Renormalization and Computation: Dyson–Schwinger equations and Information Algebras

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Ma148: Geometry of Information
Caltech, Fall 2021

This lecture based on:

- Colleen Delaney, Matilde Marcolli, *Dyson-Schwinger equations in the theory of computation*, in “Feynman amplitudes, periods and motives", pp.79–107, Contemp. Math., 648, Amer. Math. Soc., Providence, RI, 2015.
- M. Marcolli, N. Tedeschi, *Entropy algebras and Birkhoff factorization*, J. Geom. Phys. 97 (2015) 243–265
- Yuri Manin, *Renormalization and computation*, I and II, arXiv:0904.4921 and arXiv:0908.3430

Perturbative Quantum Field Theory

- Action functional in D dimensions

$$S(\phi) = \int \mathscr{L}(\phi) d^D x = S_0(\phi) + S_{int}(\phi)$$

- Lagrangian density

$$\mathscr{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathscr{L}_{int}(\phi)$$

- Perturbative expansion: Feynman rules and Feynman diagrams

$$S_{eff}(\phi) = S_0(\phi) + \sum_\Gamma \frac{\Gamma(\phi)}{\#\mathrm{Aut}(\Gamma)} \quad \text{(1PI graphs)}$$

- Generating functional $Z[J]$ of Green functions (source field J)

$$\frac{\delta^n Z}{\delta J(x_1)\cdots\delta J(x_n)}[0] = i^n Z[0]\langle\phi(x_1)\cdots\phi(x_n)\rangle$$

Algebraic renormalization in perturbative QFT

- A. Connes, D. Kreimer, *Renormalization in quantum field theory and the Riemann-Hilbert problem*, I and II, hep-th/9912092, hep-th/0003188
- A. Connes, M. Marcolli, *Renormalization, the Riemann-Hilbert correspondence, and motivic Galois theory*, hep-th/0411114
- K. Ebrahimi-Fard, L. Guo, D. Kreimer, *Integrable Renormalization II: the general case*, hep-th/0403118

Two step procedure:

• Regularization: replace divergent integral $U(\Gamma)$ by function with poles

• Renormalization: pole subtraction with consistency over subgraphs (Hopf algebra structure)

• Kreimer, Connes–Kreimer, Connes–M.: Hopf algebra of Feynman graphs and BPHZ renormalization method in terms of Birkhoff factorization and differential Galois theory

• Ebrahimi-Fard, Guo, Kreimer: algebraic renormalization in terms of Rota–Baxter algebras

Connes–Kreimer Hopf algebra $\mathcal{H} = \mathcal{H}(\mathcal{T})$ (depends on theory)

- Free commutative algebra in generators Γ 1PI Feynman graphs
- Grading: loop number (or internal lines)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

- Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

- Antipode: inductively

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

Extended to gauge theories (van Suijlekom): Ward identities as Hopf ideals

Rota–Baxter algebra of weight $\lambda = -1$

$\mathscr{R}$ commutative unital algebra
$T : \mathscr{R} \to \mathscr{R}$ linear operator with

$$T(x)T(y) = T(xT(y)) + T(T(x)y) + \lambda T(xy)$$

• Example: $T =$ projection onto polar part of Laurent series

• T determines splitting $\mathscr{R}_+ = (1 - T)\mathscr{R}$, $\mathscr{R}_- =$ unitization of $T\mathscr{R}$; both $\mathscr{R}_\pm$ are algebras

Feynman rule

• $\phi : \mathscr{H} \to \mathscr{R}$ *commutative algebra homomorphism*

from CK Hopf algebra $\mathscr{H}$ to Rota–Baxter algebra $\mathscr{R}$ weight -1

$$\phi \in \mathrm{Hom}_{\mathrm{Alg}}(\mathscr{H}, \mathscr{R})$$

• Note: ϕ does *not know* that $\mathscr{H}$ Hopf and $\mathscr{R}$ Rota-Baxter, only commutative algebras

• Birkhoff factorization $\exists \phi_\pm \in \mathrm{Hom}_{\mathrm{Alg}}(\mathscr{H}, \mathscr{R}_\pm)$

$$\phi = (\phi_- \circ S) \star \phi_+$$

where $\phi_1 \star \phi_2(X) = \langle \phi_1 \otimes \phi_2, \Delta(X) \rangle$

• Connes-Kreimer inductive formula for Birkhoff factorization:

$$\phi_-(X) = -T(\phi(X) + \sum \phi_-(X')\phi(X''))$$

$$\phi_+(X) = (1 - T)(\phi(X) + \sum \phi_-(X')\phi(X''))$$

where $\Delta(X) = 1 \otimes X + X \otimes 1 + \sum X' \otimes X''$

• Recovers what known in physics as BPHZ renormalization procedure in physics

• Bogolyubov-Parshchuk preparation

$$\tilde{\phi}(x) = \phi(x) + \sum \phi_-(x')\phi(x'')$$

Hopf algebra of rooted trees

- Rooted tree τ: data $(F_\tau, V_\tau, v_\tau, \delta_\tau, j_\tau)$
  - F_τ set of half-edges (flags)
  - V_τ set of vertices
  - distinguished $v_\tau \in V_\tau$ (the root)
  - boundary map $\partial_\tau : F_\tau \to V_\tau$
  - involution $j_\tau : F_\tau \to F_\tau$, $j_\tau^2 = 1$ gluing half-edges to edges
  - E_τ internal edges, E_τ^{ext} external edges (fixed by involution)

Orientation: root vertex as output, all edges oriented along unique path to root

Decorations: $\phi_V : V_\tau \to \mathscr{D}_V$ labels of vertices, $\phi_F : F_\tau \to \mathscr{D}_F$ labels of flags (matched by involution)

admissible cuts

• admissible cuts C of τ modify involution j_τ cutting a subset of internal edges into two flags f_i, f_i', so that every oriented path in τ from leaf to root contains at most one cut edge

• New graph is a forest

$$C(\tau) = \rho_C(\tau) \amalg \pi_C(\tau)$$

rooted tree $\rho_C(\tau)$; forest $\pi_C(\tau) = \amalg_i \pi_{C,i}(\tau)$, each tree $\pi_{C,i}(\tau)$ with single output (new roots)

Hopf algebras

• $\mathscr{H}^{nc}$ *noncommutative* Hopf algebra of *planar* rooted trees: free algebra generated by planar rooted trees, coproduct

$$\Delta(\tau) = \tau \otimes 1 + 1 \otimes \tau + \sum_C \pi_C(\tau) \otimes \rho_C(\tau)$$

grading by number of vertices, antipode

$$S(x) = -x - \sum S(x')x'', \quad \text{for } \Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$$

x', x'' lower order terms

• $\mathscr{H}$ *commutative* Hopf algebra of (planar) rooted trees: free commutative (polynomial) algebra generated by rooted trees, same form of coproduct, grading and antipode

• in Connes–Kreimer setting can equivalently work with Hopf algebra of rooted trees decorated by Feynman graphs or with Hopf algebra of Feynman graphs (coproduct: subgraphs and quotient graphs)

Dyson–Schwinger equations in QFT

• Equations of motion for Green functions (Euler–Lagrange equations)
• Infinite system of coupled differential equations
• obtained as formal Taylor series expansion at $J = 0$ of DS equation in the generating function $Z[J]$

$$\frac{\delta S}{\delta \phi(x)} \left[-i \frac{\delta}{\delta J} \right] Z[J] + J(x) Z[J] = 0$$

• in the Hopf algebraic approach to QFT, can lift the DS equations to the combinatorial level

Combinatorial Dyson–Schwinger equations

- C. Bergbauer and D. Kreimer, *Hopf algebras in renormalization theory: locality and Dyson-Schwinger equations from Hochschild cohomology*, hep-th/0506190
- K. Yeats, *Rearranging Dyson-Schwinger Equations*, AMS 2011.
- L. Foissy, *Systems of Dyson–Schwinger equations*, arXiv:0909.0358

Dyson–Schwinger equations and Hopf subalgebras

• If grafting operator satisfies *cocycle condition*, then solutions of Dyson–Schwinger equations form a *Hopf subalgebra*

Renormalization and Computation (Manin)
proposal for a “renormalization of the halting problem"

• Idea: treat noncomputable functions like infinities in QFT

• Renormalization = extraction of finite part from divergent Feynman integrals; extraction of “computable part" from noncomputables

• First step: build a Hopf algebra (flow charts, partial recursive functions) and a Feynman rule that detects the presence of noncomputability (infinities)

• Second step: BPHZ type subtraction procedure with values in a min-plus or max-plus algebra (computing time, memory size)

• Third step: meaning of the “renormalized part" and of the “divergences part" of the Birkhoff factorization in terms of theory of computation

Primitive recursive functions

- generated by *basic functions*
  - Successor $s : \mathbb{N} \to \mathbb{N}$, $s(x) = x + 1$;
  - Constant $c^n : \mathbb{N}^n \to \mathbb{N}$, $c^n(x) = 1$ (for $n \geq 0$);
  - Projection $\pi_i^n : \mathbb{N}^n \to \mathbb{N}$, $\pi_i^n(x) = x_i$ (for $n \geq 1$);
- with *elementary operations*
  - Composition
  - Bracketing
  - Recursion

Elementary operations:

- Composition $\mathfrak{c}_{(m,m,p)}$: for $f : \mathbb{N}^m \to \mathbb{N}^n$, $g : \mathbb{N}^n \to \mathbb{N}^p$,

$$g \circ f : \mathbb{N}^m \to \mathbb{N}^p, \quad \mathscr{D}(g \circ f) = f^{-1}(\mathscr{D}(g));$$

- Bracketing $\mathfrak{b}_{(k,m,n_i)}$: for $f_i : \mathbb{N}^m \to \mathbb{N}^{n_i}$, $i = 1, \dots, k$,

$$f = (f_1, \dots, f_k) : \mathbb{N}^m \to \mathbb{N}^{n_1+\cdots+n_k}, \quad \mathscr{D}(f) = \mathscr{D}(f_1) \cap \cdots \cap \mathscr{D}(f_k);$$

- Recursion $\mathfrak{r}_n$: for $f : \mathbb{N}^n \to \mathbb{N}$ and $g : \mathbb{N}^{n+2} \to \mathbb{N}$,

$$h(x_1, \dots, x_n, 1) := f(x_1, \dots, x_n),$$

$$h(x_1, \dots, x_n, k+1) := g(x_1, \dots, x_n, k, h(x_1, \dots, x_n, k)), \quad k \geq 1,$$

where recursively $(x_1, \dots, x_n, 1) \in \mathscr{D}(h)$ iff $(x_1, \dots, x_n) \in \mathscr{D}(f)$ and $(x_1, \dots, x_n, k+1) \in \mathscr{D}(h)$ iff $(x_1, \dots, x_n, k, h(x_1, \dots, x_n, k)) \in \mathscr{D}(g)$.

