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Boundary conditions of the RGE flow in the noncommutative geometry approach to particle physics and cosmology

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ABSTRACT

We investigate the effect of varying boundary conditions on the renormalization group flow in a recently developed noncommutative geometry model of particle physics and cosmology. We first show that there is a sensitive dependence on the initial conditions at unification, so that, varying a parameter even slightly can be shown to have drastic effects on the running of the model parameters. We compare the running in the case of the default and the maximal mixing conditions at unification. We then exhibit explicitly a particular choice of initial conditions at the unification scale, in the form of modified maximal mixing conditions, which have the property that they satisfy all the geometric constraints imposed by the noncommutative geometry of the model at unification, and at the same time, after running them down to lower energies with the renormalization group flow, they still agree in order of magnitude with the predictions at the electroweak scale.

1. Introduction

The recent work [26] developed cosmological models of the very early universe based on the particle physics model of [13] derived from noncommutative geometry, via the formalism of spectral triple and the spectral action functional. Other results on cosmological aspects of noncommutative geometry models of particle physics include [9,23,27–31]. In the particle physics model of [4,15,13], the Lagrangian is obtained by computing the asymptotic expansion at high energy of the spectral action functional [11] on a noncommutative space which is the product of an ordinary (commutative) spacetime manifold and extra dimensions given in the form of a noncommutative space which is metrically zero-dimensional, but K-theoretically six-dimensional. The choice of the noncommutative space determines the particle physics content of the model and the gauge symmetries. The masses and mixing angles arise geometrically as coordinates on the moduli space of Dirac operators of the spectral triple describing the extra dimensions. In the case of the model developed in [13], the particle physics content is the same as in the νMSM, namely, in addition to the particles of the Minimal Standard Model, one has right-handed neutrinos with Majorana mass terms. However, the model is significantly different from νMSM when it comes to the properties of the action functional. In fact, as proved in [13], the asymptotic expansion of the spectral action contains the full Standard Model Lagrangian, with the additional Majorana terms for the right-handed neutrinos. One has unification of the coupling constants of the three forces, hence the model has a preferred energy scale at unification. The asymptotic expansion of the spectral action also contains gravitational terms, which are the most interesting part from the point of view of applications to cosmological models. These terms contain an Einstein–Hilbert term, a cosmological term, a conformal gravity term, a nondynamical topological term, and a conformal coupling of the Higgs field to gravity.

In the approach to cosmological models developed in [26], one uses the fact that, at the unification scale, in the terms one obtains in the asymptotic expansion of the spectral action, the usual gravitational and cosmological constants are replaced by effective constants, whose expression at unification depends upon the Yukawa parameters of the particle physics content of the model. In [26], this is used to derive an early universe model in which one allows these effective gravitational and cosmological constant determined by the boundary conditions given by the asymptotic expansion of the spectral action, to run with the RGE flow of the associated particle physics model, according to the running of the Yukawa parameters and Majorana mass terms. This allows for a much more serious variability of the effective gravitational and cosmological constant in between the unification and the electroweak epochs of the very early universe (and in particular during the inflationary

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epoch) than is usually considered in other gravity models. In [26], this type of running leads to several consequences on early universe cosmology, from mechanisms for inflation to effects on the gravitational waves and the evaporation law for primordial black holes.

For the purpose of the present Letter, the specific issues of the running of the gravitational parameters and of the resulting interpretations within the model are not directly relevant, since the results we give here are specifically about the running with the RGE flow of those expressions of Yukawa parameters and Majorana masses, which enter the value at unification of the asymptotic expansion of the spectral action.

The analysis performed in [26] depends on the choice of initial boundary conditions at unification for the renormalization group flow. The results of [26] are obtained using the default boundary conditions of [1]. However, as we show in the present Letter, one obtains significantly different behaviors of the coefficients of the asymptotic expansion of the spectral action by changing boundary conditions. This implies that there will be the possibility of drawing interesting exclusion curves in the space of all possible boundary conditions, on the basis of comparing the model with cosmological data, for example through the predictions for the tensor-to-scalar ratio and the spectral index derived in [26].

For the purpose of the present Letter, we first show how one obtains significantly different curves for the running of the parameters in the action functional with different choices of the boundary conditions. This shows, as one would have expected, a sensitive dependence on the initial conditions at unification, which means that a fine-tuning problem arises within the model, in the choice of the data at unification.

The main result of the Letter is then to exhibit a specific choice of boundary conditions, which we denote *modified maximal mixing conditions*, which differ from the default one of [1], and which have the desired properties. Namely, we show that all the geometric constraints on the data at unification derived in [13] are satisfied by our choice of boundary conditions. We also show that, when running the RGE flow with those boundary conditions, one obtains values in the low energy limit that are still compatible in order of magnitude with the physical predictions and observed values at low energy, as in the case of the default conditions of [1].

An important aspect of these models is understanding how much nonperturbative effects in the spectral action may affect the low energy behavior of the model, since that is the main obstacle to extending to the more recent universe the cosmological models of [26]. Our estimates of the low energy behavior when matching geometric boundary conditions at unification may also provide some indirect evidence for the magnitude of such effects.

Recent results of [27] show that, at least in the case of sufficiently symmetric geometries, the spectral action can be fully computed nonperturbatively, using the technique of [12], and the nonperturbative effects are limited to the shape of the inflation potential.

