of regular functions and $\mathcal{Q} = z^{-1}\mathbb{C}([z^{-1}])$. Since $\mathcal{Q}$ is not unital, one needs to be more precise in defining $G(\mathcal{Q})$. Let $\tilde{\mathcal{Q}} = \mathbb{C}([z^{-1}])$ and let ε_1 be its augmentation. Then $G(\mathcal{Q})$ is the subgroup of $G(\tilde{\mathcal{Q}})$ of homomorphisms $\phi : \mathcal{H} \mapsto \tilde{\mathcal{Q}}$ such that $\varepsilon_1 \circ \phi = \varepsilon$, where ε is the augmentation of $\mathcal{H}$.

Proposition 3.2. Two connections ω_1 and ω_2 with trivial monodromy are equivalent if and only if solutions f_j of $Df = \omega_j$ have the same negative part in the Birkhoff decomposition,

$$f_1^- = f_2^-. \tag{3.9}$$

□

4 Classification of equisingular flat connections

We now modify the geometric setting of the previous section by introducing a principal $\mathbb{G}_m$-bundle

$$\mathbb{G}_m \longrightarrow B \longrightarrow \Delta \tag{4.1}$$

over the infinitesimal disk Δ. We let

$$b \longmapsto w(b) \quad \forall w \in \mathbb{C}^*, \tag{4.2}$$

denote the action of the multiplicative group $\mathbb{G}_m = \mathbb{C}^*$. We let $\pi : B \to \Delta$ be the projection, with

$$V = \pi^{-1}(\{0\}) \subset B \tag{4.3}$$

the fiber over $0 \in \Delta$ and $y_0 \in V$ a base point. We let $B^* \subset B$ denote the complement of V.

We consider again a group G as above, with grading Y. We can then view the trivial principal G-bundle $P = B \times G$ as equivariant with respect to $\mathbb{G}_m$, using the action

$$u(b, g) = \big(u(b), u^Y(g)\big) \quad \forall u \in \mathbb{C}^*, \tag{4.4}$$

where u^Y makes sense, since the grading Y is integer-valued.

Definition 4.1. Let $P^* = B^* \times G$ be the restriction to B^* of the bundle P. A connection ω on P^* is *equisingular* if it is $\mathbb{G}_m$-invariant and its restrictions to sections of the principal bundle B that agree at $0 \in \Delta$ are all equivalent in the sense of Definition 3.1 (cf. Figure 4.1).

We consider again the operator $Df = f^{-1}df$ as in (3.2) satisfying (3.3). We have the following notion of equivalence for G-differential systems on B.

Definition 4.2. Two connections ω and ω' on P^* are equivalent if and only if

$$\omega' = Dh + h^{-1}\omega h, \tag{4.5}$$

for a G-valued $\mathbb{G}_m$-invariant map h regular in B.

The main step towards the formulation of perturbative renormalization as a Riemann-Hilbert correspondence is given by the following correspondence between flat equisingular G-connections and elements in the Lie algebra $\mathfrak{g}$.

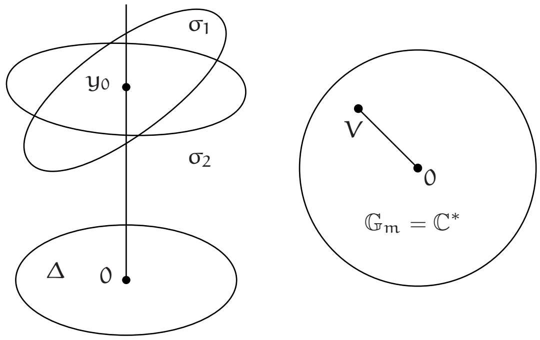

Figure 4.1

We begin by choosing a noncanonical regular section

$$\sigma : \Delta \longrightarrow B, \quad \text{with } \sigma(0) = y_0. \tag{4.6}$$

We will show later that the correspondence established in Theorem 4.3 is in fact independent of the choice of σ. To lighten notations we use σ as the local frame that trivializes the bundle B, which we identify with $\Delta \times \mathbb{C}^*$.