Manin's Hopf algebra of flow charts

• planar labelled rooted trees (bracketing and recursion are ordered: need planar)

• label set of vertices $\mathscr{D}_V = \{\mathfrak{c}_{(m,n,p)}, \mathfrak{b}_{(k,m,n_i)}, \mathfrak{r}_n\}$ (composition, bracketing, recursion)

• label set of flags $\mathscr{D}_F$ primitive recursive functions

• *admissible* labelings:

- $\phi_V(v) = \mathfrak{c}_{(m,n,p)}$: v valence 3; labels $h_1 = \phi_F(f_1)$, $h_2 = \phi_F(f_2)$ incoming flags with domains and ranges $h_1 : \mathbb{N}^m \to \mathbb{N}^n$ and $h_2 : \mathbb{N}^n \to \mathbb{N}^p$; outgoing flag composition $h_2 \circ h_1 = \mathfrak{c}_{(m,n,p)}(h_1, h_2)$.
- $\phi_V(v) = \mathfrak{r}_n$: v valence 3; labels $h_1 = \phi_F(f_1)$, $h_2 = \phi_F(f_2)$ incoming flags with domains and ranges $h_1 : \mathbb{N}^n \to \mathbb{N}$ and $h_2 : \mathbb{N}^{n+2} \to \mathbb{N}$, outgoing flag recursion $h = \mathfrak{r}_n(h_1, h_2)$.
- $\phi_V(v) = \mathfrak{b}_{(k,m,n_i)}$: v must have valence $k+1$; labels $h_i = \phi_F(f_i)$ incoming flags with domain $\mathbb{N}^m$; outgoing flag bracketing $f = (f_1, \ldots, f_k) = \mathfrak{b}_{(k,m,n_i)}(f_1, \ldots, f_k)$.

• Coproduct, grading, antipode from Hopf algebra of rooted trees

Variants on the Hopf algebra of flow charts

• noncommutative Hopf algebra $\mathcal{H}^{nc}_{\text{flow},\mathcal{P}}$

• Hopf algebra with only vertex labels $\mathcal{H}^{nc}_{\text{flow},\mathcal{V}}$

• Use only binary operations (valence 3 vertices): express bracketing as a composition of binary operations

$$\mathfrak{b}_{(k,m,n_i)} = \mathfrak{b}_{(2,m,n_1,n_2+\cdots+n_k)} \circ \cdots \circ \mathfrak{b}_{(2,m,n_{k-1},n_k)}$$

• Extend composition and recursion to k-ary operations

- k-ary compositions $\mathfrak{c}_{(k,m,n_i)}(h_i) = h_k \circ \cdots \circ h_1$ of functions $h_i : \mathbb{N}^{n_{i-1}} \to \mathbb{N}^{n_i}$, for $i = 1, \ldots, k$, with $n_0 = m$
- $(k+1)$-ary recursions with k initial conditions:

$$h(x_1, \ldots, x_n, 1) = h_1(x_1, \ldots, x_n), \ldots$$
$$h(x_1, \ldots, x_n, k) = h_k(x_1, \ldots, x_n),$$
$$h(x_1, \ldots, x_n, k+\ell) =$$
$$h_{k+1}(x_1, \ldots, x_n, h_1(x_1, \ldots, x_n), \ldots, h_k(x_1, \ldots, x_n), k+\ell-1),$$
for $\ell \geq 1$

Insertion and Hochschild 1-cocycles

• T =forest: *grafting operator* $B^+_\delta(T)$ = sum of planar trees with new root vertex added with incoming flags equal number of trees in T and a single output flag and decoration $\delta \in \{\mathfrak{b}, \mathfrak{c}, \mathfrak{r}\}$

• cocycle condition:

$$\Delta B^+_\delta = (id \otimes B^+_\delta)\Delta + B^+_\delta \otimes 1$$

equivalent to $\tilde{\Delta} B^+_\delta = (id \otimes B^+_\delta)\tilde{\Delta} + id \otimes B^+_\delta(1)$ with $\tilde{\Delta}(x) := \sum x' \otimes x''$ (non-primitive part) and $B^+_\delta(1) = v_\delta$ (single vertex, label δ): first term admissible cuts root vertex attached to $\rho_C(T)$, second term admissible cut separating root vertex.

• cocycle condition requires same type of label ($\mathfrak{b}$, $\mathfrak{c}$, or $\mathfrak{r}$) for all vertices of arbitrary valence: use version $\mathcal{H}^{nc}_{\mathrm{flow},\mathcal{V}'}$ with k-ary operations

Systems of Dyson–Schwinger equations (Foissy)

• non-constant formal power series in three variables $X = (X_\delta)$

$$F_\delta(X) = \sum_{k_1,k_2,k_3} a^{(\delta)}_{k_1,k_2,k_3} X_{\mathfrak{b}}^{k_1} X_{\mathfrak{c}}^{k_2} X_{\mathfrak{r}}^{k_3}$$

• associated system of Dyson–Schwinger equations

$$X_\delta = B_\delta^+(F_\delta(X))$$

• unique solution $X_\delta = \sum_\tau x_\tau \, \tau$ (sum over planar rooted trees root decoration δ)

$$x_\tau = \Big(\prod_{k=1}^{3} \frac{(\sum_{l=1}^{m_k} p_{\delta,l})!}{\prod_{l=1}^{m_k} p_{\delta,l}!}\Big) a^{(\delta)}_{\sum_{k=1}^3 p_{1,k}, \sum_{k=1}^3 p_{2,k}, \sum_{k=1}^3 p_{3,k}} x_{\tau_{1,1}}^{p_{1,1}} \cdots x_{\tau_{3,m_3}}^{p_{3,m_3}}$$

when

$$\tau = B^+(\tau_{1,1}^{p_{1,1}} \cdots \tau_{1,m_1}^{p_{1,m_1}} \cdots \tau_{3,1}^{p_{3,1}} \cdots \tau_{3,m_3}^{p_{3,m_3}})$$

Dyson–Schwinger equations and Hopf subalgebras

(Bergbauer–Kreimer)

• Dyson–Schwinger equations in a Hopf algebra of the form

$$X = 1 + \sum_{n=1}^{\infty} c_n B_\delta^+(X^{n+1})$$

• associative algebra $\mathscr{A}$ (subalgebra of $\mathscr{H}$) generated by components x_n of unique solution of DS equation

• using cocycle condition for B_δ^+ get

$$\Delta(x_n) = \sum_{k=0}^{n} \Pi_k^n \otimes x_k, \quad \text{where} \quad \Pi_k^n = \sum_{j_1+\cdots+j_{k+1}=n-k} x_{j_1} \cdots x_{j_{k+1}}$$

$\Rightarrow$ Hopf subalgebra

• generalized by Foissy for broader class of DS equations in Hopf algebras, including systems

Variant: Hopf ideals

- DS equation $X = 1 + \sum_{n=1}^{\infty} c_n \, B_\delta^+(X^{n+1})$
- *ideal* $\mathscr{I}$ generated by the components x_n (with $n \geq 1$) of solution
- cocycle condition for $B_\delta^+ \Rightarrow \mathscr{I}$ Hopf ideal

elements of $\mathscr{I}$ finite sums $\sum_{m=1}^{M} h_m x_{k_m}$ with $h_m \in \mathscr{H}$ and x_k components of unique solution of DS equation

Hopf ideal condition: $\Delta(\mathscr{I}) \subset \mathscr{I} \otimes \mathscr{H} \oplus \mathscr{H} \otimes \mathscr{I}$

coproduct $\Delta(x_k)$: primitive part $1 \otimes x_k + x_k \otimes 1$ in $\mathscr{H} \otimes \mathscr{I} \oplus \mathscr{I} \otimes \mathscr{H}$; other terms in $\mathscr{I} \otimes \mathscr{I}$, so coproducts $\Delta(h_m x_{k_m})$ in $\mathscr{H} \otimes \mathscr{I} \oplus \mathscr{I} \otimes \mathscr{H}$.

$\Rightarrow$ *quotient Hopf algebra* $\mathscr{H}_{\mathscr{I}} = \mathscr{H}/\mathscr{I}$

Note: commutative Hopf algebra; if noncommutative use two-sided ideals

Yanofsky's Galois theory of algorithms

- Yanofsky proposed equivalence relations on flowcharts = "implementing the same algorithm"
- algorithm as intermediate level between the flow chart (= labelled planar rooted tree) and the primitive recursive functions
- obtain "Galois correspondence"
- resulting automorphism groups are products of symmetric groups
- but there are *problems*:

Example: (Joachim Kock)

fix function f: infinitely many programs computing it; "Galois group" is symmetry group of that set; subgroup S_3 (or C_3) permuting (cyclically) three of the programs fixing others: same orbits but different groups

Proposal for a different form of Galois theory of algorithms

• *suggestion*: take the Hopf algebra structure into account in defining relations (= relations should be Hopf ideals)

• instead of the kind of groups described by Yanofsky, find a sub-group scheme $G_{\mathscr{I}} \subset G_{\text{flow}}$ corresponding to the quotient $\mathscr{H}_{\mathscr{I}} = \mathscr{H}/\mathscr{I}$, with G_{flow} group scheme dual to Hopf algebra $\mathscr{H}$ of flow charts

• in particular get a $G_{\mathscr{I}}$ from a Dyson–Schwinger equation (system)

• the groups appearing in this way have a structure more similar to the “Galois groups" playing a role in QFT

From Hopf algebras to operads

- *operad of flow charts* $\mathscr{O}_{\mathrm{flow},\mathscr{V}'}$
  - $\mathscr{O}(n) = \mathbb{K}$-vector space spanned by labelled planar rooted trees with n incoming flags
  - operad composition operations

$$\circ_{\mathscr{O}} : \mathscr{O}(n) \otimes \mathscr{O}(m_1) \otimes \cdots \otimes \mathscr{O}(m_n) \to \mathscr{O}(m_1 + \cdots + m_n)$$

    on generators $\tau \otimes \tau_1 \otimes \cdots \otimes \tau_n$ by grafting output flag of τ_i to the i-th input flag of τ

Dyson–Schwinger equations in operads

• formal series $P(t) = 1 + \sum_{k=1}^{\infty} a_k t^k$

• collection $\beta = (\beta_n)$ with $\beta_n \in \mathcal{O}(n)$

• Dyson–Schwinger equation:

$$X = \beta(P(X))$$

with $X = \sum_k x_k$ a formal sum of $x_k \in \mathcal{O}(k)$

• *self-similarity* with respect to $X \mapsto \beta(P(X))$

• right-hand-side of equation: $\beta(P(X))_1 = 1 + \beta_1 \circ x_1$, with 1 identity in $\mathcal{O}(1)$, and for $n \geq 2$

$$\beta(P(X))_n = \sum_{k=1}^{n} \sum_{j_1+\cdots+j_k=n} a_k \;\; \beta_k \circ (x_{j_1} \otimes \cdots \otimes x_{j_k})$$

with $x_{j_1} \otimes \cdots \otimes x_{j_k} \in \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k)$, composition $\beta_k \circ_{\mathcal{O}} (x_{j_1} \otimes \cdots \otimes x_{j_k}) \in \mathcal{O}(n)$, with $j_1 + \cdots + j_k = n$

