Quantum statistical mechanics, $L$-series, Anabelian Geometry

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General philosophy:

- Zeta functions are counting devices: spectra of operators with spectral multiplicities, counting ideals with given norm, number of periodic orbits, rational points, etc.
- Zeta function does not determine object: isospectral manifolds, arithmetically equivalent number fields, isogeny
- but ... sometimes a family of zeta functions does
- Zeta functions occur as partition functions of physical systems
Number fields: finite extensions $K$ of the field of rational numbers $\mathbb{Q}$.

- zeta functions: Dedekind $\zeta_K(s)$ (for $\mathbb{Q}$ Riemann zeta)
- symmetries: $G_K = Gal(\overline{K}/K)$ absolute Galois group;
  abelianized $G_K^{ab}$
- adeles $\mathbb{A}_K$ and ideles $\mathbb{A}_K^*$, Artin map $\vartheta_K : \mathbb{A}_K^* \to G_K^{ab}$
- topology: analogies with 3-manifolds (arithmetic topology)

How well do we understand them?
Analogy with manifolds: are there complete invariants?
Recovering a Number Field from invariants

- Dedekind zeta function $\zeta_K(s) = \zeta_L(s)$ arithmetic equivalence
- Gaussmann examples:
  
  $K = \mathbb{Q}(\sqrt[8]{3})$ and $L = \mathbb{Q}(\sqrt[8]{3 \cdot 2^4})$

not isomorphism $K \neq L$

- Adeles rings $\mathbb{A}_K \cong \mathbb{A}_L$ adelic equivalence $\Rightarrow$ arithmetic equivalence; Komatsu examples:
  
  $K = \mathbb{Q}(\sqrt[8]{2 \cdot 9})$ and $L = \mathbb{Q}(\sqrt[8]{2^5 \cdot 9})$

not isomorphism $K \neq L$
Abelianized Galois groups: $G_K^{ab} \cong G_L^{ab}$ also not isomorphism; Onabe examples:

$$K = \mathbb{Q}(\sqrt{-2}) \text{ and } L = \mathbb{Q}(\sqrt{-3})$$

not isomorphism $K \neq L$

But ... absolute Galois groups $G_K \cong G_L \Rightarrow$ isomorphism $K \cong L$: Neukirch–Uchida theorem
(Grothendieck’s anabelian geometry)
Question: Can combine $\zeta_K(s)$, $A_K$ and $G_{K}^{ab}$ to something as strong as $G_K$ that determines isomorphism class of $K$?

Answer: Yes! Combine as a Quantum Statistical Mechanical system

Main Idea:

- Construct a QSM system associated to a number field
- Time evolution and equilibrium states at various temperatures
- Low temperature states are related to L-series
- Extremal equilibrium states determine the system
- System recovers the number field up to isomorphism

Purely number theoretic consequence:
An identity of all $L$-functions with Großencharakter gives an isomorphism of number fields
Quantum Statistical Mechanics (minimalist sketch)

- $\mathcal{A}$ unital $C^*$-algebra of observables
- $\sigma_t$ time evolution, $\sigma : \mathbb{R} \to \text{Aut}(\mathcal{A})$
- states $\omega : \mathcal{A} \to \mathbb{C}$ continuous, normalized $\omega(1) = 1$, positive
  \[ \omega(a^*a) \geq 0 \]
- equilibrium states $\omega(\sigma_t(a)) = \omega(a)$ all $t \in \mathbb{R}$
- representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, Hamiltonian $H$
  \[ \pi(\sigma_t(a)) = e^{itH}\pi(a)e^{-itH} \]
- partition function $Z(\beta) = \text{Tr}(e^{-\beta H})$
- Gibbs states (equilibrium, inverse temperature $\beta$):
  \[ \omega_\beta(a) = \frac{\text{Tr} (\pi(a)e^{-\beta H})}{\text{Tr}(e^{-\beta H})} \]
Generalization of Gibbs states: **KMS states**
(Kubo–Martin–Schwinger) \( \forall a, b \in A, \exists \) holomorphic \( F_{a,b} \) on strip \( I_\beta = \{ 0 < \text{Im } z < \beta \} \), bounded continuous on \( \partial I_\beta \),

\[
F_{a,b}(t) = \omega(a \sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \omega(\sigma_t(b)a)
\]

- **Fixed** \( \beta > 0 \): KMS\( _\beta \) state convex simplex: extremal states
  (like points in NCG)
Isomorphism of QSM systems: \( \varphi : (\mathcal{A}, \sigma) \rightarrow (\mathcal{B}, \tau) \)

\[ \varphi : \mathcal{A} \xrightarrow{\sim} \mathcal{B}, \quad \varphi \circ \sigma = \tau \circ \varphi \]

C*-algebra isomorphism intertwining time evolution

- Algebraic subalgebras \( \mathcal{A}^\dagger \subset \mathcal{A} \) and \( \mathcal{B}^\dagger \subset \mathcal{B} \): stronger condition: QSM isomorphism also preserves “algebraic structure”

\[ \varphi : \mathcal{A}^\dagger \xrightarrow{\sim} \mathcal{B}^\dagger \]