Theorem 4.3. Let ω be a flat equisingular G-connection. There exists a unique element $\beta \in \mathfrak{g}$ of the Lie algebra of G, such that ω is equivalent to the flat equisingular connection $D\gamma$ associated to the section

$$\gamma(z, v) = \mathrm{T}e^{-(1/z)\int_0^v u^Y(\beta)(du/u)} \in G, \tag{4.7}$$

where the integral is performed on the straight path $u = tv$, $t \in [0, 1]$. □

Proof. As above, we express a connection on P^* in terms of $\mathfrak{g}$-valued 1-forms on B^*, and we use the trivialization σ to write it as

$$\omega = A dz + B \frac{dv}{v}, \tag{4.8}$$

where both $A(z, v)$ and $B(z, v)$ are $\mathfrak{g}$-valued functions and dv/v is the fundamental 1-form of the principal bundle B.

Let $\omega = A dz + B(dv/v)$ be an invariant connection. One has

$$\omega(z, uv) = u^Y\big(\omega(z, v)\big), \tag{4.9}$$

which shows that ω is determined by its restriction to the section $v = 1$. One then has

$$\omega(z, u) = u^Y(a)dz + u^Y(b)\frac{du}{u} \tag{4.10}$$

for suitable elements $a, b \in \mathfrak{g}(K)$.

The flatness of the connection means that we have

$$\frac{db}{dz} - Y(a) + [a, b] = 0. \tag{4.11}$$

The positivity of the integral grading Y shows that the connection ω extends to a flat connection on the product $\Delta^* \times \mathbb{C}$. Moreover, its restriction to $\Delta^* \times \{0\}$ is equal to 0. This suffices to show that the connection has trivial monodromy with respect to both generators of $\pi_1(B^*) = \mathbb{Z}^2$.

One can then explicitly write down a solution of the differential system

$$D\gamma = \omega \tag{4.12}$$

in the form

$$\gamma(z, v) = \mathrm{T}e^{\int_0^v u^Y(b(z))(du/u)}, \tag{4.13}$$

where integration is performed on the straight path $u = tv$, $t \in [0, 1]$ (cf. Figure 4.1).

This gives a translation invariant loop γ,

$$\gamma(z, u) = u^Y\gamma(z) \tag{4.14}$$

fulfilling

$$\gamma(z)^{-1}d\gamma(z) = a dz, \qquad \gamma(z)^{-1}Y\gamma(z) = b. \tag{4.15}$$

By hypothesis, ω is equisingular, hence the restrictions ω_s to the lines

$$\Delta_s^* = \left\{\left(z, e^{sz}\right);\ z \in \Delta^*\right\} \tag{4.16}$$

are mutually equivalent. By Proposition 3.2, using the fact that the restriction of $\gamma(z, u) = u^Y\gamma(z)$ to Δ_s^* is given by $\gamma_s(z) = \theta_{sz}\gamma(z)$, we obtain that the negative parts of the Birkhoff decomposition of the loops $\gamma_s(z)$ are independent of the parameter s.

Thus, by the results of Section 2, there exist an element $\beta \in \mathfrak{g}$ and a regular loop $\gamma_{\text{reg}}(z)$, such that

$$\gamma(z,1) = \mathrm{T}e^{-(1/z)\int_\infty^0 \theta_{-t}(\beta)dt}\gamma_{\text{reg}}(z). \tag{4.17}$$

Thus, up to an equivalence given by the regular loop $u^Y(\gamma_{\text{reg}}(z))$, we can write the solution in the form

$$\gamma(z,u) = u^Y\big(\mathrm{T}e^{-(1/z)\int_\infty^0 \theta_{-t}(\beta)dt}\big), \tag{4.18}$$

which only depends upon $\beta \in \mathfrak{g}$. Since u^Y is an automorphism, one can in fact rewrite (4.18) as

$$\gamma(z,v) = \mathrm{T}e^{-(1/z)\int_0^v u^Y(\beta)(du/u)}, \tag{4.19}$$

where the integral is performed on the straight path $u = tv$, $t \in [0,1]$.

We then need to understand in what way the class of the solution (4.18) depends upon $\beta \in \mathfrak{g}$.