Inductive construction of solutions

• $\mathcal{O} = \mathcal{O}_{\text{flow},\mathcal{V}'}$ operad of flow charts

• assume $a_1\beta_1 \neq 1 \in \mathcal{O}(1)$

• then operadic Dyson–Schwinger equation $X = \beta(P(X))$ has unique solution $X \in \prod_{n\geq 1} \mathcal{O}(n)$ given inductively by

$$(1 - a_1\beta_1) \circ x_{n+1} = \sum_{k=2}^{n+1} \sum_{j_1+\cdots j_k=n+1} a_k \ \beta_k \circ (x_{j_1} \otimes \cdots \otimes x_{j_k})$$

• $\mathcal{O}_{\beta,P}(n) = \mathbb{K}$-linear span of all compositions $x_k \circ (x_{j_1} \otimes \cdots \otimes x_{j_k})$ for $k = 1, \ldots, n$ and $j_1 + \cdots + j_k = n$, with x_k coordinates of solution $X \Rightarrow \mathcal{O}_{\beta,P}(n)$ is a sub-operad

• choosing $a_1 \neq 1$ and β_k single vertex k incoming flags, label δ gives operadic version of DS equation with B^+_δ, but more general DS equations in operadic setting (without cocycle condition)

Operads and Properads

• Manin: extend Hopf algebra of flow charts to graphs (not trees) with acyclic orientations

• replace operad with *properad*: compositions grafting outputs and inputs of acyclic graphs

• *properad* (Valette): operations with varying numbers of inputs and outputs labelled by connected acyclic graphs; (*operads*: trees varying number of inputs and single output; *props*: allow disconnected graphs)

• composition operations: m inputs, n outputs

$$\mathscr{P}(m,n)\otimes\mathscr{P}(j_1,k_1)\otimes\cdots\otimes\mathscr{P}(j_\ell,k_\ell)\to\mathscr{P}(j_1+\cdots+j_\ell,n)$$

for $k_1+\cdots+k_\ell=m$

• $\mathscr{P}_{\text{flow},\mathscr{V}'}$ properad of flow charts

• $\mathscr{P}(m, n) = \mathbb{K}$-vector space spanned by planar connected directed (acyclic) graphs with m incoming flags and n outgoing flags

• vertices decorated by operations including $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{r}$ (m inputs, one output) and *macros* with m inputs and n outputs

Dyson–Schwinger equations in properads

- formal power series $P(t) = 1 + \sum_k a_k t^k$
- collection $\beta = (\beta_{m,n})$ with $\beta_{m,n} \in \mathscr{P}(m,n)$
- DS equation $X = \beta(P(X))$ (self-similarity)
- in components

$$\beta(P(X))_{m,n} = \sum_{k=1}^{m} a_k \sum_{\substack{j_1+\ldots j_k=m \\ i_1+\cdots i_k=\ell}} \beta_{\ell,n} \circ (x_{j_1,i_1} \otimes \cdots \otimes x_{j_k,i_k})$$

Construction of solutions in properads

• transformations $\Lambda_n = \Lambda_n(a,\beta)$

$\Lambda_n(a,\beta) : \oplus_{k=1}^n \mathscr{P}(n,k) \to \oplus_{k=1}^n \mathscr{P}(n,k)$, with $\Lambda_n(a,\beta)_{ij} = a_j \beta_{j,i}$

• assume $I - \Lambda_n(a,\beta)$ invertible for all n (not always satisfied)

• then unique solution to DS equation $X = \beta(P(X))$

• inductive construction: $x_{1,1} = \Lambda_1^{-1}$ and for $m < n$

$$x_{m,n} = \sum_{k=1}^{m} a_k \beta_{k,n} \circ \left(\sum_{\ell=1}^{k} \sum_{\substack{j_1+\cdots+j_\ell=m \\ i_1+\cdots+i_\ell=k}} x_{j_1,i_1} \otimes \cdots \otimes x_{j_\ell,i_\ell} \right)$$

remaning components $m \geq n$ determined by

$$Y_n(x) = (I - \Lambda_n)^{-1} \Lambda_n V^{(n)}(x)$$

with $Y_n(x)^t = (x_{n,1}, \ldots, x_{n,n})$ and $V^{(n)}(x)^t = (V^{(n)}(x)_j)_{j=1,\ldots,n}$

$$V^{(n)}(x)_j = \sum_{k=2}^{n} \sum_{\substack{r_1+\cdots+r_k=n \\ s_1+\cdots+s_k=j}} x_{r_1,s_1} \otimes \cdots \otimes x_{r_k,s_k}$$

Partial recursive functions and the Hopf algebra

• enlarge from primitive recursive to partial recursive: same elementary operations $\mathfrak{c}, \mathfrak{b}, \mathfrak{r}$ of composition, bracketing and recursion but additional μ operation

• μ operation: input function $f : \mathbb{N}^{n+1} \to \mathbb{N}$, output

$$h : \mathbb{N}^n \to \mathbb{N}, \quad h(x_1, \ldots, x_n) = \min\{x_{n+1} \mid f(x_1, \ldots, x_{n+1}) = 1\},$$

with domain $\mathscr{D}(h)$ those $(x_1, \ldots, x_n)$ such that $\exists x_{n+1} \geq 1$

$$f(x_1, \ldots, x_{n+1}) = 1, \text{ with } (x_1, \ldots, x_n, k) \in \mathscr{D}(f), \forall k \leq x_{n+1}$$

• Church's thesis: get all semi-computable functions, for which $\exists$ program computing $f(x)$ for $x \in \mathscr{D}(f)$ and computed zero or never stops for $x \notin \mathscr{D}(f)$

• Hopf algebra: additional vertex decoration by μ operations, extended to arbitrary valence by combining with bracketing; edge decorations by partial recursive functions

Feynman rule for computation (Manin)

• $\mathscr{B}$ algebra of functions $\Phi : \mathbb{N}^k \to \mathscr{M}(D)$ from $\mathbb{N}^k$, for some k, to algebra $\mathscr{M}(D)$ of analytic functions in unit disk $D = \{z \in \mathbb{C} \,:\, |z| < 1\}$.

• Rota–Baxter operator T on $\mathscr{B}$ componentwise projection onto polar part at $z = 1$

• For any tree τ that computes f set

$$\Phi_\tau(\underline{k}, z) = \Phi(\underline{k}, f, z) := \sum_{n \geq 0} \frac{z^n}{(1 + n\bar{f}(\underline{k}))^2}$$

$\bar{f} : \mathbb{N}^m \to \mathbb{Z}_{\geq 0}$ computes $f(x)$ at $x \in \mathscr{D}(f)$ and 0 at $x \notin \mathscr{D}(f)$.

• $\Phi_\tau(\underline{k}, z)$ pole at $z = 1$ iff $\underline{k} \notin \mathscr{D}(f)$

• this Φ is algebraic Feynman rule: commutative algebra homomorphism from enlarged Hopf algebra of flow charts to Rota–Baxter algebra $\mathscr{B}$

apply BPHZ

• negative part of Birkhoff factorization becomes

$$\Phi_-(\underline{k}, f_\tau, z) = -T(\Phi(\underline{k}, f_\tau, z) + \sum_C \Phi_-(\underline{k}, f_{\pi_C(\tau)}, z)\Phi(\underline{k}, f_{\rho_C(\tau)}, z))$$

• Note: $f = f_\tau$ label of outgoing flag of τ: then $f_{\rho_C(\tau)} = f_\tau$

$$\Phi_-(\underline{k}, f_\tau, z) = -T\left(\Phi(\underline{k}, f_\tau, z)(1 + \Phi_-(\underline{k}, \sum_C f_{\pi_C(\tau)}, z))\right)$$

• What is happening here? Like in QFT, looking not only at "divergences" of program τ but also of *all subprograms* $\pi_C(\tau)$ and $\rho_C(\tau)$ determined by admissible cuts (the problem of subdivergences in renormalization)

Why subdivergences in computation?

• $\Phi_-(\underline{k}, f_\tau, z)$ detects not only if τ has infinities but if any subroutine does

• Note: $\Phi(\underline{k}, f_\tau, z)$ only depends on $f = f_\tau$ not on τ, but $\Phi_-(\underline{k}, f_\tau, z)$ really *depends on* τ

• Unlike QFT there are programs without divergences that do have subdivergences

• *Example:* (Joachim Kock)

identity function computed as composite of successor function followed by partial predecessor function $\mu(|y + 1 - x|)$ (undefined at 0, and $x - 1$ for $x > 0$), τ with a $\mathfrak{c}$ node and a μ node

Renormalized part   What does it measure?

$$\Phi_+(\underline{k}, f_\tau, z) = (1-T)(\Phi(\underline{k}, f_\tau, z) + \sum_C \Phi_-(\underline{k}, f_{\pi_C(\tau)}, z)\Phi(\underline{k}, f_{\rho_C(\tau)}, z))$$

- Main question: is there a new $f_{\rm ren}$, now *primitive recursive*, such that $\Phi_+(\underline{k}, f_\tau, z) = \Phi(\underline{k}, f_{\rm ren}, z)$?
- in general not true simply as stated, but in QFT there is an *equivalence relation* on Feynman rules and renormalized values, a kind of gauge transformation by germs of holomorphic functions (Connes–Marcolli): correct statement of question is up to such an equivalence?
- *Useful viewpoint*: every partial recursive function can be computed by a Hopf-primitive program: Kleene normal form as μ of a total function

Min-Plus Algebra (same setting used for Tropical Semirings)

min-plus (or tropical) semiring $\mathbb{T} = \mathbb{R} \cup \{\infty\}$

• operations $\oplus$ and $\odot$

$$x \oplus y = \min\{x, y\} \quad \text{with identity } \infty$$

$$x \odot y = x + y \quad \text{with identity } 0$$

• operations $\oplus$ and $\odot$ satisfy:

- associativity
- commutativity
- left/right identity
- distributivity of product $\odot$ over sum $\oplus$

Note: can work equivalently with $(\mathbb{R} \cup \{\infty\}, \min, +)$ or with $(\mathbb{R}_+, \max, *)$ isomorphic under $-\log$ map

Thermodynamic semirings $\mathbb{T}_{\beta,S} = (\mathbb{R} \cup \{\infty\}, \oplus_{\beta,S}, \odot)$

• deformation of the tropical addition $\oplus_{\beta,S}$

$$x \oplus_{\beta,S} y = \min_p \{px + (1-p)y - \frac{1}{\beta} S(p)\}$$

β thermodynamic inverse temperature parameter
$S(p) = S(p, 1-p)$ binary information measure, $p \in [0,1]$

• for $\beta \to \infty$ (zero temperature) recovers unperturbed idempotent addition $\oplus$

• multiplication $\odot = +$ is undeformed

von Neumann entropy and the tropical trace

- convex set of density matrices

$$\mathcal{M}^{(N)} = \{\rho \in M_{N\times N}(\mathbb{C}) \,|\, \rho^* = \rho,\ \rho \geq 0,\ \mathrm{Tr}(\rho) = 1\}$$