- Pullback of a state: \( \varphi^* \omega(a) = \omega(\varphi(a)) \)
Why QSM and Number theory? (a historical note)

1995: Bost–Connes QSM system $\mathcal{A}_{BC} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$

- generators $e(r), r \in \mathbb{Q}/\mathbb{Z}$ and $\mu_n, n \in \mathbb{N}$ and relations

\[
\mu_n \mu_m = \mu_m \mu_n, \quad \mu_m^* \mu_m = 1
\]

\[
\mu_n \mu_m^* = \mu_m^* \mu_n \quad \text{if} \quad (n, m) = 1
\]

\[
e(r + s) = e(r)e(s), \quad e(0) = 1
\]

\[
\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s)
\]

- time evolution $\sigma_t(f) = f$ and $\sigma_t(\mu_n) = n^{it} \mu_n$
• representations $\pi_\rho : \mathcal{A}_{BC} \to L^2(\mathbb{N})$, $\rho \in \hat{\mathbb{Z}}^*$

$$\pi_\rho(\mu_n)\epsilon_m = \epsilon_{nm}, \quad \pi_\rho(e(r))\epsilon_m = \zeta_r^m\epsilon_m$$

$\zeta_r = \rho(e(r))$ root of unity

• Hamiltonian $H\epsilon_m = \log(m)\epsilon_m$, partition function

$$Z(\beta) = \text{Tr}(e^{-\beta H}) = \zeta_Q(\beta)$$

Riemann zeta function

• Low temperature KMS states: L-series normalized by zeta

• Galois action on zero temperature states (class field theory)
Further generalizations: other QSM’s with similar properties

- Bost-Connes as $GL_1$-case of QSM for moduli spaces of $\mathbb{Q}$-lattices up to commensurability (Connes-M.M. 2006) $\Rightarrow GL_2$-case, modular curves and modular functions

- QSM systems for imaginary quadratic fields (class field theory): Connes-M.M.-Ramachandran

- B.Jacob and Consani-M.M.: QSM systems for function fields (Weil and Goss $L$-functions as partition functions)

- Ha-Paugam: QSM systems for Shimura varieties $\Rightarrow$ QSM systems for arbitrary number fields (Dedekind zeta function) further studied by Laca-Larsen-Neshveyev

We use these QSM systems for number fields
The Noncommutative Geometry viewpoint:
• Equivalence relation $R$ on $X$: quotient $Y = X/R$. Even for very good $X \Rightarrow X/R$ pathological!
• Functions on the quotient $A(Y) := \{ f \in A(X) \mid fR \text{ invariant} \}$ ⇒ often too few functions: $A(Y) = \mathbb{C}$ only constants
• NCG: $A(Y)$ noncommutative algebra $A(Y) := A(\Gamma_R)$ functions on the graph $\Gamma_R \subset X \times X$ of the equivalence relation with involution $f^*(x, y) = f(y, x)$ and convolution product

$$(f_1 \ast f_2)(x, y) = \sum_{x \sim u \sim y} f_1(x, u)f_2(u, y)$$

• $A(\Gamma_R)$ associative noncommutative $\Rightarrow Y = X/R$ noncommutative space (as good as $X$ to do geometry, but new phenomena: time evolutions, thermodynamics, quantum phenomena)
In the various cases QSM system semigroup action on a space: Bost–Connes revisited (Connes–M.M. 2006)

- **Q-lattices**: $(\Lambda, \phi)$ Q-lattice in $\mathbb{R}^n$: lattice $\Lambda \subset \mathbb{R}^n$ + group homomorphism
  \[
  \phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda
  \]

  - Commensurability: $(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2)$ iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$
    and $\phi_1 = \phi_2 \pmod{\Lambda_1 + \Lambda_2}$
  - Quotient Q-lattices/Commensurability $\Rightarrow$ NC space
  - 1-dimensional Q-lattices up to scaling $C(\mathbb{Z})$

    \[(\Lambda, \phi) = (\lambda \mathbb{Z}, \lambda \rho) \quad \lambda > 0\]

- $\rho \in \text{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim_n \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}$
  - with action of semigroup $\mathbb{N}$ commensurability

    \[\alpha_n(f)(\rho) = f(n^{-1}\rho) \text{ or zero}\]

$C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$ Bost–Connes algebra: moduli space
QSM systems for number fields: algebra and time evolution \((A, \sigma)\)

\[
A_K := C(X_K) \rtimes J^+_K, \quad \text{with} \quad X_K := G^a_K \times \hat{\mathcal{O}}_K^* \hat{\mathcal{O}}_K,
\]

\(\hat{\mathcal{O}}_K = \) ring of finite integral adeles, \(J^+_K = \) is the semigroup of ideals, acting on \(X_K\) by Artin reciprocity