2. The spectral action and the renormalization group flow

In noncommutative geometry one models the analog of a Riemannian manifold through the notion of a *spectral triple*, consisting of data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ of an involutive algebra, a Hilbert space representation, and a Dirac operator, which has the compatibility condition of having bounded commutators with elements of the algebra. Additional structure, in the form of grading γ and real involution J with compatibility conditions with the data $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ are also introduced. In the particle physics context, γ corresponds to the two chiralities of fermions and J to the involution that exchanges particles and antiparticles. See [13] for a more detailed account of the underlying mathematical structure, which we do not recall here. The action functional considered in noncommutative geometry models for particle physics is based on the spectral action [11] for the Dirac operator of a spectral triple, with additional fermionic terms. In the model of [13] this takes the form

$$\mathrm{Tr}\big(f(D_A/\Lambda)\big) + \frac{1}{2}\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle. \tag{2.1}$$

Here $D_A = D + A + \varepsilon' J A J^{-1}$ is the Dirac operator with inner fluctuations given by the gauge potentials of the form $A = A^\dagger = \sum_k a_k[D, b_k]$, for elements $a_k, b_k \in \mathcal{A}$. The ε' is just a function of $n \bmod 8$ that gives -1 for n congruent to $1 \bmod 4$ and 1 for all other values of n. The fermionic term $\langle J\tilde{\xi}, D_A\tilde{\xi}\rangle$ should be seen as a pairing of classical fields $\tilde{\xi} \in \mathcal{H}^+ = \{\xi \in \mathcal{H} \mid \gamma\xi = \xi\}$, viewed as Grassmann variables. For the purpose of cosmological applications, the most important part of this action functional is the one that comes from the asymptotic expansion at high energy Λ of the spectral action $\mathrm{Tr}(f(D_A/\Lambda))$, since this contains the gravitational terms and their coupling to matter.

2.1. The asymptotic form of the spectral action

The asymptotic expansion of the spectral action is obtained in the form (see [11,13])

$$\mathrm{Tr}\big(f(D/\Lambda)\big) \sim \sum_{k\in \mathrm{DimSp}+} f_k \Lambda^k \int\!\!\!\!\!\!- |D|^{-k} + f(0)\zeta_D(0) + o(1), \tag{2.2}$$

where $f_k = \int_0^\infty f(v)v^{k-1}\,dv$ are the momenta of the function f and the noncommutative integration is defined in terms of residues of zeta functions

$$\zeta_{a,D}(s) = \mathrm{Tr}\big(a|D|^{-s}\big). \tag{2.3}$$

The sum in (2.2) is over points in the *dimension spectrum* of the spectral triple, which is a refined notion of dimension for noncommutative spaces, consisting of the set of poles of the zeta functions (2.3). More explicitly, as proved in [13], when applied to a noncommutative space of the form $X \times F$, with X an ordinary 4-dimensional (Euclidean) spacetime and F the noncommutative space whose algebra of coordinates is $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with $\mathbb{H}$ the algebra of quaternions, the expansion (2.2) of $\mathrm{Tr}(f(D_A/\Lambda))$ gives terms of the form

$$\begin{aligned} S = {} & \frac{1}{2\kappa_0^2}\int R\sqrt{g}\,d^4x + \gamma_0\int \sqrt{g}\,d^4x \\ & + \alpha_0 \int C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma}\sqrt{g}\,d^4x + \tau_0\int R^*R^*\sqrt{g}\,d^4x \\ & + \frac{1}{2}\int |DH|^2\sqrt{g}\,d^4x - \mu_0^2\int |H|^2\sqrt{g}\,d^4x \\ & - \xi_0\int R|H|^2\sqrt{g}\,d^4x + \lambda_0\int |H|^4\sqrt{g}\,d^4x \\ & + \frac{1}{4}\int \big(G^i_{\mu\nu}G^{\mu\nu i} + F^\alpha_{\mu\nu}F^{\mu\nu\alpha} + B_{\mu\nu}B^{\mu\nu}\big)\sqrt{g}\,d^4x. \end{aligned} \tag{2.4}$$

The coefficients of these terms are functions

$$\frac{1}{2\kappa_0^2} = \frac{96 f_2\Lambda^2 - f_0\mathfrak{c}}{24\pi^2},$$

$$\gamma_0 = \frac{1}{\pi^2}\left(48 f_4\Lambda^4 - f_2\Lambda^2\mathfrak{c} + \frac{f_0}{4}\mathfrak{d}\right),$$

$$\alpha_0 = -\frac{3f_0}{10\pi^2},$$

$$\tau_0 = \frac{11 f_0}{60\pi^2},$$

$$\mu_0^2 = 2\frac{f_2\Lambda^2}{f_0} - \frac{\mathfrak{e}}{\mathfrak{a}},$$
$$\xi_0 = \frac{1}{12},$$
$$\lambda_0 = \frac{\pi^2\mathfrak{b}}{2f_0\mathfrak{a}^2}. \tag{2.5}$$

These depend upon the three parameters f_0, f_2, f_4, where $f_0 = f(0)$ and for $k > 0$

$$f_k = \int_0^\infty f(v)v^{k-1}\,dv,$$

where f_0 depends upon the common value of the coupling constants at unification energy and f_2 and f_4 are free parameters of the model. The expressions (2.5) also depend upon the energy scale Λ and the running of these parameters is the main topic of our present investigation. In addition to the explicit dependence on Λ of the coefficients (2.5) there is also an additional and very interesting dependence on Λ through the coefficients $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$ and $\mathfrak{e}$. These are functions of the Yukawa parameters and Majorana masses of the particle physics content of the model, in the form

$$\mathfrak{a} = \mathrm{Tr}\big(Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e + 3\big(Y_u^\dagger Y_u + Y_d^\dagger Y_d\big)\big),$$
$$\mathfrak{b} = \mathrm{Tr}\big(\big(Y_\nu^\dagger Y_\nu\big)^2 + \big(Y_e^\dagger Y_e\big)^2 + 3\big(Y_u^\dagger Y_u\big)^2 + 3\big(Y_d^\dagger Y_d\big)^2\big),$$
$$\mathfrak{c} = \mathrm{Tr}\big(MM^\dagger\big),$$
$$\mathfrak{d} = \mathrm{Tr}\big(\big(MM^\dagger\big)^2\big),$$
$$\mathfrak{e} = \mathrm{Tr}\big(MM^\dagger Y_\nu^\dagger Y_\nu\big). \tag{2.6}$$