- von Neumann entropy

$$\mathcal{N}(\rho) = -\mathrm{Tr}(\rho \log \rho), \quad \text{for} \quad \rho \in \mathcal{M}^{(N)}$$

Shannon entropy in diagonal case

- matrices $M_{N\times N}(\mathbb{T})$ over $\mathbb{T} = (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$

$$(A \oplus B)_{ij} = \min\{A_{ij}, B_{ij}\} \ \text{ and } \ (A \odot B)_{ij} = \oplus_k A_{ik} \odot B_{kj} = \min_k \{A_{ik} + B_{kj}\}$$

- tropical trace $\mathrm{Tr}^{\oplus}(A) = \min_i \{A_{ii}\}$
- also consider

$$\widetilde{\mathrm{Tr}}^{\oplus}(A) := \min_{U \in U(N)} \min_i \{(UAU^*)_{ii}\} \leq \mathrm{Tr}^{\oplus}(A)$$

Entropical trace: thermodynamic deformation of tropical trace

$$\mathrm{Tr}^{\oplus}_{\beta,S}(A) := \min_{\rho\in\mathcal{M}^{(N)}} \{\mathrm{Tr}(\rho A) - \beta^{-1} S(\rho)\}$$

Tr in the right-hand-side is the *ordinary* trace

• in particular $S(\rho) = \mathcal{N}(\rho)$ von Neumann entropy, but also other entropy functionals (e.g. quantum versions of Rényi and Tsallis)

• Note: $\mathrm{Tr}(\rho A) = \langle A \rangle$ expectation value of observable A

• zero temperature limit

$$\lim_{\beta\to\infty} \mathrm{Tr}^{\oplus}_{\beta,S}(A) = \widetilde{\mathrm{Tr}}^{\oplus}(A)$$

Kullback–Leibler divergence and von Neumann entropical trace

• relative entropy (Kullback–Leibler divergence)

$$S(\rho||\sigma) = \mathrm{Tr}(\rho(\log\rho - \log\sigma))$$

• von Neumann deformation and relative entropy

(for $A = A^*$, $A \geq 0$)

$$\mathrm{Tr}(\rho A) - \beta^{-1}\mathcal{N}(\rho) = \frac{1}{\beta} S(\rho||\sigma_{\beta,A}) - \frac{1}{\beta}\log Z_A(\beta)$$

$$\sigma_{\beta,A} = \frac{e^{-\beta A}}{Z_A(\beta)} \quad \text{with} \quad Z_A(\beta) = \mathrm{Tr}(e^{-\beta A})$$

• von Neumann entropical trace (for $A = A^*$, $A \geq 0$)

$$\mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(A) = -\frac{\log Z_A(\beta)}{\beta}$$

with $Z_A(\beta) = \mathrm{Tr}(e^{-\beta A})$: rhs above is Helmholtz free energy

- if for $A = A^*$, $A \geq 0$ is direct sum of two matrices A_1 and A_2

$$\mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(A) = \mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(A_1) \odot \mathrm{Tr}^{\oplus}_{\beta,\mathcal{N}}(A_2)$$

$$= -\beta^{-1}\left(\log \mathrm{Tr}(e^{-\beta A_1}) + \log \mathrm{Tr}(e^{-\beta A_2})\right)$$

Relative entropies

- The quantum relative entropy: for $\rho, \sigma \in \mathcal{M}^{(N)}$

$$S(\rho||\sigma) = \mathrm{Tr}(\rho(\log \rho - \log \sigma))$$

- The Belavkin–Staszewski relative entropy: for $\rho, \sigma \in \mathcal{M}^{(N)}$

$$S_{BS}(\rho||\sigma) = \mathrm{Tr}(\rho \log(\rho^{1/2}\sigma^{-1}\rho^{1/2}))$$

- The Umegaki deformed relative entropy: for $\rho, \sigma \in \mathcal{M}^{(N)}$

$$S_\alpha(\rho||\sigma) = \frac{4}{1-\alpha^2}\mathrm{Tr}((I - \sigma^{(\alpha+1)/2}\rho^{(\alpha-1)/2})\rho)$$

- given a fixed density matrix σ set $S_\sigma(\rho) = S(\rho||\sigma)$, so that

$$\mathrm{Tr}^{\oplus}_{\beta,S_\sigma}(A) = \min_{\rho\in\mathcal{M}^{(N)}} \{\mathrm{Tr}(\rho A) - \beta^{-1}S(\rho||\sigma)\}$$

deformation of states on C^*-algebras

- states $\mathscr{M} = \{\varphi : \mathscr{A} \to \mathbb{C} \text{ linear} \,|\, \varphi(1) = 1 \text{ and } \varphi(a^* a) \geq 0\}$
- relative entropy of states: in case of Gibbs states $\varphi(a) = \tau(a\xi)$, $\psi(a) = \tau(a\eta)$

$$S(\varphi||\psi) = \tau(\xi(\log \xi - \log \eta))$$

in general more complicated

- thermodynamic deformation of a state $\psi \in \mathscr{M}$

$$\psi_{\beta,S}(a) = \min_{\varphi \in \mathscr{M}} \{\varphi(a) + \beta^{-1} S(\varphi||\psi)\}$$

Example:

• noncommutative torus: C^*-algebra generated by two unitaries U, V with $VU = e^{2\pi i\theta} UV$

• canonical trace, $\tau(U^n V^m) = 0$ for $(n, m) \neq (0, 0)$ and $\tau(1) = 1$

• Gibbs states $\varphi(a) = \tau(a\xi)$ positive elements $\xi \in \mathscr{A}_\theta$

• thermodynamic deformation of canonical trace

$$\tau_{\beta,S}(a) = \min_{\varphi \in \mathscr{M}_\tau} \{\varphi(a) + \beta^{-1} S(\varphi || \tau)\}$$

• KMS state $\varphi_{\beta,a}(b) = \frac{\tau(be^{-\beta a})}{\tau(e^{\beta a})}$ of time evolution $\sigma_t(b) = e^{ita} b e^{-ita}$

$$\tau_{\beta,S}(a) = \min_{\varphi \in \mathscr{M}_\tau} \{\beta^{-1} S(\varphi || \varphi_{\beta,a}) - \beta^{-1} \log \tau(e^{-\beta a})\} = -\beta^{-1} \log \tau(e^{-\beta a})$$

Helmholtz free energy

• $\lim_{\beta\to\infty} \tau_{\beta,S}(a)$ a notion of “tropicalization" of the von Neumann trace τ of the NC torus

Symbolic dynamics and Dynamical and Topological Entropy

- locally compact Hausdorff space X, with a dynamical system $\sigma : X \to X$ (continuous function)
- X shift space of sequences $w = w_0 w_1 \ldots w_i w_{i+1} \ldots$ in a finite alphabet $w_i \in \mathfrak{A}$, with $\#\mathfrak{A} = n$
- X topologically Cantor set with topology generated by cylinder sets

$$\mathscr{C}(a_0, \ldots, a_N) = \{ w \in X \,|\, w_i = a_i,\ 0 \leq i \leq N \}$$

- one-sided shift $\sigma : X \to X$, defined by $\sigma(w)_i = w_{i+1}$

- Bernoulli measure μ_P on X shift-invariant, defined by probability $P = (p_1, \dots, p_n)$ on alphabet $\mathfrak{A}$
- assigns measure $\mu_P(\mathscr{C}(a_0, \dots, a_N)) = p_{a_0} \cdots p_{a_N}$ to cylinder sets
- Markov measure $\mu_{P,\rho}$ on X shift-invariant measure defined by a pair (P, ρ) of probability $P = (p_1, \dots, p_n)$ on $\mathfrak{A}$ and stochastic matrix ρ satisfying $P\rho = P$
- assigns measure $\mu_{P,\rho}(\mathscr{C}(a_0, \dots, a_N)) = p_{a_0} \rho_{a_0 a_1} \cdots \rho_{a_{N-1} a_N}$
- Markov measure $\mu_{P,\rho}$ is supported on a subshift of finite type $X_A \subset X$, given by shift-invariant $X_A = \{w \in X \,|\, A_{w_i w_{i+1}} = 1, \forall i \geq 0\}$ with matrix A_{ij} entries 0 or 1 when $\rho_{ij} = 0$ or $\rho_{ij} \neq 0$

- for μ a σ-invariant probability measure on X, entropy $S(\mu, \sigma)$ is μ-almost everywhere value of local entropy

$$h_{\mu,\sigma}(x) = \lim_{\delta \to 0} \lim_{n \to \infty} -\frac{1}{n} \log \mu(B_\sigma(x, n, \delta)),$$

where $B_\sigma(x, n, \delta) = \{y \in X \mid d(\sigma^j(x), \sigma^j(y)) < \delta, \forall 0 \leq j \leq n\}$ are the Bowen balls

- for Bernoulli measure $\mu = \mu_P$, dynamical entropy agrees with Shannon entropy of P,

$$S(\mu_P, \sigma) = -\sum_{i=1}^{N} p_i \log p_i$$

- for Markov measure dynamical entropy

$$S(\mu_{P,\rho}, \sigma) = -\sum_{i=1}^{N} p_i \sum_{j=1}^{N} \rho_{ij} \log \rho_{ij}$$

- analogous to thermodynamic deformations of trace, deformation of integration of functions $f \in C(X, \mathbb{R})$

$$\int_X^{(\beta,S)} f(x)dx := \inf_{\mu} \{ \int_X f(x) d\mu(x) - \beta^{-1} S(\mu, \sigma) \}$$

- infimum is taken over a specific class of σ-invariant measures
  - Bernoulli measures
  - Markov measures
  - σ-invariant ergodic measures
- topological entropy of shift σ is

$$h(X, \sigma) = \sup_{\mu} \{ S(\mu, \sigma) \},$$

with supremum over all σ-invariant ergodic measures

Semirings of functions

• min-plus semirings $\mathbb{S} = C(X, \mathbb{T})$ with pointwise $\oplus$, $\odot$

• thermodynamic deformations $\mathbb{S}_{\beta,S} = C(X, \mathbb{T}_{\beta,S})$ with pointwise $\oplus_{\beta,S}$, $\odot$

Logarithmically related pairs $(\mathscr{R}, \mathbb{S})$

• $\mathscr{R}$ commutative ring (algebra); $\mathbb{S}$ min-plus semiring; with formal logarithm bijective map $\mathscr{L} : \mathrm{Dom}(\mathscr{L}) \subset \mathscr{R} \to \mathbb{S}$

$$\mathscr{L}(ab) = \mathscr{L}(a) + \mathscr{L}(b) = \mathscr{L}(a) \odot \mathscr{L}(b)$$

• thermodynamic deformation (Shannon entropy)

$$f_1 \oplus_{\beta,S} f_2 = -\beta^{-1} \log(E(-\beta f_1) + E(-\beta f_2))$$

with $E : \mathbb{S} \to \mathrm{Dom}(\mathscr{L}) \subset \mathscr{R}$ inverse of $\mathscr{L}$