- Crossed product algebra \(A_K := C(X_K) \rtimes J^+_K\), generators and relations: \(f \in C(X_K)\) and \(\mu_n, n \in J^+_K\)

\[
\mu_n \mu_n^* = e_n; \quad \mu_n^* \mu_n = 1; \quad \rho_n(f) = \mu_n f \mu_n^*; \quad \sigma_n(f) e_n = \mu_n^* f \mu_n; \quad \sigma_n(\rho_n(f)) = f; \quad \rho_n(\sigma_n(f)) = f e_n
\]
Artin reciprocity map $\vartheta_K : A^*_K \to G_{K}^\text{ab}$, write $\vartheta_K(n)$ for ideal $n$ seen as idele by non-canonical section $s$ of

$$A^*_K, f \xrightarrow{s} J_K$$

$$: (x_p)_p \mapsto \prod_{p \text{ finite}} p^{\nu_p(x_p)}$$

semigroup action: $n \in J_K^+$ acting on $f \in C(X_K)$ as

$$\rho_n(f)(\gamma, \rho) = f(\vartheta_K(n)\gamma, s(n)^{-1}\rho)e_n,$$

$$e_n = \mu_n\mu_n^*$$ projector onto $[(\gamma, \rho)]$ with $s(n)^{-1}\rho \in \hat{\mathcal{O}}_K$

partial inverse of semigroup action:

$$\sigma_n(f)(x) = f(n * x) \text{ with } n *[\gamma, \rho] = [\vartheta_K(n)^{-1}\gamma, n\rho]$$

Time evolution $\sigma_K$ acts on $J_K^+$ as a phase factor $N(n)^{it}$

$$\sigma_K, t(f) = f \text{ and } \sigma_K, t(\mu_n) = N(n)^{it}\mu_n$$

for $f \in C(G_{K}^\text{ab} \times \hat{\mathcal{O}}_K^*)$ and for $n \in J_K^+$
Algebraic structure: covariance algebra

Algebraic subalgebra $A^\dagger_K$ of $C^*$-algebra $A_K := C(X_K) \rtimes J^+_K$:

$A^\dagger_K$ unital, non-involutive algebra generated by $C(X_K)$ and the $\mu_n$, $n \in J^+_K$ (but not $\mu_n^*$), with relations

\[
(\text{using } \mu_n^*\mu_n = 1) \quad f\mu_n = \mu_n\sigma_n(f), \quad \mu_n f = \rho_n(f)\mu_n
\]

Comment: presence of an algebraic subalgebra also in previous examples of arithmetic QSM

Comment: similar NCG interpretation as moduli spaces of $\mathbb{K}$-lattices up to commensurability
QSM isomorphism: two number fields $\mathbb{K}$ and $\mathbb{L}$

$$\varphi : A_{\mathbb{K}} \xrightarrow{\sim} A_{\mathbb{L}}$$

$C^*$-algebra isomorphism

$$\varphi \circ \sigma_{\mathbb{K}} = \sigma_{\mathbb{L}} \circ \varphi$$

intertwines the time evolutions

$$\varphi : A_{\mathbb{K}}^\dagger \xrightarrow{\sim} A_{\mathbb{L}}^\dagger$$

preserves the covariance algebras
Theorem  The following are equivalent:

1. \( K \cong L \) are isomorphic number fields
2. Quantum Statistical Mechanical systems are isomorphic

\[ (A_K, \sigma_K) \cong (A_L, \sigma_L) \]

\( C^* \)-algebra isomorphism \( \varphi : A_K \to A_L \) compatible with time evolution, \( \sigma_L \circ \varphi = \varphi \circ \sigma_K \) and covariance \( \varphi : A_K^\dagger \to A_L^\dagger \)

3. There is a group isomorphism \( \psi : \hat{G}_{ab}^K \to \hat{G}_{ab}^L \) of Pontrjagin duals of abelianized Galois groups with

\[ L_K(\chi, s) = L_L(\psi(\chi), s) \]

identity of all \( L \)-functions with Großenencharakter

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Generalization of arithmetic equivalence:
\[ \chi = 1 \] gives \[ \zeta_K(s) = \zeta_L(s) \]

Now also a purely number theoretic proof of \((3) \Rightarrow (1)\) available by Hendrik Lenstra and Bart de Smit

\(L\)-functions \(L(\chi, s)\), for \(s = \beta > 1\) is product of \(\zeta_K(\beta)\) and evaluation of an extremal KMS\(_{\beta}\) state of the QSM system \((A_K, \sigma_K)\) at a test function \(f_\chi \in C(X_K)\)
Scheme of proof: (2) ⇒ (1)

QSM isomorphism ⇒ arithmetic equivalence \( \zeta_K(s) = \zeta_L(s) \)

\( A_K^+ \cong A_L^+ \) gives homeomorphism \( X_K \cong X_L \) and compatible semigroup isomorphism \( J_K^+ \cong J_L^+ \)

Group isomorphism \( G_{ab}^K \cong G_{ab}^L \)