An equivalence between two equisingular flat connections generates a relation between solutions of the form

$$\gamma_2(z,u) = \gamma_1(z,u)h(z,u) \tag{4.20}$$

with h regular. Thus, the negative parts of the Birkhoff decomposition of both

$$\gamma_j(z,1) = \mathrm{T}e^{-(1/z)\int_\infty^0 \theta_{-t}(\beta_j)dt} \tag{4.21}$$

have to be the same, and this gives $\beta_1 = \beta_2$.

Finally, we need to show that, for any $\beta \in \mathfrak{g}$, the connection $\omega = D\gamma$ with γ given by (4.7) is equisingular. The equivariance follows from the invariance of the section γ. Let then $v(z) \in \mathbb{C}^*$ be a regular function with $v(0) = 1$ and consider the section $v(z)\sigma(z)$ instead of $\sigma(z)$. The restriction of $\omega = D\gamma$ to this new section is now given by $D\gamma_v$, where

$$\gamma_v(z) = \mathrm{T}e^{-(1/z)\int_0^{v(z)} u^Y(\beta)(du/u)} \in G. \tag{4.22}$$

We claim that the Birkhoff decomposition of γ_v is given by $\gamma_v(z) = \gamma_v^-(z)^{-1}\gamma_v^+(z)$ with

$$\gamma_v^-(z)^{-1} = \mathrm{T}e^{-(1/z)\int_0^1 u^Y(\beta)(du/u)}, \qquad \gamma_v^+(z) = \mathrm{T}e^{-(1/z)\int_1^{v(z)} u^Y(\beta)(du/u)}. \tag{4.23}$$

Indeed, the first term in (4.23) is a regular function of z^{-1} and gives a polynomial in z^{-1} when paired with any element of $\mathcal{H}$. The second term is a regular function of z, using the Taylor expansion of $\nu(z)$ at $z = 1$. ■

By a similar argument, one gets the independence on the choice of the section, as follows.

Theorem 4.4. The above correspondence between flat equisingular G-connections and elements $\beta \in \mathfrak{g}$ of the Lie algebra of G is independent of the choice of the local regular section $\sigma : \Delta \to B$, with $\sigma(0) = y_0$.

Given two choices $\sigma_2 = \alpha\sigma_1$ of local sections, the regular values $\gamma_{\mathrm{reg}}(y_0)_j$ of solutions of the differential system above, in the corresponding singular frames, are related by

$$\gamma_{\mathrm{reg}}(y_0)_2 = e^{-s\beta}\gamma_{\mathrm{reg}}(y_0)_1, \tag{4.24}$$

where

$$s = \left(\frac{d\alpha(z)}{dz}\right)_{z=0}. \tag{4.25}$$

□

It is this second statement that controls the ambiguity inherent to the renormalization group action in the physics setting, where there is no preferred choice of local regular section σ. In that context the principal bundle B over an infinitesimal disk of complexified dimensions around D admits as fiber over $z \in \Delta$ the set of all possible normalizations for the integration in complexified dimension $D - z$. Moreover, the choice of the base point in the fiber V over D corresponds to the choice of the Planck constant, while the choice of the section σ (up to order one) corresponds to the choice of a "unit of mass."

5 The universal singular frame

We will now reformulate the results of Section 4 as a Riemann-Hilbert correspondence. At the representation-theoretic level, we want to encode the data classifying equivalence classes of equisingular flat connections (Theorem 4.3) by a homomorphism

$$U^* \longrightarrow G^* \tag{5.1}$$

from some universal group U^* to G^*.

Viewed in this perspective, the group $\mathbb{U}^*$ can be thought of as an analog of the Ramis exponential torus in the wild fundamental group that gives the local Riemann-Hilbert correspondence in the context of differential Galois theory (cf. [19, 20]). In fact, here the equisingular flat connections have trivial monodromy and one does not see the Stokes phenomenon, as we are only dealing with perturbative renormalization. Thus, the group $\mathbb{U}^*$ resembles most the remaining part of the wild fundamental group, given by the exponential torus, which appears in the formal local theory (cf. [19, 20] and [21, Section 3]). We will analyze more closely the relation to the wild fundamental group in [11].