Examples

• $\mathscr{R} = C(X,\mathbb{R})$ and $\mathrm{Dom}(\mathscr{L}) \subset \mathscr{R}$ given by $C(X,\mathbb{R}_+^*)$ with $\mathscr{L}(a) = -\beta^{-1}\log(a)$

$$f_1 \oplus_{\beta,S} f_2 = -\beta^{-1}\log(e^{-\beta f_1} + e^{-\beta f_2})$$

is $\mathbb{S}_{\beta,S} = C(X,\mathbb{T}_{\beta,S})$ with $S = \mathrm{Sh}$

• $\mathscr{R} = \mathbb{Q}[[t]]$ ring of formal power series, $\mathrm{Dom}(\mathscr{L}) \subset \mathscr{R}$ power series $\alpha(t) = \sum_{k\geq 0} a_k t^k$ with $a_0 = 1$, with formal log

$$\mathscr{L}(1+\alpha) = \alpha - \frac{1}{2}\alpha^2 + \frac{1}{3}\alpha^3 + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}\alpha^k$$

$$\alpha_1 \oplus_{\beta,S} \alpha_2 = \beta^{-1}\mathscr{L}(E(-\beta\alpha_1) + E(-\beta\alpha_2))$$

formal exponential $E(\gamma) = \sum_{k\geq 0} \gamma^k/k!$

min-plus valued characters (algebraic Feynman rules)

• $\mathscr{H}$ commutative Hopf algebra; $\mathbb{S}$ be a min-plus semiring

• $\psi : \mathscr{H} \to \mathbb{S}$ satisfying $\psi(1) = 0$ and

$$\psi(xy) = \psi(x) + \psi(y), \quad \forall x, y \in \mathscr{H}$$

• main idea: “arithmetic of orders of magnitude" $\epsilon \to 0$
- leading term in $\epsilon^{\alpha} + \epsilon^{\beta}$ is $\epsilon^{\min\{\alpha,\beta\}}$
- leading term of $\epsilon^{\alpha}\epsilon^{\beta}$ is $\epsilon^{\alpha+\beta}$

• model characters and Birkhoff factorization on “order of magnitude" version of usual ones convolution of min-plus characters

$$(\psi_1 \star \psi_2)(x) = \min\{\psi_1(x^{(1)}) + \psi_2(x^{(2)})\} = \bigoplus (\psi_1(x^{(1)}) \odot \psi_2(x^{(2)}))$$

minimum over all pairs $(x^{(1)}, x^{(2)})$ in coproduct
$\Delta(x) = \sum x^{(1)} \otimes x^{(2)}$ in Hopf algebra $\mathscr{H}$

Birkhoff factorization of a min-plus character ψ

$$\psi_+ = \psi_- \star \psi$$

$\star$ convolution product, $\psi_\pm$ satisfying $\psi_\pm(xy) = \psi_\pm(x) + \psi_\pm(y)$

Note: does not require antipode, works also for $\mathscr{H}$ bialgebra

Rota–Baxter semirings

• $\mathbb{S}$ be a min-plus semiring, map $T : \mathbb{S} \to \mathbb{S}$ is $\oplus$-additive if monotone, $T(a) \leq T(b)$ for $a \leq b$ (pointwise)

• Rota–Baxter semiring $(\mathbb{S}, \oplus, \odot)$ weight $\lambda > 0$:
exists $\oplus$-additive map $T : \mathbb{S} \to \mathbb{S}$ with

$$T(f_1) \odot T(f_2) = T(T(f_1) \odot f_2) \oplus T(f_1 \odot T(f_2)) \oplus T(f_1 \odot f_2) \odot \log \lambda$$

• Rota–Baxter semiring $(\mathbb{S}, \oplus, \odot)$ weight $\lambda < 0$:
exists $\oplus$-additive map $T : \mathbb{S} \to \mathbb{S}$ with

$$T(f_1) \odot T(f_2) \oplus T(f_1 \odot f_2) \odot \log(-\lambda) = T(T(f_1) \odot f_2) \oplus T(f_1 \odot T(f_2))$$

Birkhoff factorization in min-plus semirings (weight $+1$)

- Bogolyubov-Parashchuk preparation

$$\tilde{\psi}(x) = \min\{\psi(x), \psi_-(x') + \psi(x'')\} = \psi(x) \oplus \bigoplus \psi_-(x') \odot \psi(x'')$$

(x', x'') ranges over non-primitive part of coproduct
$\Delta(x) = x \otimes 1 + 1 \otimes x + \sum x' \otimes x''$

- ψ_- defined inductively on lower degree x' in Hopf algebra

$$\psi_-(x) := T(\tilde{\psi}(x)) = T(\min\{\psi(x), \psi_-(x') + \psi(x'')\})$$

$$= T\left(\psi(x) \oplus \bigoplus \psi_-(x') \odot \psi(x'')\right)$$

• by $\oplus$-linearity of T same as

$$\psi_-(x) = \min\{T(\psi(x)), T(\psi_-(x') + \psi(x''))\}$$

$$= T(\psi(x)) \oplus \bigoplus T(\psi_-(x') \odot \psi(x''))$$

• then ψ_+ by convolution

$$\psi_+(x) := (\psi_- \star \psi)(x) = \min\{\psi_-(x), \psi(x), \psi_-(x') + \psi(x'')\}$$

$$= \min\{\psi_-(x), \tilde{\psi}(x)\} = \psi_-(x) \oplus \tilde{\psi}(x)$$

• key step: associativity and commutativity of $\oplus$ and $\oplus$-additivity of T, plus Rota-Baxter identity weight $+1$ gives

$$\psi_-(xy) = \psi_-(x) + \psi_-(y)$$

hence ψ_+ also as convolution

- to check that $\psi_\pm$ satisfy $\psi_\pm(xy) = \psi_\pm(x) + \psi_\pm(y)$
- have $\psi_-(xy) = T \min\{\psi(x) + \psi(y), \psi_-((xy)') + \psi((xy)'')\}$
- decompose the terms $(xy)'$ and $(xy)''$ in terms of x, y, x' and x'', y' and y''

$$\psi_-(xy) = T \min \left\{ \begin{array}{l} \psi(x) + \psi(y), \\ \psi_-(x) + \psi(y), \\ \psi_-(y) + \psi(x), \\ \psi_-(y') + \psi(xy''), \\ \psi_-(x') + \psi(x''y), \\ \psi_-(xy') + \psi(y''), \\ \psi_-(x'y) + \psi(x''), \\ \psi_-(x'y') + \psi(x''y'') \end{array} \right\}$$

- by associativity and commutativity of $\oplus$ and $\oplus$-additivity of T can group these terms together into

$$\psi_-(xy) = \min\{\alpha(x,y,x',y'), \beta(x,y,x',y')\}$$

$$\alpha(x,y,x',y') = T \min \left\{ \begin{array}{l} \psi_-(x) + \psi(y), \\ \psi(x) + \psi_-(y), \\ \psi_-(xy') + \psi(y''), \\ \psi_-(x'y) + \psi(x'') \end{array} \right\}$$

$$\beta(x,y,x',y') = T \min \left\{ \begin{array}{l} \psi(x) + \psi(y), \\ \psi_-(y') + \psi(xy''), \\ \psi_-(x') + \psi(x''y), \\ \psi_-(x'y') + \psi(x''y'') \end{array} \right\}$$

- assume inductively that

$$\psi_-(uv) = \psi_-(u) + \psi_-(v),$$

  for all terms u and v in $\mathscr{H}$ of degrees $\deg(u) + \deg(v) < \deg(xy)$
- use the fact that T is $\oplus$-additive
- rewrite $\alpha(x, y, x', y')$ as

$$\begin{aligned}\alpha(x, y, x', y') &= T \min\{\psi_-(x) + \tilde{\psi}(y), \tilde{\psi}(x) + \psi_-(y)\} \\ &= \min\{T(T(\tilde{\psi}(x)) + \tilde{\psi}(y)), T(\tilde{\psi}(x) + T(\tilde{\psi}(y)))\}\end{aligned}$$

- write the term $\beta(x, y, x', y')$ as

$$\beta(x, y, x', y') = T \min\{\tilde{\psi}(x) + \tilde{\psi}(y)\} = \min\{T(\tilde{\psi}(x) + \tilde{\psi}(y))\}.$$

- have then

$$\psi_-(xy) = \min \left\{ \begin{array}{l} T(\tilde{\psi}(x) + \tilde{\psi}(y)), \\ T(T(\tilde{\psi}(x)) + \tilde{\psi}(y)), \\ T(\tilde{\psi}(x) + T(\tilde{\psi}(y))) \end{array} \right\}$$

$$= T(\tilde{\psi}(x) \odot \tilde{\psi}(y)) \oplus T(T(\tilde{\psi}(x)) \odot \tilde{\psi}(y))$$
$$\oplus \; T(\tilde{\psi}(x) \odot T(\tilde{\psi}(y))).$$

- the operator T satisfies the Rota–Baxter identity with $\lambda = 1$,

$$T(f_1) \odot T(f_2) = T(T(f_1) \odot f_2) \oplus T(f_1 \odot T(f_2)) \oplus T(f_1 \odot f_2) \odot \log \lambda$$

- so can rewrite the above as

$$\psi_-(xy) = T(\tilde{\psi}(x)) \odot T(\tilde{\psi}(y))$$

$$= T(\tilde{\psi}(x)) + T(\tilde{\psi}(y)) = \psi_-(x) + \psi_-(y)$$

- the fact that $\psi_+(xy) = \psi_+(x) + \psi_+(y)$ then follows from $\psi_+ = \psi_- \star \psi$

Birkhoff factorization in min-plus semirings (weight -1)

- $\psi : \mathscr{H} \to \mathbb{S}$ min-plus character, and $T : \mathbb{S} \to \mathbb{S}$ Rota-Baxter weight -1: there is a Birkhoff factorization $\psi_+ = \psi_- \star \psi$; if T satisfies $T(f_1 + f_2) \geq T(f_1) + T(f_2)$, then ψ_- and ψ_+ are also min-plus characters: $\psi_\pm(xy) = \psi_\pm(x) + \psi_\pm(y)$

- as before

$$\psi_-(x) := T(\tilde{\psi}(x)) \quad \text{and} \quad \psi_+(x) := (\psi_- \star \psi)(x) = \min\{\psi_-(x), \tilde{\psi}(x)\}$$

- Rota–Baxter identity of weight -1 gives

$$\psi_-(xy) = \min\{T(\tilde{\psi}(x) + \tilde{\psi}(y)), T(\tilde{\psi}(x)) + T(\tilde{\psi}(y)))\}$$

if $T(f_1 + f_2) \geq T(f_1) + T(f_2)$ then

$$\psi_-(xy) = T(\tilde{\psi}(x)) + T(\tilde{\psi}(y))) = \psi_-(x) + \psi_-(y)$$

Thermodynamic Rota–Baxter structures

• $\mathbb{S}_{\beta,S}$ thermodynamic Rota–Baxter semiring weight $\lambda > 0$: there is $\oplus_{\beta,S}$-additive map $T : \mathbb{S}_{\beta,S} \to \mathbb{S}_{\beta,S}$

$$T(f_1) \odot T(f_2) = T(T(f_1) \odot f_2) \oplus_{\beta,S} T(f_1 \odot T(f_2)) \oplus_{\beta,S} T(f_1 \odot f_2) \odot \log \lambda$$