2.2. Renormalization group flow

The particle physics models based on the spectral action functional of noncommutative geometry as in [11,13] are (at present) entirely a classical theory. In particular, this means that whenever physical predictions are derived in these models using renormalization group techniques to lower the energy scale from unification, where the model naturally lives, to ordinary energies, one uses beta functions and renormalization group equations that are imported from the ordinary QFT of the specific particle physics Lagrangian that is obtained from the asymptotic expansion of the spectral action. This is a delicate issue, since in fact the asymptotic expansion includes both matter and gravitational terms. The beta functions and RGE flow adopted here (as in [26]) is the one for the extension of the Minimal Standard Model that includes right-handed neutrinos with Majorana mass terms, while the gravitational effects are not taken into account in the form of RGE. This is an approximation, since the nonminimal coupling of the Higgs to gravity in the model means that one no longer has a clear separation between the particle and gravitational sectors. Consequences of modified RGE flows coming from nonminimal couplings to the Higgs can be found for instance in [7,8], and in [32], while effects from gravity terms are considered in [18]. For the Minimal Standard Model, there is an extensive literature on the form of the beta functions and the RGE flow, see for instance [24] and references therein. In the case of the noncommutative geometry model of particle physics of [14], which did not yet include right-handed neutrinos and Majorana mass terms, predictions of the Higgs mass were obtained based on using the RGE of the Minimal Standard Model.

The RGE analysis of the model of [13] considered in [26], which we also work with in this Letter, differs from the usual RGE analysis of the Standard Model in the following ways:

- Instead of the RGE of the Minimal Standard Model, one considers the equations for the extension with right-handed neutrinos and Majorana masses, as in [1]. As in [1] these are treated by considering different effective field theories in between the different see–saw scales (see also [2,3]).
- We vary the initial conditions at unification, by imposing the geometric constraints derived in [13] and at the same time requiring that the low energy values remain close to the expected physical values.

The specific information on the NCG model of [13] enters here in two ways: first in selecting the appropriate matter content of the model (the presence of the extra right-handed neutrinos with Majorana mass terms in addition to the usual Standard Model), hence the use of the RGE flow of [1], and also in the geometric constraints imposed on the boundary conditions at unification.

We use, as in [26] the renormalization group equations for the Standard Model with right-handed neutrinos and Majorana mass terms of [1]. The numerical results described here are obtained with a Mathematica code based on the REAP program of [1] adapted to our model by the first author.

We recall here that the RGE for this particle physics model is given (at one loop) by the beta functions [1]

$$16\pi^2\beta_{g_i} = b_i g_i^3 \quad \text{with } (b_{SU(3)}, b_{SU(2)}, b_{U(1)}) = \left(-7, -\frac{19}{6}, \frac{41}{10}\right),$$
$$16\pi^2\beta_{Y_u} = Y_u\left(\frac{3}{2}Y_u^\dagger Y_u - \frac{3}{2}Y_d^\dagger Y_d + \mathfrak{a} - \frac{17}{20}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right),$$
$$16\pi^2\beta_{Y_d} = Y_d\left(\frac{3}{2}Y_d^\dagger Y_d - \frac{3}{2}Y_u^\dagger Y_u + \mathfrak{a} - \frac{1}{4}g_1^2 - \frac{9}{4}g_2^2 - 8g_3^2\right),$$
$$16\pi^2\beta_{Y_\nu} = Y_\nu\left(\frac{3}{2}Y_\nu^\dagger Y_\nu - \frac{3}{2}Y_e^\dagger Y_e + \mathfrak{a} - \frac{9}{20}g_1^2 - \frac{9}{4}g_2^2\right),$$
$$16\pi^2\beta_{Y_e} = Y_e\left(\frac{3}{2}Y_e^\dagger Y_e - \frac{3}{2}Y_\nu^\dagger Y_\nu + \mathfrak{a} - \frac{9}{4}g_1^2 - \frac{9}{4}g_2^2\right),$$
$$16\pi^2\beta_M = Y_\nu Y_\nu^\dagger M + M\big(Y_\nu Y_\nu^\dagger\big)^T,$$
$$16\pi^2\beta_\lambda = 6\lambda^2 - 3\lambda\left(3g_2^2 + \frac{3}{5}g_1^2\right) + 3g_2^4 + \frac{3}{2}\left(\frac{3}{5}g_1^2 + g_2^2\right)^2 + 4\lambda\mathfrak{a} - 8\mathfrak{b}.$$

Notice that we use here the normalization of the coupling constants used in [1], which is different from the one of [13].

In particular, as in [1], we solve numerically these equations using different effective field theories in the intervals of energies between the three see–saw scales, with matching boundary conditions. Namely, starting from assigned boundary conditions at unification, one runs the RGE flow down until the first see–saw scale (the top eigenvalue of the Majorana mass matrix M). Then one integrates out the higher modes by introducing a first effective theory with Yukawa parameters $Y_\nu^{(3)}$ obtained by removing the last row of Y_ν in the basis where M is diagonal and with Majorana mass matrix $M^{(3)}$ obtained by removing the last row and column. One then restarts the RGE flow for these new variables with matching boundary conditions at the top see–saw scale, until the second see–saw scale, and so on. One has in this way effective field theories $(Y_\nu^{(k)}, M^{(k)})$, $k = 3, 2, 1$.

We study the effect on this RGE flow of changing boundary conditions at unification scale, and we then derive consequences for the running of the coefficients $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$ of (2.6). In the next section we show, as could have been expected, that the running is highly sensitive to the choice of the initial conditions at unification. This shows that there is an important fine-tuning issue in the model related to the assigned values at unification. We then present in the following section a specific choice of boundary conditions that meets all the geometric constraints on the model and that produces realistic values at low energies.