This preserves ramification ⇒ isomorphism of local units \( \hat{O}_K^* \cong \hat{O}_L^* \) and products \( \varphi : \hat{O}_K^* \cong \hat{O}_L^* \)

Semigroup isomorphism \( A_{K,f}^* \cap \hat{O}_K^* \cong A_{L,f}^* \cap \hat{O}_L^* \)

Endomorphism action of these ⇒ inner: \( \hat{O}_{K,+}^x \cong \hat{O}_{L,+}^x \)

(tot pos non-zero integers)

Recover additive structure (mod any totally split prime)

\( \varphi(x + y) = \varphi(x) + \varphi(y) \mod p \)

⇒ \( \mathcal{O}_K \cong \mathcal{O}_L \Rightarrow K \cong L \)
Scheme of proof: (2) $\Rightarrow$ (3)

- QSM isomorphism $\Rightarrow G_{ab}^K \simeq G_{ab}^L$ preserving ramification (as above)
- character groups $\psi : \hat{G}_{ab}^K \rightarrow \hat{G}_{ab}^L$
- character $\chi$ to function $f_\chi \in C(X_K)$, matching $\varphi(f_\chi) = f_\psi(\chi)$
- $\chi(\psi_K(n)) = \psi(\chi)(\psi_L(\varphi(n)))$
- Matching KMS$_\beta$ states: $\omega_{\gamma,\beta}^L(\varphi(f)) = \omega_{\tilde{\gamma},\beta}^K(f)$
- using arithmetic equivalence: $L_K(\chi, s) = L_{L}(\psi(\chi), s)$

QSM isomorphism $\Rightarrow$ matching of L-series
Scheme of proof: $(3) \Rightarrow (1)$

- need compatible isomorphisms $J_{\mathbb{K}}^+ \sim J_{\mathbb{L}}^+$ and $C(X_{\mathbb{K}}) \sim C(X_{\mathbb{L}})$
- know same number of primes $\wp$ above same $p$ with inertia degree $f$ want to match compatibly with Artin map
- use combinations of $L$-series as counting functions: on finite quotients $\pi_G : G_{ab}^{\mathbb{K}} \to G$

\[
\sum_{n \in J_{\mathbb{K}}^+ \atop N_{\mathbb{K}}(n)} \left( \sum_{\hat{G}} \chi(\pi_G(\gamma)^{-1}) \chi(\vartheta_{\mathbb{K}}(n)) \right) = b_{\mathbb{K},G,n}(\gamma)
\]

\[b_{\mathbb{K},G,n}(\gamma) = \# \{ n \in J_{\mathbb{K}}^+ : N_{\mathbb{K}}(n) = n \text{ and } \pi_G(\vartheta_{\mathbb{K}}(n)) = \pi_G(\gamma) \}\]

- For $G_{ab}^{\mathbb{L},n} = \text{Gal of max ab ext unram over } n$, get unique $m \in J_{\mathbb{L}}^+$ with $N_{\mathbb{L}}(m) = N_{\mathbb{K}}(n)$ and
  \[\pi_{G_{ab}^{\mathbb{K},n}}(\vartheta_{\mathbb{L}}(m)) = \pi_{G_{ab}^{\mathbb{L},n}}((\psi^{-1})^*(\vartheta_{\mathbb{K}}(n)))\]

- Use stratification of $X_{\mathbb{K}}$ to extend $\psi : C(G_{ab}^{\mathbb{K}}) \sim C(G_{ab}^{\mathbb{L}})$ to $\varphi : C(X_{\mathbb{K}}) \sim C(X_{\mathbb{L}})$ compatibly with semigroup actions
One more equivalent formulation: $\mathbb{K}$ and $\mathbb{L}$ isomorphic iff $\exists$

- topological group isomorphism $\hat{\psi} : G_{ab}^{\mathbb{K}} \sim \rightarrow G_{ab}^{\mathbb{L}}$
- semigroup isomorphism $\Psi : J_{\mathbb{K}}^+ \sim \rightarrow J_{\mathbb{L}}^+$

with compatibility conditions

- Norm compatibility: $N_{\mathbb{L}}(\Psi(n)) = N_{\mathbb{K}}(n)$ for all $n \in J_{\mathbb{K}}^+$
- Artin map compatibility: for every finite abelian extension $\mathbb{K}' = (\mathbb{K}_{ab})^N / \mathbb{K}$, with $N \subset G_{ab}^\mathbb{K}$: prime $p$ of $\mathbb{K}$ unramified in $\mathbb{K}'$

$\Rightarrow$ prime $\Psi(p)$ unramified in $\mathbb{L}' = (\mathbb{L}_{ab})^{\hat{\psi}(N)} / \mathbb{L}$ and

$$\hat{\psi}({\text{Frob}}_{p}) = \text{Frob}_{\Psi(p)}$$
Conclusions

- Is Quantum Statistical Mechanics a “noncommutative version" of anabelian geometry?
- What about function fields? QSM systems exist, purely NT proof seems not to work, but this QSM proof may work