• $\mathbb{S}_{\beta,S}$ thermodynamic Rota–Baxter semiring weight $\lambda < 0$: there is $\oplus_{\beta,S}$-additive map $T : \mathbb{S}_{\beta,S} \to \mathbb{S}_{\beta,S}$

$$T(f_1) \odot T(f_2) \oplus_{\beta,S} T(f_1 \odot f_2) \odot \log(-\lambda) = T(T(f_1) \odot f_2) \oplus_{\beta,S} T(f_1 \odot T(f_2))$$

like previous case but with $\oplus$ replaced with deformed $\oplus_{\beta,S}$

• $(\mathscr{R}, \mathbb{S})$ logarithmically related pair: $T : \mathbb{S} \to \mathbb{S}$ determines $\mathscr{T} : \mathscr{R} \to \mathscr{R}$ with $\mathscr{T}(e^{-\beta f}) := e^{-\beta T(f)}$, for $a = e^{-\beta f}$ in $\mathrm{Dom}(\log) \subset \mathscr{R}$

$\mathscr{T}$ Rota-Baxter weight λ_β on $\mathscr{R}$ $\Leftrightarrow$ T Rota-Baxter weight λ on $\mathbb{S}_{\beta,S}$

with $S = \mathrm{Sh}$ and $\lambda_\beta = \lambda^{-\beta}$, for $\lambda > 0$, or $\lambda_\beta = -|\lambda|^{-\beta}$ for $\lambda < 0$

$$\mathscr{T}(e^{-\beta f_1})\mathscr{T}(e^{-\beta f_2}) = \mathscr{T}(\mathscr{T}(e^{-\beta f_1})e^{-\beta f_2}) + \mathscr{T}(e^{-\beta f_1}\mathscr{T}(e^{-\beta f_2}))$$

$$+\lambda_\beta \, \mathscr{T}(e^{-\beta f_1} e^{-\beta f_2})$$

• $\mathscr{T}$ is $\mathbb{R}$-linear iff T is $\oplus_{\beta,S}$-linear

Birkhoff factorization in thermodynamic Rota–Baxter semirings (weight $+1$)

- $T : \mathbb{S}_{\beta,S} \to \mathbb{S}_{\beta,S}$ Rota–Baxter of weight $\lambda = +1$
- Bogolyubov–Parashchuk preparation of $\psi : \mathscr{H} \to \mathbb{S}_{\beta,S}$

$$\tilde{\psi}_{\beta,S}(x) = \psi(x) \oplus_{\beta,S} \bigoplus_{\beta,S} \psi_-(x') + \psi(x'')$$

$$= -\beta^{-1} \log \left(e^{-\beta\psi(x)} + \sum e^{-\beta(\psi_-(x')+\psi(x''))} \right)$$

- $\phi_\beta(x) := e^{-\beta\psi(x)}$ in $\mathscr{R}$: Bogolyubov–Parashchuk preparation $\tilde{\phi}_\beta(x) = e^{-\beta\tilde{\psi}(x)}$

$$\tilde{\phi}_\beta(x) := \phi_\beta(x) + \sum \mathscr{T}(\tilde{\phi}_\beta(x'))\phi_\beta(x'')$$

with $\mathscr{T}(e^{-\beta f}) := e^{-\beta T(f)}$ and $\mathscr{T}(-e^{-\beta f}) := -\mathscr{T}(e^{-\beta f})$

• Birkhoff factorization $\psi_{\beta,+} = \psi_{\beta,-} \star_\beta \psi$

$$\psi_{\beta,-}(x) = T(\tilde{\psi}_\beta(x)) = -\beta^{-1} \log \left(e^{-\beta T(\psi(x))} + \sum e^{-\beta T(\psi_-(x')+\psi(x''))} \right)$$

$$\psi_{\beta,+}(x) = -\beta^{-1} \log \left(e^{-\beta \psi_{\beta,-}(x)} + e^{-\beta \tilde{\psi}_\beta(x)} \right)$$

satisfying $\psi_{\beta,\pm}(xy) = \psi_{\beta,\pm}(x) + \psi_{\beta,\pm}(y)$

• in limit $\beta \to \infty$ thermodynamic Birkhoff factorization converges to min-plus Birkhoff factorization

Entropical von Neumann trace and Rota–Baxter identity

• $(\mathscr{R}, \mathscr{T})$ ordinary Rota–Baxter algebra weight λ; same weight on matrices $M_n(\mathscr{R})$ by $\mathscr{T}(A) = (\mathscr{T}(a_{ij}))$, for $A = (a_{ij})$

• for $(M_n(\mathscr{R}), M_n(\mathbb{S}))$ logarithmically related, with $\mathscr{T}$ Rota–Baxter weight $+1$ on $\mathscr{R} \Rightarrow T : M_n(\mathbb{S}) \to M_n(\mathbb{S})$ with $\mathscr{T}(e^{-\beta A}) = e^{-\beta T(A)}$ satisfying

$$\begin{aligned} \mathrm{Tr}^{\oplus}_{\beta,\mathscr{N}}(T(A) \boxplus T(B)) &= \mathrm{Tr}^{\oplus}_{\beta,\mathscr{N}}(T(T(A) \boxplus B)) \\ &\oplus_{\beta,S} \mathrm{Tr}^{\oplus}_{\beta,\mathscr{N}}(T(A \boxplus T(B))) \\ &\oplus_{\beta,S} \mathrm{Tr}^{\oplus}_{\beta,\mathscr{N}}(T(A) \boxplus T(B)) \end{aligned}$$

where $\boxplus$ = direct sum of matrices, $\mathscr{N}$ = von Neumann entropy

Example: partial sums

- $\mathbb{R}$-algebra $\mathscr{R}$ of $\mathbb{R}$-valued sequences $a = (a_1, a_2, a_3, \cdots) = (a_n)_{n=1}^{\infty}$, with coordinate-wise addition and multiplication
- $\mathscr{T} : \mathscr{R} \to \mathscr{R}$ be the linear operator that maps the sequence $(a_1, a_2, a_3, \cdots, a_n, \cdots)$ to $(0, a_1, a_1 + a_2, \cdots, \sum_{k=1}^{n-1} a_k, \cdots)$
- $\mathscr{T}$ is a Rota–Baxter operator of weight $+1$

- this Rota–Baxter algebra $(\mathscr{R}, \mathscr{T})$ of weight $+1$ determines a Rota–Baxter structure of weight $+1$ on the thermodynamic semi-rings $\mathbb{S}_{\beta,S}$ of functions $f : \mathbb{N} \to \mathbb{T} = \mathbb{R} \cup \{\infty\}$, with the pointwise operations $\oplus_{\beta,S}$ and $\odot$

$$(Tf)(n) = \bigoplus_{\beta,S \; k=1,\dots,n-1} f(k)$$

  for $n \geq 2$ and $(Tf)(1) = \infty$
- here $\mathscr{D} \subset \mathscr{R}$ be the subset of sequences with values in $\mathbb{R}_+$, which we can write as $a_n = e^{-\beta c_n}$, when $a_n > 0$ and zero otherwise
- have $(\mathscr{T} a)_1 = 0$ and $(\mathscr{T} a)_n = \sum_{k=1}^{n-1} a_k$ for $n \geq 2$
- set $(Tc)_n = -\beta^{-1} \log(\mathscr{T} a)_n$, so that

$$(Tc)_n = \begin{cases} \infty & n = 1 \\ -\beta^{-1} \log \left(\sum_{k=1}^{n-1} e^{-\beta c_k} \right) & n \geq 2. \end{cases}$$

Example: q-integral

- $\mathscr{R} = \mathbb{R}[[t]]$ be the ring of formal power series with real coefficients
- for q not a root of unity, $\mathscr{T}$ linear operator

$$(\mathscr{T}\alpha)(t) = \sum_{k=1}^{\infty} \alpha(q^n t)$$

- $\mathscr{T}$ is a Rota–Baxter operator of weight $+1$
- $\mathscr{T}$ maps a single power t^n to $q^n t^n/(1-q^n)$, hence it restricts to a Rota–Baxter operator of weight $+1$ on the subring of polynomials $\mathbb{R}[t]$

- $\mathbb{S}$ thermodynamic semiring of formal power series $\mathbb{S}_{\beta,S} = \mathbb{R}[[t]] \cup \{\infty\}$ with operations

$$(\gamma_1 \oplus_{\beta,S} \gamma_2)(t) = -\beta^{-1} \log(e^{-\beta\gamma_1(t)} + e^{-\beta\gamma_2(t)})$$

$$(\gamma_1 \odot \gamma_2)(t) = \gamma_1(t) + \gamma_2(t)$$

- the Rota–Baxter algebra $(\mathscr{R}, \mathscr{T})$ of weight $+1$ induces a Rota–Baxter structure of weight $+1$ on $\mathbb{S}_{\beta,S}$

$$(T\gamma)(t) = \bigoplus_{\beta,S}{}_{k=1}^{\infty} \gamma(q^k t)$$

- here $\mathscr{D} \subset \mathscr{R}$ formal series with $a_0 = 1$, namely $\mathscr{D} = 1 + t\mathbb{R}[[t]]$
- for $\alpha \in \mathscr{D}$ and $\gamma(t) = \log \alpha(t)$ define T by the relation $\mathscr{T}(e^{-\beta\gamma(t)}) = e^{-\beta(T\gamma)(t)}$
- this gives $e^{-\beta(T\gamma(t))} = \sum_{k=1}^{\infty} e^{-\beta\gamma(q^k t)}$

$$(T\gamma)(t) = -\beta^{-1} \log\left(\sum_{k=1}^{\infty} e^{-\beta\gamma(q^k t)}\right) = \bigoplus_{\beta,S}{}_{k=1}^{\infty} \gamma(q^k t)$$

Example: Witt rings

- commutative ring R, Witt ring $W(R) = 1 + tR[[t]]$: addition is product of formal power series, multiplication determined by

$$(1-at)^{-1} \star (1-bt)^{-1} = (1-abt)^{-1} \quad a, b \in R$$

- injective ring homomorphism $g : W(R) \to R^{\mathbb{N}}$, ghost coordinates coefficients of

$$t \frac{1}{\alpha} \frac{d\alpha}{dt} = \sum_{r \geq 1} \alpha_r t^r \quad \text{for} \quad \alpha = \exp(\sum_{r \geq 1} \alpha_r t^r / r)$$