In (5.1) we need to get both Z_0 and β in the range at the Lie algebra level. Thus, working with Lie algebras, it is natural to consider first the free Lie algebra generated by Z_0 and β. It is important, though, to keep track of the properties one needs so that the formulae above make sense, such as positivity and integrality of the grading.

By these properties, we can write β as an infinite formal sum

$$\beta = \sum_{1}^{\infty} \beta_n, \tag{5.2}$$

where, for each n, β_n is homogeneous of degree n for the grading,

$$Y(\beta_n) = n\beta_n. \tag{5.3}$$

Thus, assigning β and the action of the grading on it is the same as giving a collection of homogeneous elements β_n fulfilling no restriction besides $Y(\beta_n) = n\beta_n$. In particular, there is no condition on their Lie brackets. Thus, these data are the same as giving a homomorphism from the following affine group scheme $\mathbb{U}$ to G.

At the Lie algebra level, $\mathbb{U}$ comes from the free graded Lie algebra

$$\mathcal{F}(1, 2, 3, \ldots)_{\bullet} \tag{5.4}$$

generated by elements e_{-n} of degree n, $n > 0$. At the Hopf algebra level, we therefore take the graded dual of the enveloping algebra $\mathcal{U}(\mathcal{F})$, so that

$$\mathcal{H}_u = \mathcal{U}\big(\mathcal{F}(1, 2, 3, \ldots)_{\bullet}\big)^{\vee}. \tag{5.5}$$

It is well known that $\mathcal{H}_u$, as an algebra, is isomorphic to the linear space of noncommutative polynomials in variables f_n, $n \in \mathbb{N}_{>0}$, with the shuffle product.

We have obtained this way a prounipotent affine group scheme $\mathbb{U}$ which is graded in positive degree.

We can now reformulate the main theorem of Section 4 as follows.

Theorem 5.1. Let G be a positively graded prounipotent Lie group. There exists a canonical bijection between equivalence classes of flat equisingular G-connections and graded representations

$$\rho : \mathcal{U} \longrightarrow G \tag{5.6}$$

in G of the group scheme $\mathcal{U}$ defined above. □

We can consider the semidirect product $\mathcal{U}^*$ of $\mathcal{U}$ by the grading as an affine group scheme with a natural homomorphism $\mathcal{U}^* \to \mathbb{G}_m$ to the multiplicative group. The compatibility with the grading means that ρ extends to a homomorphism

$$\rho^* : \mathcal{U}^* \longrightarrow G^*. \tag{5.7}$$

Theorem 5.1 shows that the group $\mathcal{U}^*$ plays, in the formal theory, a role analogous to that of the Ramis exponential torus of differential Galois theory. The conceptual reason for considering the group $\mathcal{U}^*$ rather than $\mathcal{U}$ will become clear in the next section.

The equality

$$e = \sum_1^\infty e_{-n} \tag{5.8}$$

defines an element of the Lie algebra of $\mathcal{U}$. Since $\mathcal{U}$ is, by construction, a prounipotent affine group scheme, we can lift e to a morphism of affine group schemes

$$rg : \mathbb{G}_a \longrightarrow \mathcal{U} \tag{5.9}$$

from the additive group $\mathbb{G}_a$ to $\mathcal{U}$.

The morphism (5.9) represents the renormalization group in our context. The corresponding ambiguity is generated, as explained above in Theorem 4.4, by the absence of a canonical trivialization for the $\mathbb{G}_m$-bundle corresponding to integration in complexified dimensions around D.

The formulae considered in the previous sections still make sense in the universal case where $G^* = \mathcal{U}^*$, hence we can define the universal singular frame by the equality

$$\gamma(z, v) = Te^{-(1/z)\int_0^v u^Y(e)(du/u)} \in \mathcal{U}. \tag{5.10}$$

This is easily computed in terms of iterated integrals and one obtains the following expression.