General philosophy $L$-functions as coordinates determining underlying geometry

Examples:
- Cornelissen-M.M.: zeta functions of a spectral triple on limit set of Schottky uniformized Riemann surface determine conformal structure
- Cornelissen–J.W.de Jong: family of zeta functions of spectral triple of Riemannian manifold determine manifold up to isometry
Anabelian versus Noncommutative

- Anabelian geometry describes a number field $K$ in terms of the absolute Galois group $G_K$
- But... no description of $G_K$ in terms of internal data of $K$ only (Kronecker’s hope)
- Langlands: relate to internal data via automorphic forms
- For abelian extensions yes: $G_K^{ab}$ in terms of internal data: adeles, ideles (class field theory)
- But... $G_K^{ab}$ does not recover $K$
- Noncommutative geometry replaces $G_K$ with the QSM system $(A_K, \sigma_K)$ to reconstruct $K$
- $A_K = C(X_K) \rtimes J_K^+$ is built only from internal data of $K$ (primes, adeles, $G_K^{ab}$)
More details on the proof of (2) ⇒ (1): Stratification of \( X_K \)

- \( \hat{\partial}_{K,n} := \prod_{p|n} \hat{\partial}_{K,p} \) and

\[
X_{K,n} := G_{K}^{ab} \times \hat{\partial}_{K,n}^{*} \quad \text{with} \quad X_K = \lim_{\rightarrow n} X_{K,n}
\]

- Topological groups

\[
G_{K}^{ab} \times \hat{\partial}_{K,n}^{*} \quad G_{K}^{ab}/\vartheta_K(\hat{\partial}_{K,n}^{*}) = G_{K}^{ab,n}
\]

- Gal of max ab ext \textit{unramified} at primes dividing \( n \)

- \( J_{K,n}^{+} \subset J_{K}^{+} \) subsemigroup gen by prime ideals dividing \( n \)

- Decompose \( X_{K,n} = X_{K,n}^{1} \coprod X_{K,n}^{2} \)

\[
X_{K,n}^{1} := \bigcup_{\vartheta_K(n)G_{K,n}^{ab}} \quad \text{and} \quad X_{K,n}^{2} := \bigcup_{p|n} Y_{K,p}
\]

where \( Y_{K,p} = \{(\gamma, \rho) \in X_{K,n} : \rho_p = 0\} \)

- \( X_{K,n}^{1} \) dense in \( X_{K,n} \) and \( X_{K,n}^{2} \) has \( \mu_K \)-measure zero

- Algebra \( C(X_{K,n}) \) is generated by functions

\[
f_{\chi,n} : \gamma \mapsto \chi(\vartheta_K(n))\chi(\gamma), \quad \chi \in \hat{G}_{K,n}^{ab}, \quad n \in J_{K,n}^{+}
\]
First Step of (2) ⇒ (1): \((A_K, \sigma_K) \simeq (A_L, \sigma_L) \Rightarrow \zeta_K(s) = \zeta_L(s)\)

- QSM \((A, \sigma)\) and representation \(\pi : A \to B(\mathcal{H})\) gives Hamiltonian

\[
\pi(\sigma_t(a)) = e^{itH}\pi(a)e^{-itH}
\]

\[
H_{\sigma_K} \varepsilon_n = \log N(n) \varepsilon_n
\]

Partition function \(\mathcal{H} = \ell^2(\mathcal{J}_K^+)\)

\[
Z(\beta) = \text{Tr}(e^{-\beta H}) = \zeta_K(\beta)
\]

- Isomorphism \(\varphi : (A_K, \sigma_K) \simeq (A_L, \sigma_L) \Rightarrow \) homeomorphism of sets of extremal KMS\(_\beta\) states by pullback \(\omega \mapsto \varphi^*(\omega)\)

- KMS\(_\beta\) states for \((A_K, \sigma_K)\) classified [LLN]: \(\beta > 1\)

\[
\omega_{\gamma, \beta}(f) = \frac{1}{\zeta_K(\beta)} \sum_{m \in \mathcal{J}_K^+} \frac{f(\vartheta_K(m)\gamma)}{N_K(m)^\beta}
\]

parameterized by \(\gamma \in \hat{G}_K^{ab}/\vartheta_K(\hat{\Omega}_K^*)\)
Comparing GNS representations of $\omega \in \text{KMS}_\beta(A_L, \sigma_L)$ and $\varphi^*(\omega) \in \text{KMS}_\beta(A_K, \sigma_K)$ find Hamiltonians

$$H_K = U H_L U^* + \log \lambda$$

for some $U$ unitary and $\lambda \in \mathbb{R}^*_+$

Then partition functions give

$$\zeta_L(\beta) = \lambda^{-\beta} \zeta_K(\beta)$$

identity of Dirichlet series

$$\sum_{n \geq 1} \frac{a_n}{n^\beta} \quad \text{and} \quad \sum_{n \geq 1} \frac{b_n}{(\lambda n)^\beta}$$

with $a_1 = b_1 = 1$, taking limit as $\beta \to \infty$

$$a_1 = \lim_{\beta \to \infty} b_1 \lambda^{-\beta} \Rightarrow \lambda = 1$$
Conclusion of first step: arithmetic equivalence $\zeta_L(\beta) = \zeta_K(\beta)$

Consequences:
From arithmetic equivalence already know $K$ and $L$ have same degree over $\mathbb{Q}$, discriminant, normal closure, unit groups, archimedean places.