2.3. A remark on gravitational and Yukawa parameters in the NCG models

This subsection is not directly relevant to the main result of the Letter, which is simply a statement about the running of the parameters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$ of (2.6), subject to different choices of boundary conditions at unification, with particular attention to those dictated by the geometric constraints imposed by the model of [13] at unification. However, we include it here to discuss briefly and compare different existing points of view on the role of the parameters (2.6) in the coefficients (2.5) of the spectral action expansion.

In the NCG model of [13], the relation (2.5) between the coefficients of the asymptotic expansion of the spectral action and the Yukawa coupling and Majorana mass terms of the particle physics sector holds only at unification energy. In particular, the dependence of the effective gravitational and effective cosmological constants upon the parameters $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$ of (2.6) only sets the boundary conditions at unification. In [13] (see also the exposition in Chapter 1 of [16]), consequently, the running of the gravitational terms of the model is deduced from the usual approach as in [18], see also [19], by which one obtains only a very moderate (or essential lack of) running of the gravitational parameters. The running of the particle physics sector is then ruled, in the NCG models, only by the RGE flow of the matter Lagrangian, neglecting gravitational effects (with the caveat mentioned above on the nonminimal coupling with the Higgs).

However, there are cosmological models that include the possibility of a much more drastic variability of the gravitational parameters in the very early universe, including in particular the inflationary epoch. Scenarios with variable gravitational constant had been considered early on in Jordan–Brans–Dicke gravity, where the variability happens through the nonminimal coupling of gravity to a scalar field, and more recently within other modified gravity models, and in terms of RGE running [21], as well as in the context of primordial black holes with gravitational memory (see for instance [5], or the recent [10] and references therein). Similarly, a variable cosmological constant plays a role in various models (see, for example [6,20,25,33]).

In [26], therefore, a different viewpoint on the effective gravitational and cosmological constant in the asymptotic expansion of the spectral action in the NCG models is proposed, and a possible early universe model is investigated, which only covers the epochs in between the unification and the electroweak eras, a period which is expected to include the inflationary epoch. It is shown that, if one considers an effective action where the gravitational and cosmological constant are allowed to run according to the RGE flow of the coefficients (2.6) through the expressions (2.5) and with the assigned boundary conditions at unification, then one recovers many of the scenarios predicted by other models with variable gravitational and cosmological constant, as [10,17,22], and several different mechanism for inflation, with predictions about parameters such as the spectral index and tensor-to-scalar ratio.

Other recent cosmological applications of [13], such as those in [9,27–31], follow the more conventional point of view on the asymptotic expansion of the spectral action and the form of the coefficients (2.5). These different viewpoints do not directly affect in any way the results of the present Letter, and we only mention them here for the reader's information.

3. Effects of changing boundary conditions

The REAP program from [1] allows the user to adjust the boundary conditions. These changes are generally made at Λ_{unif}, taken here to be 2×10^{16} GeV. As we understand that only fine tuned initial conditions for the universe allowed its current form, we expect the boundary conditions at unification energy to drastically effect the development of our model parameters. We show here, as an example, the different running of the coefficients $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$ of (2.6) for the default boundary conditions and for the maximal mixing case. We also show explicitly the changing behavior of the running of one of these coefficients when one of the parameters varies at unification, in order to illustrate the significant dependence on the initial conditions.

3.1. The default boundary conditions

The boundary conditions at unification used in [26] are the default boundary conditions of [1] (see Fig. 1). These have the following values:

$$\lambda(\Lambda_{unif}) = \frac{1}{2},$$

$$Y_u(\Lambda_{unif}) = \begin{pmatrix} 5.40391 \times 10^{-6} & 0 & 0 \\ 0 & 0.00156368 & 0 \\ 0 & 0 & 0.482902 \end{pmatrix}.$$

For $Y_d(\Lambda_{unif}) = (y_{ij})$ they have

$$y_{11} = 0.0000482105 - 3.382 \times 10^{-15} i,$$
$$y_{12} = 0.000104035 + 2.55017 \times 10^{-7} i,$$
$$y_{13} = 0.0000556766 + 6.72508 \times 10^{-6} i,$$
$$y_{21} = 0.000104035 - 2.55017 \times 10^{-7} i,$$
$$y_{22} = 0.000509279 + 3.38205 \times 10^{-15} i,$$
$$y_{23} = 0.00066992 - 4.91159 \times 10^{-8} i,$$
$$y_{31} = 0.000048644 - 5.87562 \times 10^{-6} i,$$
$$y_{32} = 0.000585302 + 4.29122 \times 10^{-8} i,$$
$$y_{33} = 0.0159991 - 4.21364 \times 10^{-20} i,$$

$$Y_e(\Lambda_{unif}) = \begin{pmatrix} 2.83697 \times 10^{-6} & 0 & 0 \\ 0 & 0.000598755 & 0 \\ 0 & 0 & 0.0101789 \end{pmatrix},$$

$$Y_{nu}(\Lambda_{unif}) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.1 \end{pmatrix},$$

$$M(\Lambda_{unif}) = \begin{pmatrix} -6.01345 \times 10^{14} & 3.17771 \times 10^{12} & -6.35541 \times 10^{11} \\ 3.17771 \times 10^{12} & -1.16045 \times 10^{14} & 5.99027 \times 10^{12} \\ -6.35541 \times 10^{11} & 5.99027 \times 10^{12} & -4.6418 \times 10^{12} \end{pmatrix}.$$

3.2. Maximal mixing example

To look at the maximal mixing case, we simply change Y_{nu} at unification energy. With maximal mixing, our parameters will take these values:

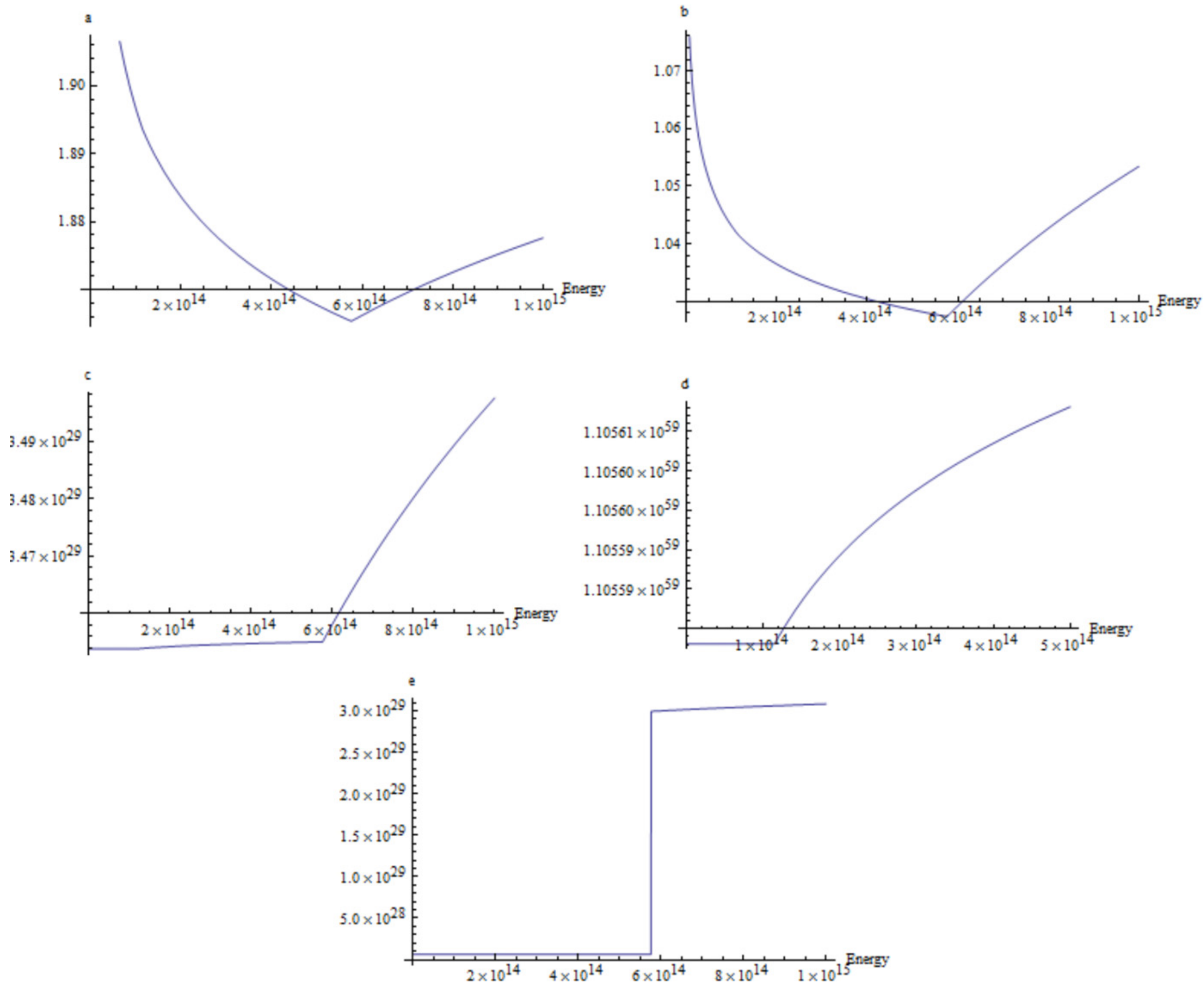

Fig. 1. The running coefficients with default boundary conditions near top see–saw scale.

$$\zeta = \exp(2\pi i/3),$$

$$U_{PMNS}(\Lambda_{unif}) = \frac{1}{3}\begin{pmatrix} 1 & \zeta & \zeta^2 \\ \zeta & 1 & \zeta \\ \zeta^2 & \zeta & 1 \end{pmatrix}.$$

From the available estimates of the neutrino masses, we get the diagonal mass matrix

$$\delta_{(\uparrow 1)} = \frac{1}{246}\begin{pmatrix} 12.2\times 10^{-9} & 0 & 0 \\ 0 & 170\times 10^{-6} & 0 \\ 0 & 0 & 15.5\times 10^{-3} \end{pmatrix}.$$

Finally,

$$Y_\nu = U^\dagger_{PMNS}\delta_{(\uparrow 1)}U_{PMNS}.$$

Using this form for Y_ν and the default boundary conditions on all the other parameters, we can look at the running coefficients. From Fig. 2, we see that there are vast differences in the development of the parameters with this boundary condition change.

3.3. Running coefficients with changing boundary conditions

It is possible to get even more interesting behavior by using less standard boundary conditions. By changing just one parameter we can examine how it affects the flow of our running parameters. A specific example is the Y_ν matrix. Using our standard boundary conditions, this matrix is diagonal at unification energy. We can adjust each of these elements on the diagonal, which correspond to our neutrino masses, to affect our flow. Using animation functions in mathematica, it is possible to get a nearly continuous idea of how the flow of our parameters develops with our boundary conditions. Fig. 3 illustrate such a development discretely.