- component-wise addition and multiplication on $R^{\mathbb{N}}$
- writing elements of the Witt ring $W(R)$ in exponential form

$$\exp\left(\sum_{r \geq 1} \alpha_r \frac{t^r}{r} \right)$$

the ghost coordinates are the coefficients of

$$t \frac{1}{\alpha} \frac{d\alpha}{dt} = \sum_{r \geq 1} \alpha_r t^r$$

• linear operator $\mathscr{T} : R^{\mathbb{N}} \to R^{\mathbb{N}}$ is Rota–Baxter weight λ iff $\mathscr{T}_W : W(R) \to W(R)$ defined by $g(\mathscr{T}_W(\alpha)) = \mathscr{T}(g(\alpha))$ satisfies Rota–Baxter on $W(R)$

$$\mathscr{T}_W(\alpha_1) \star \mathscr{T}_W(\alpha_2) = \mathscr{T}_W(\alpha_1 \star \mathscr{T}_W(\alpha_2)) +_W \ \mathscr{T}_W(\mathscr{T}_W(\alpha_1) \star \alpha_2)$$
$$+_W \ \lambda \ \mathscr{T}_W(\alpha_1 \star \alpha_2)$$

- Rota–Baxter identity for $\mathscr{T}_W$ (with $+_W$ the sum in $W(R)$)

$$\mathscr{T}_W(\alpha_1) \star \mathscr{T}_W(\alpha_2) = \mathscr{T}_W(\alpha_1 \star \mathscr{T}_W(\alpha_2)) +_W \mathscr{T}_W(\mathscr{T}_W(\alpha_1) \star \alpha_2)$$
$$+_W \lambda \star \mathscr{T}_W(\alpha_1 \star \alpha_2))$$

- taking ghost components

$$g(\mathscr{T}_W(\alpha_1) \star \mathscr{T}_W(\alpha_2))) = g(\mathscr{T}_W(\alpha_1 \star \mathscr{T}_W(\alpha_2))) + g(\mathscr{T}_W(\mathscr{T}_W(\alpha_1) \star \alpha_2))$$
$$+ \lambda \, g(\mathscr{T}_W(\alpha_1 \star \alpha_2))$$

- gives the Rota–Baxter identity for $\mathscr{T}$

$$\mathscr{T}(g(\alpha_1))\mathscr{T}(g(\alpha_2)) = \mathscr{T}(g(\alpha_1)\mathscr{T}(g(\alpha_2))) + \mathscr{T}(\mathscr{T}(g(\alpha_1))g(\alpha_2))$$
$$+ \lambda \mathscr{T}(g(\alpha_1)g(\alpha_2))$$

- injectivity of ghost map: can run the implication backward

- $W(R)$ also has a convolution product (with same $+_W$ addition)

$$\alpha \circledast \gamma := \exp\left(\sum_{n\geq 1} (\sum_{r+\ell=n} \alpha_r \gamma_\ell) \frac{t^n}{n} \right)$$

for $\alpha = \exp(\sum_{r\geq 1} \alpha_r t^r / r)$ and $\gamma = \exp(\sum_{r\geq 1} \gamma_r t^r / r)$

- $\alpha \circledast \gamma$ is defined so that the ghost $g(\alpha \circledast \gamma) = \sum_{n\geq 1} \sum_{r+\ell=n} \alpha_r \gamma_\ell \, t^n$ is the product as power series $g(\alpha) \bullet g(\gamma)$ of the ghosts $g(\alpha) = \sum_{r\geq 1} \alpha_r t^r$ and $g(\gamma) = \sum_{r\geq 1} \gamma_r t^t$

• linear operator $\mathscr{T} : R[[t]] \to R[[t]]$ is Rota–Baxter weight λ iff $\mathscr{T}_W : W(R) \to W(R)$ defined by $g(\mathscr{T}_W(\alpha)) = \mathscr{T}(g(\alpha))$ satisfies Rota–Baxter on $W(R)$ with convolution product

$$\mathscr{T}_W(\alpha_1) \circledast \mathscr{T}_W(\alpha_2) = \mathscr{T}_W(\alpha_1 \circledast \mathscr{T}_W(\alpha_2)) +_W \, \mathscr{T}_W(\mathscr{T}_W(\alpha_1) \circledast \alpha_2)$$

$$+_W \, \lambda \, \mathscr{T}_W(\alpha_1 \circledast \alpha_2)$$

• Example: $\mathscr{R} = R^{\mathbb{N}}$ Rota–Baxter weight $+1$

$$\mathscr{T} : (a_1, a_2, \ldots, a_n, \ldots) \mapsto (0, a_1, a_1 + a_2, \ldots, \sum_{k=1}^{n-1} a_k, \ldots)$$

resulting Rota–Baxter $\mathscr{T}_W$ weight $+1$ on Witt ring $W(R)$

$$\mathscr{T}_W(\alpha) = \alpha \circledast \mathbb{I}$$

convolution product with multiplicative unit $\mathbb{I} = (1 - t)^{-1}$

Example: $\mathscr{R} = R[[t]]$ with the Rota–Baxter operator $\mathscr{T}_q$ of weight $+1$ given by the q-integral (where $q \in R$ is not a root of unity)

- operator $\mathscr{T}_{W,q}$ on $W(R)$ defined by $g(\mathscr{T}_{W,q}(\alpha)) = \mathscr{T}_q(g(\alpha))$ is a Rota–Baxter operator of weight one with respect to the convolution product
- explicitly given by

$$\mathscr{T}_W(\alpha)(t) = \prod_{k\geq 1} \alpha(q^k t)$$

$$\mathscr{T}_{W,q}(\exp(\sum_{r\geq 1} \alpha_r \frac{t^r}{r})) = \exp(\sum_{r\geq 1}\sum_{k\geq 1} \alpha_r \frac{q^{kr} t^r}{r})) = \prod_{k\geq 1} \exp(\sum_{r\geq 1} \alpha_r \frac{(q^k t)^r}{r})$$

- product $\prod_k \alpha(q^k t)$ (product as power series) is *addition* in Witt ring $W(R)$
- so $\mathscr{T}_W$ has same form as q-integral $\mathscr{T}$ replacing sum in $R[[t]]$ with sum in $W(R)$ so for same reason $\mathscr{T}_{W,q}$ satisfies RB

$$\mathscr{T}_{W,q}(\alpha_1) \circledast \mathscr{T}_{W,q}(\alpha_2) = \mathscr{T}_{W,q}(\alpha_1 \circledast \mathscr{T}_{W,q}(\alpha_2)) +_W \mathscr{T}_{W,q}(\mathscr{T}_{W,q}(\alpha_1) \circledast \alpha_2)$$
$$+_W \mathscr{T}_{W,q}(\alpha_1 \circledast \alpha_2)$$

Example: Zeta functions of algebraic varieties over finite fields

• Hasse–Weil zeta functions of varieties over $\mathbb{F}_q$

$$Z(X,t) = \exp\left(\sum_{r\geq 1} \#X(\mathbb{F}_{q^r})\, \frac{t^r}{r}\right)$$

elements in Witt ring:

$$Z(X \sqcup Y, t) = Z(X,t) Z(Y,t) \quad \text{and} \quad Z(X \times Y, t) = Z(X,t) \star Z(Y,t)$$

Rota–Baxter operator weight $+1$

$$\mathscr{T}_W(Z(X,t)) = Z(X,t) \circledast Z(\mathrm{Spec}(\mathbb{F}_q), t)$$

- $K_0(\mathscr{V})_{\mathbb{F}_q}$ Grothendieck ring of varieties over $\mathbb{F}_q$ generated by isomorphism classes $[X]$ with inclusion-exclusion relationz $[X] = [Y] + [X \setminus Y]$ for closed $Y \subset X$ and product $[X \times Y] = [X]\,[Y]$
- the zeta function $Z(X,t) = Z([X],t)$ factors as a ring homomorphism from the Grothendieck ring to the Witt ring
- Lefschetz motive: class of the affine line $\mathbb{L} = [\mathbb{A}^1]$; Tate motive formal inverse $\mathbb{L}^{-1}$
- Rota–Baxter operator $\mathscr{T}_{W,q}$ or $\mathscr{T}_{W,q^{-1}}$

$$\mathscr{T}_{W,q}(Z(X,t)) = \prod_{k\geq 1} Z([X]\,\mathbb{L}^k, t), \quad \mathscr{T}_{W,q^{-1}}(Z(X,t)) = \prod_{k\geq 1} Z([X]\,\mathbb{L}^{-k}, t)$$

with $\mathbb{L}$ Lefschetz motive and $\mathbb{L}^{-1}$ Tate motive

- operators $\tilde{\mathscr{T}}_{W,q^{\pm 1}} := -_W id -_W \mathscr{T}_{W,q^{\pm 1}}$ are also Rota–Baxter operators of weight $+1$

$$\tilde{\mathscr{T}}_{W,q^{\pm 1}}(Z(X,t)) = \prod_{k\geq 0} Z([X]\mathbb{L}^{\pm k}, t)^{-1}$$

min-plus characters from inclusion–exclusion functions on graphs

- real valued functions τ on a set of graphs with inclusion-exclusion
- for $\Gamma = \Gamma_1 \cup \Gamma_2$ with intersection $\gamma = \Gamma_1 \cap \Gamma_2$

$$\tau(\Gamma) = \tau(\Gamma_1) + \tau(\Gamma_2) - \tau(\gamma)$$

- typical examples a “cost function" to the sets of vertices and edges
- $F_E = \{f_e \,:\, e \in E(\Gamma)\}$ and $F_V = \{f_v \,:\, v \in V(\Gamma)\}$ setting

$$\tau(\Gamma) = \sum_{v \in V(\Gamma)} f_v + \sum_{e \in E(\Gamma)} f_e$$

  gives function satisfying inclusion-exclusion
- on disjoint union $\Gamma = \Gamma_1 \sqcup \Gamma_2$ have $\tau(\Gamma) = \tau(\Gamma_1) + \tau(\Gamma_2)$, hence morphism $\tau : \mathscr{H} \to \mathbb{T}$, with $\mathscr{H}$ (Hopf) algebra of graphs and $\mathbb{T}$ tropical semiring
- examples from Random Markov Fields

Computation examples

• inclusion–exclusion “cost functions": $\Gamma = \Gamma_1 \cup \Gamma_2$, $\gamma = \Gamma_1 \cap \Gamma_2$

$$\psi(\Gamma) = \psi(\Gamma_1) + \psi(\Gamma_2) - \psi(\gamma)$$

determine $\psi : \mathscr{H} \to \mathbb{T}$ character $\psi(xy) = \psi(x) + \psi(y)$

• class of machines $\psi_n(\Gamma)$ step-counting function of n-th machine: when it outputs on computation Γ (Hopf algebra of flow charts)

• Rota–Baxter operator weight $+1$ of partial sums: Bogolyubov–Parashchuck preparation

$$\tilde{\psi}_n(\Gamma) = \min\{\psi_n(\Gamma), \psi_n(\Gamma/\gamma) + \sum_{k=1}^{n-1} \tilde{\psi}_k(\gamma)\}$$

• a graph Γ with $\psi_n(\Gamma) = \infty$ (n-th machine does not halt) can have $\tilde{\psi}_n(\Gamma) < \infty$ if both

- source of infinity was localized in $\gamma \smallsetminus \partial\gamma$, so $\psi_n(\Gamma/\gamma) < \infty$
- $\psi_k(\gamma) < \infty$ for all previous machines

“renormalization of computational infinities" in Manin's sense

Application to QFT

- expansion of perturbative QFT into Feynman diagrams
- each a graph with an integral on momentum variables
- reformulated in Schwinger–Feynman parameters: integral of an algebraic differential form on a cycle in the complement of an algebraic hypersurface defined over $\mathbb{Z}$ (period integrals)
- divergence issues from intersections of cycle and hypersurface
- question on the arithmetic nature of the hypersurfaces (graph hypersurfaces) and the resulting periods
- original conjecture: mixed Tate motives and periods mutliple zeta values (conjecture proved false)

- the mixed Tate condition closely related to polynomial countability condition
  - *Question*: are classes in the Grothendieck ring of the graph hypersurfaces polynomials in the Lefschetz motive $\mathbb{L}$ with $\mathbb{Z}$-coefficients?
  - *Question*: is the counting of points over a finite field $\mathbb{F}_q$ a polynomial function of q with $\mathbb{Z}$-coefficients?
- answer to both is *no* but are there interesting families of graphs for which it holds? can one extract from a graph the subgraphs and quotient graphs for which it holds?