But... not class group (or class number)
Second Step of (2) ⇒ (1): unraveling the crossed product

\[ \varphi : C(X_K) \times J_K^+ \xrightarrow{\sim} C(X_L) \times J_L^+ \] with \( \sigma_L \circ \varphi = \varphi \circ \sigma_K \)

and preserving the covariance algebra \( \varphi : A_K^+ \xrightarrow{\sim} A_L^+ \)

- Restrict to finitely many isometries \( \mu_\varphi, N_K(\varphi) = p \)
- \( A_K \) generated by \( \mu_n f \mu_m^* \); in \( A_K^+ \) only \( \mu_n f \)
- Eigenspaces of time evolution in \( A_K^+ \) preserved:
  so \( C(X_K) \xrightarrow{\sim} C(X_L) \) and \( \varphi(\mu_n) = \sum \mu_m f_{n,m} \)
- Commutators \( [f, \mu_n] = (f - \rho_n(f))\mu_n \): match maximal ideals
  (mod commutators) so that homeomorphism \( \Phi : X_K \xrightarrow{\sim} X_L \)
  compatible with semigroup actions \( \gamma_{\alpha_x(n)}(\Phi(x)) = \Phi(\gamma_n(x)) \)
  with locally constant \( \alpha_x : J_K^+ \to J_L^+ \)
  (that is, \( \varphi(\mu_n) = \sum \delta_{m,\alpha_x(n)} \mu_n \))
- \( \alpha_x = \alpha \) constant: know [LLN] ergodic action of \( J_K^+ \) on \( X_K \), level
  sets would be clopen invariant subsets
Third Step of (2) $\Rightarrow$ (1): isomorphism $G^{ab}_K \sim G^{ab}_L$

- Projectors $e_{K,n} = \mu_n \mu_n^*$ mapped to projector $e_{L,\varphi(n)}$
- Fix $m \in J^+_K$ and $\hat{O}_{K,m} = \prod_{p|m} \hat{O}_{K,p}$, then

$$V_{K,m} := \bigcap_{(m,n)=1} \text{Range}(e_{K,n}) = G^{ab}_K \times \hat{O}^*_K \{(0, \ldots, 0, \hat{O}_{K,m}, 0, \ldots, 0)\}$$

$$\Phi(V_{K,m}) = \bigcap_{(m,n)=1} \Phi(\text{Range}(e_{K,n})) = \bigcap_{(\varphi(m), \varphi(n))=1} \text{Range}(e_{L,\varphi(n)})$$

$$= G^{ab}_L \times \hat{O}^*_L \{(0, \ldots, 0, \hat{O}_{L,\varphi(m)}, 0, \ldots, 0)\} = V_{L,\varphi(m)}$$

- $1_m$ integral adele $= 1$ at the prime divisors of $m$, zero elsewhere

$$H_{K,m} := G^{ab}_K \times \hat{O}^*_K \{1_m\} \subseteq X_K \xrightarrow{\varphi} G^{ab}_L \times \hat{O}^*_L \{y_{\varphi(m)}\} \subseteq X_L$$

- check that $y \in \hat{O}^*_L,m$ is a unit
- then $H_{K,m}$ classes $[(\gamma, 1_m)] \sim [(\gamma', 1_m)]$ $\iff$ $\exists u \in \hat{O}^*_K$ with $\gamma' = \vartheta_K(u)^{-1} \gamma$ and $1_m = u1_m$
then for $\hat{G}^{ab}_{K,m}$ Gal of max ab ext unram *outside* prime div of m

$$H_{K,m} \cong \frac{G^{ab}_K}{\vartheta_K} \left( \prod_{q \nmid m} \hat{\vartheta}^*_q \right) \cong \hat{G}^{ab}_{K,m}$$

$\hat{G}^{ab}_{K,m}$ has dense subgroup gen by $\vartheta_K(n)$, ideals coprime to m

$\Rightarrow H_{K,m}$ gen by these $\gamma_n := [(\vartheta_K(n)^{-1}, 1_m)]$

with $1_m = [(1, 1_m)]$ and $\Phi(1_m) = [(x_m, y_m)]$ get

$$\Phi(\gamma_{n_1} \cdot \gamma_{n_2}) = \Phi([(\vartheta_K(n_1 n_2)^{-1}, 1_m)])$$

$$= \Phi([(\vartheta_K(n_1 n_2)^{-1}, n_1 n_2 1_m)]) \text{ (since } n_1, n_2 \text{ coprime to } m)$$

$$= \Phi(n_1 n_2 \ast 1_m) = \varphi(n_1 n_2) \ast \Phi(1_m) = [(\vartheta_L(\varphi(n_1 n_2))^{-1} x_m, \varphi(n_1 n_2) y_m)]$$