In these diagrams, we notice the transition changing as the upper neutrino mass varies. The sharp transition at the upper see–saw scale comes from the program integrating out the heavy neutrino at this scale. The second plot shows the behavior we expect from the standard conditions. In the first plot we can see the upper and middle transitions are much closer together than in our second plot. The final plot shows the transition at a much higher energy, corresponding to the higher neutrino mass. From these and other such plots, we learn how the running develops independently by changing different parameters. Of course, chang-

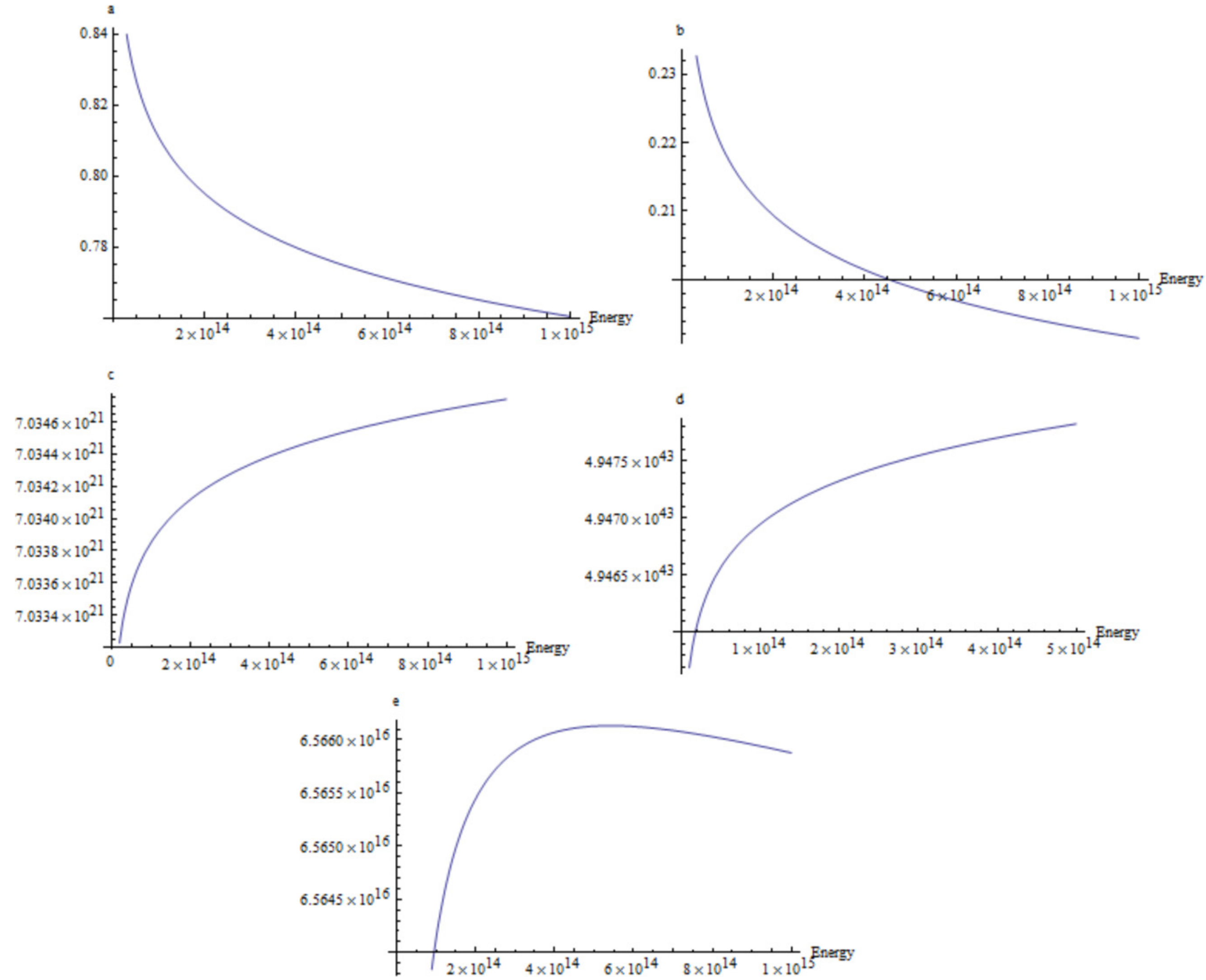

Fig. 2. The running coefficients with maximal mixing boundary conditions near top see–saw scale.

ing multiple parameters complicates this development and is dealt with in more detail when matching specific boundary conditions.

4. Geometric constraints at unification

There are some constraints on the boundary conditions at unification that are imposed by the underlying geometry of the model. These are derived in [13], see also the discussion in Section 1 of [16]. We recall them here. Not all of these constraints are satisfied by the default boundary conditions of [1], so a first improvement on the model of [26] is to identify choices of boundary conditions that satisfy these constraints, and then, among them, eliminate those that produce nonphysical predictions.

We show here how to obtain a choice of boundary conditions that satisfy all the constraints by modifying the maximal mixing conditions.

4.1. Constraint on λ

A first constraint imposed by the geometry is on the value of the Higgs self-coupling λ at unification. This satisfies

$$\lambda(\Lambda_{unif}) = \frac{\pi^2}{2f_0}\frac{\mathfrak{b}(\Lambda_{unif})}{\mathfrak{a}(\Lambda_{unif})^2}. \tag{4.1}$$

Looking at our maximal mixing boundary conditions we can calculate that $\lambda(\Lambda_{unif}) = 2.989$. By setting it to this value at unification energy in our flow we can ensure that this requirement is met.

4.2. The $\mathfrak{a}$ *parameter and the Higgs vacuum*

The model of [13] also relates the parameter $\mathfrak{a}$ to the Higgs vacuum through the relation

$$\frac{\sqrt{\mathfrak{a} f_0}}{\pi} = \frac{2M_W}{g}, \tag{4.2}$$

where g is the common value of the coupling constants at unification and M_W is the W-boson mass. As M_W is directly proportional to $\sqrt{\mathfrak{a}}$, this condition is a statement of the equality of f_0 and the coupling constants at unification energy.