Feynman rules for $I_\Gamma(k_1, \ldots, k_\ell, p_1, \ldots, p_N)$:

- Internal lines $\Rightarrow$ propagator = quadratic form q_i

$$\frac{1}{q_1 \cdots q_n}, \quad q_i(k_i) = k_i^2 + m^2$$

- Vertices: conservation (valences = monomials in $\mathscr{L}$)

$$\sum_{e_i \in E(\Gamma): s(e_i) = v} k_i = 0$$

- Integration over k_i, internal edges

$$U(\Gamma) = \int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j)}{q_1 \cdots q_n} \, d^D k_1 \cdots d^D k_n$$

$n = \#E_{int}(\Gamma)$, $N = \#E_{ext}(\Gamma)$

$$\epsilon_{e,v} = \begin{cases} +1 & t(e) = v \\ -1 & s(e) = v \\ 0 & \text{otherwise,} \end{cases}$$

Parametric Feynman integrals

• Schwinger parameters $q_1^{-k_1} \cdots q_n^{-k_n} =$

$$\frac{1}{\Gamma(k_1)\cdots\Gamma(k_n)} \int_0^\infty \cdots \int_0^\infty e^{-(s_1 q_1 + \cdots + s_n q_n)} \, s_1^{k_1 - 1} \cdots s_n^{k_n - 1} \, ds_1 \cdots ds_n .$$

• Feynman trick

$$\frac{1}{q_1 \cdots q_n} = (n-1)! \int \frac{\delta(1 - \sum_{i=1}^n t_i)}{(t_1 q_1 + \cdots + t_n q_n)^n} \, dt_1 \cdots dt_n$$

then change of variables $k_i = u_i + \sum_{k=1}^{\ell} \eta_{ik} x_k$

$$\eta_{ik} = \begin{cases} \pm 1 & \text{edge } \pm e_i \in \text{ loop } \ell_k \\ 0 & \text{otherwise} \end{cases}$$

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{n - D\ell/2}}$$

$\sigma_n = \{ t \in \mathbb{R}_+^n \,|\, \sum_i t_i = 1 \}$, vol form ω_n

Graph polynomials

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e \notin T} t_e \quad \text{with} \quad (M_\Gamma)_{kr}(t) = \sum_{i=0}^{n} t_i \eta_{ik} \eta_{ir}$$

Massless case $m = 0$:

$$V_\Gamma(t,p) = \frac{P_\Gamma(t,p)}{\Psi_\Gamma(t)} \quad \text{and} \quad P_\Gamma(p,t) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e$$

cut-sets C (complement of spanning tree plus one edge)
$s_C = (\sum_{v \in V(\Gamma_1)} P_v)^2$ with $P_v = \sum_{e \in E_{ext}(\Gamma), t(e)=v} p_e$ for $\sum_{e \in E_{ext}(\Gamma)} p_e = 0$
with $\deg \Psi_\Gamma = b_1(\Gamma) = \deg P_\Gamma - 1$

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2} \omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

stable range $-n + D\ell/2 \geq 0$; log divergent $n = D\ell/2$:

$$\int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2}}$$

Graph hypersurfaces

Residue of $U(\Gamma)$ (up to divergent Gamma factor)

$$\int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

Graph hypersurfaces $\hat{X}_\Gamma = \{t \in \mathbb{A}^n \,|\, \Psi_\Gamma(t) = 0\}$

$$X_\Gamma = \{t \in \mathbb{P}^{n-1} \,|\, \Psi_\Gamma(t) = 0\} \quad \deg = b_1(\Gamma)$$

• Relative cohomology: (range $-n + D\ell/2 \geq 0$)

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma)) \quad \text{with} \quad \Sigma_n = \{\prod_i t_i = 0\} \supset \partial\sigma_n$$

• Periods: $\int_\sigma \omega$ integrals of algebraic differential forms ω on a cycle σ defined by algebraic equations in an algebraic variety

Graph hypersurfaces

Residue of $U(\Gamma)$ (up to divergent Gamma factor)

$$\int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

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- Relative cohomology: (range $-n + D\ell/2 \geq 0$)

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma)) \quad \text{with} \quad \Sigma_n = \{\prod_i t_i = 0\} \supset \partial\sigma_n$$

- Periods: $\int_\sigma \omega$ integrals of algebraic differential forms ω on a cycle σ defined by algebraic equations in an algebraic variety

Feynman integrals and periods

Parametric Feynman integral: algebraic differential form on cycle in algebraic variety

But... divergent: where $X_\Gamma \cap \sigma_n \neq \emptyset$, inside divisor $\Sigma_n \supset \sigma_n$ of coordinate hyperplanes

- Blowups of coordinate linear spaces defined by edges of 1PI subgraphs (toric variety $P(\Gamma)$)
- Iterated blowup $P(\Gamma)$ separates strict transform of X_Γ from non-negative real points
- Deform integration chain: monodromy problem; lift to $P(\Gamma)$
- Subtraction of divergences: Poincaré residuces and limiting mixed Hodge structure

• S. Bloch, E. Esnault, D. Kreimer, *On motives associated to graph polynomials*, arXiv:math/0510011.

• S. Bloch, D. Kreimer, *Mixed Hodge Structures and Renormalization in Physics*, arXiv:0804.4399.

Motives and the Grothendieck ring of varieties

• Difficult to determine explicitly the motive of X_Γ (singular variety!) in the triangulated category of *mixed motives*

• Simpler invariant (universal Euler characteristic for motives): class $[X_\Gamma]$ in the Grothendieck ring of varieties $K_0(\mathscr{V})$

- generators $[X]$ isomorphism classes
- $[X] = [X \smallsetminus Y] + [Y]$ for $Y \subset X$ closed
- $[X] \cdot [Y] = [X \times Y]$

Tate motives: $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}] \subset K_0(\mathscr{M})$
(K_0 group of category of pure motives: virtual motives)

Universal Euler characteristics:

Any additive invariant of varieties: $\chi(X) = \chi(Y)$ if $X \cong Y$

$$\chi(X) = \chi(Y) + \chi(X \smallsetminus Y), \quad Y \subset X$$

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

values in a commutative ring $\mathscr{R}$ is same thing as a ring homomorphism

$$\chi : K_0(\mathscr{V}) \to \mathscr{R}$$

Examples:

- Topological Euler characteristic
- Couting points over finite fields
- Gillet–Soulé motivic $\chi_{mot}(X)$:

$$\chi_{mot} : K_0(\mathscr{V})[\mathbb{L}^{-1}] \to K_0(\mathscr{M}), \quad \chi_{mot}(X) = [(X, id, 0)]$$

for X smooth projective; complex $\chi_{mot}(X) = W^{\cdot}(X)$

Graph hypersurfaces and polynomial countability

- graph hypersurfaces X_Γ
- classes in the Grothendieck ring $[X_\Gamma] \in K_0(\mathscr{V})$
- Conjecture (Kontsevich 1997): Graph hypersurfaces have classes $[X_\Gamma] \in \mathbb{Z}[\mathbb{L}]$ with $\mathscr{L} = [\mathbb{A}^1]$ (Tate motives)
- Conjecture was first verified for all graphs up to 12 edges:
  • J. Stembridge, *Counting points on varieties over finite fields related to a conjecture of Kontsevich*, 1998

- But ... Conjecture is false!
  - P. Belkale, P. Brosnan, *Matroids, motives, and a conjecture of Kontsevich*, arXiv:math/0012198
  - Dzmitry Doryn, *On one example and one counterexample in counting rational points on graph hypersurfaces*, arXiv:1006.3533
  - Francis Brown, Oliver Schnetz, *A K3 in phi4*, arXiv:1006.4064.
  - Francis Brown, Dzmitry Doryn, *Framings for graph hypersurfaces*, arXiv:1301.3056
- Belkale–Brosnan: general argument shows "motives of graph hypersurfaces can be arbitrarily complicated"
- Doryn, Brown–Schnetz, Brown–Doryn: explicit counterexamples (14 edges)
- a dichotomy
  - After localization (Belkale-Brosnan): the graph hypersurfaces X_Γ *generate* the Grothendieck ring *localized* at $\mathbb{L}^n - \mathbb{L}$, $n > 1$
  - Stable birational equivalence: the graph hypersurfaces span $\mathbb{Z}$ inside $\mathbb{Z}[SB] = K_0(\mathscr{V})|_{\mathbb{L}=0}$
- P. Aluffi, M.M. *Graph hypersurfaces and a dichotomy in the Grothendieck ring*, arXiv:1005.4470

Polynomial countability

• in perturbative quantum field theory: graph hypersurfaces

$$X_\Gamma = \{\Psi_\Gamma = 0\} \subset \mathbb{A}^{\#E_\Gamma}$$

$$\Psi_\Gamma(t) = \sum_T \prod_{e \notin E(T)} t_e$$

sum over spanning trees

• X variety over $\mathbb{Z}$, reductions X_p over $\mathbb{F}_p$

$$\text{counting function} \quad N(X, q) := \#X_p(\mathbb{F}_q)$$

Polynomially countable X if counting function polynomial $P_X(q)$

• Question: when are graph hypersurfaces X_Γ polynomially countable? or equivalently complements $Y_\Gamma = \mathbb{A}^{\# E_\Gamma} \setminus X_\Gamma$

• max-plus character $\psi : \mathscr{H} \to \mathbb{T}_{max}$ with $N(Y_\Gamma, q) \sim q^{\psi(\Gamma)}$ leading order if Y_Γ polynomially countable or $\psi(\Gamma) := -\infty$ if not

• when Y_Γ not polynomially countable

$$\tilde{\psi}(\Gamma) = \max\{\psi(\Gamma), \tilde{\psi}(\gamma) + \psi(\Gamma/\gamma)\}$$

$$= \max\{\psi(\Gamma), \sum_{j=1}^{N} \psi(\gamma_j) + \psi(\gamma_{j-1}/\gamma_j)\}$$

identifies chains of subgraphs and quotient graphs whose hypersurfaces are polynomially countable