$\lim_{m \to +\infty} 1_m = 1 \Rightarrow \lim_{m \to +\infty} \Phi(1_m) = \Phi(1)$ and get

$$\tilde{\Phi}(\gamma_1 \gamma_2) = \Phi(\gamma_1 \cdot \gamma_2) \Phi(1)^{-1} = \Phi(\gamma_1) \Phi(\gamma_2) \Phi(1)^{-2} = \tilde{\Phi}(\gamma_1) \cdot \tilde{\Phi}(\gamma_2)$$
Fourth step of (2): Preserving ramification

$N \subset G_{abK}$ subgroup, $G_{abK}/N \sim \rightarrow G_{abL}/\Phi(N)$

$p$ ramifies in $K'/K \iff \varphi(p)$ ramifies in $L'/L$

where $K' = (K^{ab})^N$ finite extension and $L' := (L^{ab})^{\Phi(N)}$

- seen have isomorphism $\Phi : \hat{G}_{abK,m} \sim \rightarrow \hat{G}_{abL,\varphi(m)}$ (Gal of max ab ext $K_m$ unram outside prime div of $m$)
- $K' = (K^{ab})^N$ fin ext ramified precisely above $p_1, \ldots, p_r \in J_{K}^+$
- By previous $L' := (L)^{\Phi(N)}$ contained in $L_{\varphi(p_1)\ldots\varphi(p_r)}$ but not in any $L_{\varphi(p_1)\ldots\varphi(p_i)\ldots\varphi(p_r)} \Rightarrow L'/L$ ramified precisely above $\varphi(p_1), \ldots, \varphi(p_r)$
Fifth Step of \((2) \Rightarrow (1)\): from QSM isomorphism get also

- Isomorphism of local units

\[
\varphi : \hat{O}_p^* \xrightarrow{\sim} \hat{O}^*_\varphi(p)
\]

max ab ext where \(p\) unramified = fixed field of inertia group \(I_{p}^{ab}\),

by ramification preserving

\[
\Phi(I_{p}^{ab}) = I_{\varphi(p)}^{ab}
\]

and by local class field theory \(I_{p}^{ab} \simeq \hat{O}_p^*\)

- by product of the local units: isomorphism

\[
\varphi : \hat{O}_K^* \xrightarrow{\sim} \hat{O}_L^*
\]

- Semigroup isomorphism

\[
\varphi : (A_{K,f}^* \cap \hat{O}_K, \times) \xrightarrow{\sim} (A_{L,f}^* \cap \hat{O}_L, \times)
\]

by exact sequence

\[
0 \rightarrow \hat{O}_K^* \rightarrow A_{K,f}^* \cap \hat{O}_K \rightarrow J_{K}^{+} \rightarrow 0
\]

(non-canonically) split by choice of uniformizer \(\pi_p\) at every place
Recover multiplicative structure of the field

- **Endomorphism action of** \( A^{*}_{K,f} \cap \hat{\mathcal{O}}_{K} \)

\[
\epsilon_s(f)(\gamma, \rho) = f(\gamma, s^{-1} \rho) e_{\tau}, \quad \epsilon_s(\mu_n) = \mu_n e_{\tau}
\]

- \( e_{\tau} \) char function of set \( s^{-1} \rho \in \hat{\mathcal{O}}_{K} \)

- \( \hat{\mathcal{O}}_{K}^{*} \) = part acting by automorphisms

- \( \overline{\mathcal{O}_{K,+}^{*}} \) (closure of tot pos units): trivial endomorphisms

- \( \mathcal{O}_{K,+}^{\times} = \mathcal{O}_{K,+} - \{0\} \) (non-zero tot pos elements of ring of integers): *inner endomorphisms* (isometries in \( A_{K}^{\dagger} \) eigenv of time evolution)

- \( \varphi(\epsilon_s) = \epsilon_{\varphi(s)} \) for all \( s \in A^{*}_{K,f} \cap \hat{\mathcal{O}}_{K} \)

**Conclusion:** isom of multiplicative semigroups of tot pos non-zero elements of rings of integers

\[
\varphi : (\mathcal{O}_{K,+}^{\times}, \times) \xrightarrow{\sim} (\mathcal{O}_{L,+}^{\times}, \times)
\]
Last Step of (2) ⇒ (1): Recover additive structure of the field
Extend by \( \varphi(0) = 0 \) the map \( \varphi : (O_K^\times, +, \times) \xrightarrow{\sim} (O_L^\times, +, \times) \), Claim: it is additive