4.3. Constraint on $\mathfrak{c}$

The see–saw mechanism is implemented in [13] geometrically, through the fact that the restriction of the Dirac operator $D(Y)$ to the subspace of $\mathcal{H}_F$ spanned by ν_R, ν_L, $\bar{\nu}_R$, $\bar{\nu}_L$ is of the form

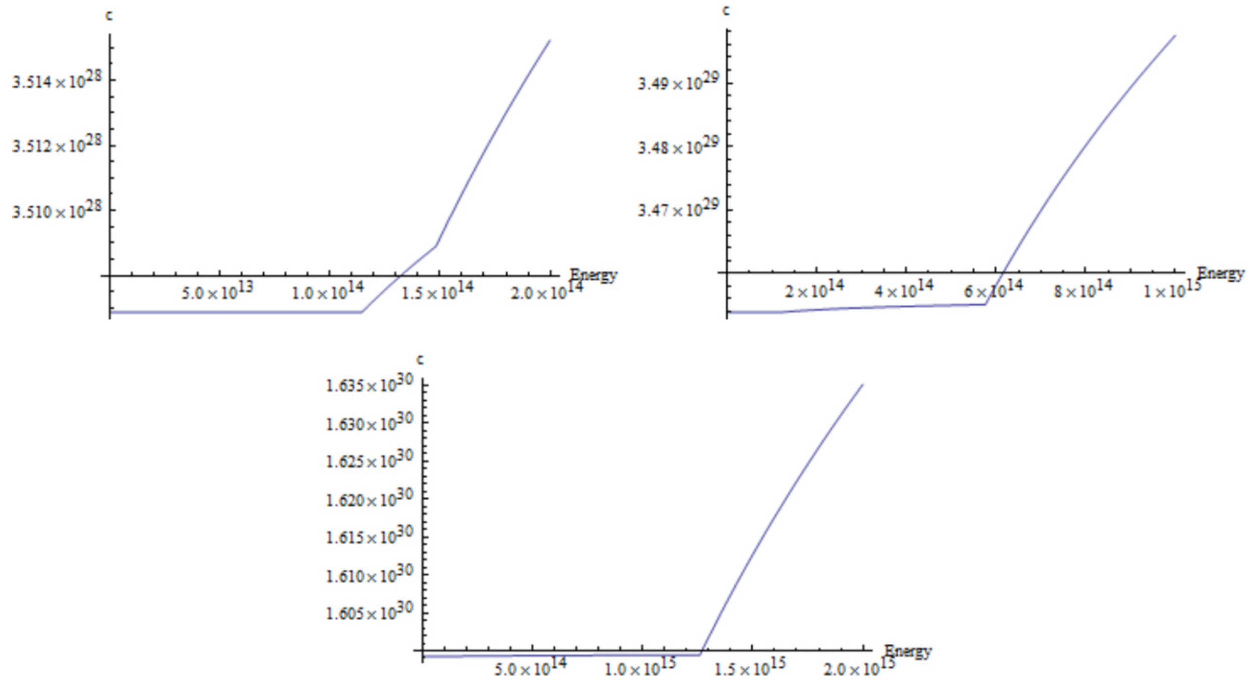

Fig. 3. Coefficient $\mathfrak{c}$ at the upper see–saw scale with the first term of Y_ν as 0.5, 1.0, and 1.5 respectively.

$$\begin{pmatrix} 0 & M_\nu^\dagger & \bar{M}_R^\dagger & 0 \\ M_\nu & 0 & 0 & 0 \\ \bar{M}_R & 0 & 0 & \bar{M}_\nu^\dagger \\ 0 & 0 & \bar{M}_\nu & 0 \end{pmatrix}, \tag{4.3}$$

where M_ν is the neutrino mass matrix, see Lemma 1.225 of [16]. This imposes a constraint at unification on the coefficient $\mathfrak{c}$, of the form

$$\frac{2 f_2 \Lambda_{unif}^2}{f_0} \leqslant \mathfrak{c}(\Lambda_{unif}) \leqslant \frac{6 f_2 \Lambda_{unif}^2}{f_0}. \tag{4.4}$$

By setting our Majorana mass matrix to 10 times its default value, the inequality can be matched. In this particular case, the f_2 that is used is in the range given in [26]. f_0 is calculated from the coupling constants at unification energy.

4.4. The mass relation at unification

Another prediction which is specific to the model of [13] is a quadratic relation between the masses at unification scale, of the form

$$\sum_{generations} \left(m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2\right)\Big|_{\Lambda=\Lambda_{unif}} = 8M_W^2\Big|_{\Lambda=\Lambda_{unif}}, \tag{4.5}$$

where m_ν, m_e, m_u, and m_d are the masses of the leptons and quarks, that is, the eigenvectors of the matrices $\delta_{\uparrow 1}$, $\delta_{\downarrow 1}$, $\delta_{\uparrow 3}$ and $\delta_{\downarrow 3}$, respectively, and M_W is the W-boson mass. We use the fact that M_W is given as a function of the model parameters by

$$\frac{\sqrt{\mathfrak{a}}}{2\sqrt{2}} = M_W. \tag{4.6}$$

So, our equation becomes

$$\sum_{generations} \left(m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2\right)\Big|_{\Lambda=\Lambda_{unif}} = \mathfrak{a}|_{\Lambda=\Lambda_{unif}}. \tag{4.7}$$

In our maximal mixing boundary conditions, we get

$$\sum_{generations} \left(m_\nu^2 + m_e^2 + 3m_u^2 + 3m_d^2\right)\Big|_{\Lambda=\Lambda_{unif}} = 0.6698 = \mathfrak{a}|_{\Lambda=\Lambda_{unif}}. \tag{4.8}$$

This value of $\mathfrak{a}$, when converted to conventional units, gives a value of M_W of 72 GeV. The expected value on M_W is around 80 GeV so these boundary conditions are believable.