Proposition 5.2. The universal singular frame is given by

$$\gamma(z,v) = \sum_{n\geq 0} \sum_{k_j>0} \frac{e(-k_1)e(-k_2)\cdots e(-k_n)}{k_1(k_1+k_2)\cdots(k_1+k_2+\cdots+k_n)} v^{\sum k_j} z^{-n}. \tag{5.11}$$

□

It is interesting to notice that exactly the same expression occurs in the local index formula of [12]. The renormalization group idea is also used in that context, in the case of higher poles in the dimension spectrum.

Adopting this universal singular frame in the dimensional regularization technique and the minimal subtraction scheme has the effect of removing all divergences. One obtains a finite theory, which depends only upon the choice of local trivialization of the principal $\mathbb{G}_m$-bundle B, whose base Δ is the space of complexified dimensions around D and whose fibers correspond to normalizations of the integral in complex dimensions, as used by physicists in the Dim-Reg scheme.

6 The classifying affine group scheme as a motivic Galois group

In this section we construct a category of equivalence classes of equisingular flat vector bundles. This allows us to reformulate the Riemann-Hilbert correspondence in terms of finite-dimensional linear representations of U^*. The relation to the formulation given in the previous section is given by passing to the finite-dimensional representations of the group G^*. Since G^* is an affine group scheme, there are enough such representations, and they are specified (cf. [13]) by assigning the following data:

(i) a graded vector space $E = \oplus_{n\in\mathbb{Z}} E_n$,

(ii) a graded representation π of G in E.

Notice that a graded representation of G in E can be equivalently described as a graded representation of $\mathfrak{g}$ in E. Moreover, since the Lie algebra $\mathfrak{g}$ is positively graded, both representations are compatible with the *weight* filtration given by

$$W^{-n}(E) = \oplus_{m\geq n} E_m. \tag{6.1}$$

At the group level, the corresponding representation in the associated graded

$$\mathrm{Gr}_n^W = W^{-n}(E)/W^{-n-1}(E) \tag{6.2}$$

is the identity.

We now proceed to construct a Tannakian category of equivalence classes of equisingular flat vector bundles, independent of the group G.

Definition 6.1. Let (E, W) be a filtered vector bundle with a given trivialization of the associated graded $\mathrm{Gr}^W(E)$.

(1) A W-connection on E is a connection ∇ on E, which is compatible with the filtration (i.e., restricts to all $W^k(E)$) and induces the trivial connection on the associated graded $\mathrm{Gr}^W(E)$.

(2) Two W-connections on E are W-equivalent if and only if there exists an automorphism of E preserving the filtration, inducing the identity on $\mathrm{Gr}^W(E)$, and conjugating the connections.

We now define the category $\mathcal{E}$ of equisingular flat bundles.

Let B be the principal $\mathbb{G}_m$-bundle considered in Section 4. The *objects* of $\mathcal{E}$ are the equivalence classes of pairs

$$\Theta = (E, \nabla), \tag{6.3}$$

where

(i) E is a $\mathbb{Z}$-graded finite-dimensional vector space,

(ii) ∇ is an equisingular flat W-connection on B^*, defined on the $\mathbb{G}_m$-equivariant filtered vector bundle $(\tilde{E}, W)$ induced by E with its weight filtration (6.1).

By construction, $\tilde{E}$ is the trivial bundle $B \times E$ endowed with the action of $\mathbb{G}_m$ given by the grading. The trivialization of the associated graded $\mathrm{Gr}^W(\tilde{E})$ is simply given by the identification with the trivial bundle with fiber E. The equisingularity of ∇ here means that it is $\mathbb{G}_m$-invariant and that all restrictions to sections σ of B with $\sigma(0) = y_0$ are W-equivalent.

We refer to such pairs $\Theta = (E, \nabla)$ as *flat equisingular bundles*. We only retain the datum of the W-equivalence class of the connection ∇.