- Start with induced multipl map of local units \( \varphi : \hat{O}_{K,p}^* \xrightarrow{\sim} \hat{O}_{L,\varphi(p)}^* \) (from ramification preserving)
- set \( 1_p = (0, \ldots, 0, 1, 0, \ldots, 0) \) and \( 1_p := [(1, 1_p)] \in X_K \); for \( u \in \hat{O}_{K,p}^* \), integral idele \( u_p := (1, \ldots, 1, u, 1, \ldots, 1) \):
  \[
  [(1, u_p)] = [(\nu_K(u_p)^{-1}, 1)] \mapsto \varphi([(\nu_K(u_p)^{-1}), 1)]) =: [(1, \varphi(u)\varphi(p))]
  \]
- Group isom to image \( \lambda_{K,p} : \hat{O}_{K,p}^* \xrightarrow{\sim} X_K \xrightarrow{\cdot 1_p} Z_{K,p} \subset X_K \)
  \[
  u \mapsto [(1, u_p)] \mapsto [(1, u_p \cdot 1_p)] = [(1, (0, \ldots, 0, u, 0, \ldots, 0)]
  \]
- Commutative diagram
Fix rational prime $p$ totally split in $\mathbb{K}$ (hence unramified) $\Rightarrow$ arithm equiv: $p$ tot split in $\mathbb{L}$

Set $\mathbb{Z}_{(p\Delta)}$ integers coprime to $p\Delta$ with $\Delta = \Delta_{\mathbb{K}} = \Delta_{\mathbb{L}}$ discriminant

map $\varpi_{\mathbb{K},p} : \mathbb{Z}_{(p\Delta)} \leftrightarrow \hat{\mathcal{O}}^*_{\mathbb{K},p} \to \mathbb{Z}_{\mathbb{K},p}$ with $\varpi_{\mathbb{K},p} : a \mapsto [(1, a \cdot 1_p)]$

$a = p_1 \ldots p_r$ rational prime unramified $\Rightarrow$ permute factors $\alpha_x((a)) = p_{\sigma(1)} \ldots p_{\sigma(r)}$ so $\alpha_x((a)) = (a)$ fixes ideals $(a) \in J^+_\mathbb{Q}$

$\Phi(\varpi_{\mathbb{K},p}(a)) = \Phi((a)*1_p) = \alpha_{1_p}((a))*\Phi(1_p) = (a)*1_{\varphi(p)} = \varpi_{\mathbb{L},\varphi(p)}(a)$

so $\varphi : \hat{\mathcal{O}}^*_{\mathbb{K},p} \sim \hat{\mathcal{O}}^*_{\mathbb{L},\varphi(p)}$ constant on $\mathbb{Z}_{(p\Delta)}$
As above fix ational prime $p$ totally split in $\mathbb{K}$ (hence in $\mathbb{L}$) and $p \in J_{\mathbb{K}}^+$ above $p$ with $f(p|\mathbb{K}) = 1$ (hence $f(\varphi(p)|\mathbb{L}) = 1$)

Use $\varphi: \hat{\mathcal{O}}_{\mathbb{K},p}^* \xrightarrow{\sim} \hat{\mathcal{O}}_{\mathbb{L},\varphi(p)}^*$ to get multiplicative map of residue fields by Teichmüller lift $\tau_{\mathbb{K},p}: \overline{\mathbb{K}}_p^* \cong \mathbb{F}_p^* \hookrightarrow \hat{\mathcal{O}}_{\mathbb{K},p}^* \cong \mathbb{Q}_p^*$

Show its extension by zero additive (hence identity map $\tilde{\varphi}: \mathbb{F}_p^* \to \mathbb{F}_p^*$) by extending $\tau_{\mathbb{K},p}: \hat{\mathcal{O}}_{\mathbb{K},p}^* \to \hat{\mathcal{O}}_{\mathbb{K},p}^*$ with $x \mapsto \lim_{n \to +\infty} x^{p^n}$

for $\tilde{a}$ residue class in $\overline{\mathbb{K}}_p^* \cong \mathbb{F}_p$, choose integer $a$ congruent to $\tilde{a}$ mod $p$ and coprime to discriminant $\Delta$ (Chinese remainder thm)

$$\varphi(\tau_{\mathbb{K},p}(a)) = \varphi \left( \lim_{n \to +\infty} a^{p^n} \right) = \lim_{n \to +\infty} \varphi(a)^{p^n} = \tau_{\mathbb{L},p}(\varphi(a)) = \tau_{\mathbb{L},p}(a)$$

$$\tilde{\varphi}(\tilde{a}) = \varphi(\tau_{\mathbb{K},p}(a)) \mod \varphi(p) = \tau_{\mathbb{L},p}(a) \mod \varphi(p) = \tilde{a} \mod \varphi(p)$$

So $\varphi$ identity mod any tot split prime, so for any $x, y \in \mathcal{O}_{\mathbb{K},+}$

$$\varphi(x + y) = \varphi(x) + \varphi(y) \mod \varphi(p)$$

totally split primes of arbitrary large norm (Chebotarev)

$\Rightarrow \varphi$ additive