4.5. Modified maximal mixing

Thus, the conclusion of this analysis is that we obtain a choice of boundary conditions that satisfies all the geometric constraints of the geometric model at unification by using our maximal mixing boundary conditions as described in the previous section, but with a modified Majorana mass matrix and Higgs parameter, as explained here. We refer to the resulting boundary conditions as the *modified maximal mixing conditions*.

We then need to check that, when we run the RGE flow with these initial conditions at unification, we obtain values at low energies that are compatible, within order of magnitude, with the required physical values. We discus this in the next section.

5. Low energy physical constraints

At the electroweak scale, physical data impose other boundary conditions on some of the Yukawa matrices. Finding the unification scale conditions that can also match these lower energy requirements is crucial to the theory.

We look at the conditions that are expected from physical data and compare to the results from the running of the model parameters. We show that our modified maximal mixing boundary conditions also satisfy the required constraints at low energy.

5.1. Boundary conditions at the electroweak scale

Current predictions at the electroweak scale tell us that the CKM matrix at Λ_0 can be taken to be of the form

$$U_{CKM}(\Lambda_0) = \begin{pmatrix} 0.97419 & 0.2257 & 0.00359 \\ 0.2256 & 0.97334 & 0.0415 \\ 0.00874 & 0.0407 & 0.999133 \end{pmatrix}.$$

Combined with

$$\delta_{(\downarrow 3)}(\Lambda_0) = \frac{1}{246}\begin{pmatrix} 0.00475 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4.25 \end{pmatrix}$$

we get that the Yukawa parameters for the quarks are given by

$$Y_d = U_{CKM}(\Lambda_0)\delta_{(\downarrow 3)}(\Lambda_0)U_{CKM}(\Lambda_0)^\dagger$$

and

$$Y_u(\Lambda_0) = \frac{1}{246}\begin{pmatrix} 0.0024 & 0 & 0 \\ 0 & 1.25 & 0 \\ 0 & 0 & 173 \end{pmatrix}.$$

Similarly, for the matrix of charged leptons, the known values and low energy are

$$Y_e(\Lambda_0) = \frac{1}{246}\begin{pmatrix} 0.000511 & 0 & 0 \\ 0 & 0.1056 & 0 \\ 0 & 0 & 1.777 \end{pmatrix}.$$

The conditions for the other parameters are all given at the unification scale.

5.2. Comparison of expected and measured values

We use the modified maximal mixing boundary conditions to run the parameters and compare to the physical boundary conditions at low energy. From this analysis, we get that the measured Yukawa parameters for quarks are

$$Y_{d,\text{measured}}(\Lambda_0) = \frac{1}{246}\begin{pmatrix} 0.0121 & 0 & 0 \\ 0 & 0.128 & 0 \\ 0 & 0 & 4.032 \end{pmatrix}$$

and

$$Y_{u,\text{measured}}(\Lambda_0) = \frac{1}{246}\begin{pmatrix} 0.0032 & 0 & 0 \\ 0 & 0.9223 & 0 \\ 0 & 0 & 248 \end{pmatrix}.$$

For the charged leptons, we get the mass matrix

$$Y_{e,\text{measured}}(\Lambda_0) = \frac{1}{246}\begin{pmatrix} 0.000699 & 0 & 0 \\ 0 & 0.147 & 0 \\ 0 & 0 & 2.51 \end{pmatrix}.$$

Comparing these to the expected values at low energies, we see that the order of magnitude and form of the matrices agree. While the agreement is not exact, it seems that this is the closest we can get while maintaining the geometric constraints of the model. In order to make the agreement more exact, further fine tuning is required.

6. Conclusions

In this Letter we investigate the RGE running of the coefficients $\mathfrak{a}$, $\mathfrak{b}$, $\mathfrak{c}$, $\mathfrak{d}$, $\mathfrak{e}$ of (2.6), which appear in the asymptotic expansion of the spectral action functional of the noncommutative geometry model of particle physics of [13]. The equations used in the renormalization group analysis are based on the beta function calculation of [1], for the extension of the Standard Model that includes right-handed neutrinos with Majorana mass terms.

We showed that the running is very sensitive to the fine tuning of the initial conditions at unification energy. We exhibited, as significant examples, the different running for the default boundary conditions of [1] and the maximal mixing conditions, and we also showed the effect on the running of the coefficients of changing a single parameter in the initial conditions at unification.

We then showed that a choice of boundary conditions based on the maximal mixing, with a modified Majorana mass matrix and Higgs parameter at unification, satisfies all the geometric constraints on the model described in [13], while at the same time gives rise to low energy values that are, within order of magnitude, in agreement with the expected physical values.

We consider here the asymptotic expansion of the spectral action in the range of energies from the unification scale down to the electroweak scale. Within this range of energies, replacing the nonperturbative form of the spectral action with its asymptotic expansion is justified, since the error term is at worse of the order of Λ^{-2}. However, it is known that interesting nonperturbative effects do arise in the spectral action, as shown in the recent results of [12], for example, in the form of a slow-roll inflation potential. Cosmological implications of these effects are discussed in [27]. In terms of the RGE analysis considered here, we find that with our choice of modified maximal mixing conditions at unification, one obtains low energy values that are in agreement with the physical data within order of magnitude, which is not yet as good an agreement as one could hope for. This may be an indication that further fine tuning of the initial conditions may achieve a better fit of the low energy data, or else that nonperturbative effects may play a role. This is not completely unlikely, considering that the nonperturbative effects identified in [12] essentially appear in the coupling of Higgs and gravity and this in turn can affect the form of the RGE running, as observed in [7,8,32]. These questions will require further investigation.

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