Given two flat equisingular bundles Θ, Θ', we define the *morphisms*

$$T \in \mathrm{Hom}(\Theta, \Theta') \tag{6.4}$$

in the category $\mathcal{E}$ as linear maps $T : E \to E'$, compatible with the grading, fulfilling the condition that the following W-connections ∇_j, $j = 1, 2$, on $\tilde{E}' \oplus \tilde{E}$ are W-equivalent:

$$\nabla_1 = \begin{bmatrix} \nabla' & T\nabla - \nabla' T \\ 0 & \nabla \end{bmatrix} \sim \nabla_2 = \begin{bmatrix} \nabla' & 0 \\ 0 & \nabla \end{bmatrix}. \tag{6.5}$$

In (6.5), ∇_1 is obtained by conjugating ∇_2 by the unipotent matrix

$$\begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}. \tag{6.6}$$

This shows that condition (6.5) is well defined, independently of the choice of representatives for the connections ∇ and ∇'.

For $\Theta = (E, \nabla)$, we set $\omega(\Theta) = E$ and we view ω as a functor from the category of equisingular flat bundles to the category of vector spaces. We then have the following result.

Theorem 6.2. Let $\mathcal{E}$ be the category of equisingular flat bundles defined above.

(1) $\mathcal{E}$ is a Tannakian category.

(2) The functor ω is a fiber functor.

(3) $\mathcal{E}$ is equivalent to the category of finite-dimensional representations of U^*. □

In all the above we worked over $\mathbb{C}$, with convergent Laurent series. However, much of it can be rephrased with formal Laurent series. Since the universal singular frame is given in rational terms by Proposition 5.2, the result of Theorem 6.2 holds over any field of characteristic zero and in particular over $\mathbb{Q}$.

For each integer $n \in \mathbb{Z}$, we then define an object $\mathbb{Q}(n)$ in the category $\mathcal{E}$ of equisingular flat bundles as the trivial bundle given by a one-dimensional $\mathbb{Q}$-vector space placed in degree n, endowed with the trivial connection on the associated bundle over B.

For any flat equisingular bundle Θ, let

$$\omega_n(\Theta) = \mathrm{Hom}\left(\mathbb{Q}(n), \mathrm{Gr}^{W}_{-n}(\Theta)\right) \tag{6.7}$$

and notice that $\omega = \oplus \omega_n$.

The group U^* can be regarded as a motivic Galois group. One has, for instance, the following identification (see [13, 15]).

Proposition 6.3. There is a (noncanonical) isomorphism

$$U^* \sim G_{\mathcal{M}_T}(\mathcal{O}) \tag{6.8}$$

of the affine group scheme U^* with the motivic Galois group $G_{\mathcal{M}_T}(\mathcal{O})$ of the scheme S_4 of 4-cyclotomic integers. □

It is important here to stress the fact (cf. [13, Section 2.4]) that there is so far no "canonical" choice of a free basis in the Lie algebra of the above motivic Galois group so that the above isomorphism still requires making a large number of noncanonical choices. In particular, it is premature to assert that the above category of equisingular flat bundles is directly related to the category of 4-cyclotomic Tate motives. The isomorphism (6.8) does not determine the scheme S_4 uniquely. In fact, a similar isomorphism holds with S_3 the scheme of 3-cyclotomic integers.

On the other hand, when considering the category $\mathcal{M}_T$ in relation to physics, inverting the prime 2 is relevant to the definition of geometry in terms of K-homology, which is at the center stage in noncommutative geometry. We recall, in that respect, that it is only after inverting the prime 2 that (in sufficiently high dimension) a manifold structure on a simply connected homotopy type is determined by the K-homology fundamental class.

Moreover, passing from $\mathbb{Q}$ to a field with a complex place, such as the above cyclotomic fields k, allows for the existence of nontrivial regulators for all algebraic K-theory groups $K_{2n-1}(k)$. It is also noteworthy that algebraic K-theory and regulators have already appeared in the context of quantum field theory and noncommutative geometry in [7]. The appearance of multiple polylogarithms in the coefficients of divergences in quantum field theory, discovered by Broadhurst and Kreimer (see [2, 3]), as well as recent considerations of Kreimer on analogies between residues of quantum fields and variations of mixed Hodge-Tate structures associated to polylogarithms (cf. [18]), suggest the existence, for the above category of equisingular flat bundles, of suitable Hodge-Tate realizations given by a specific choice of quantum field theory.

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