Quantum statistical mechanics, L-series, Anabelian Geometry

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Recovering a Number Field from invariants

- Dedekind zeta function $\zeta_{\mathbb{K}}(s) = \zeta_{\mathbb{L}}(s)$ arithmetic equivalence
  Gaßmann examples:

$$\mathbb{K} = \mathbb{Q}(\sqrt[8]{3}) \text{ and } \mathbb{L} = \mathbb{Q}(\sqrt[8]{3 \cdot 2^4})$$

  not isomorphism $\mathbb{K} \neq \mathbb{L}$

- Adeles rings $\mathbb{A}_{\mathbb{K}} \cong \mathbb{A}_{\mathbb{L}}$ adelic equivalence $\Rightarrow$ arithmetic equivalence; Komatsu examples:

$$\mathbb{K} = \mathbb{Q}(\sqrt[8]{2 \cdot 9}) \text{ and } \mathbb{L} = \mathbb{Q}(\sqrt[8]{2^5 \cdot 9})$$

  not isomorphism $\mathbb{K} \neq \mathbb{L}$

- Abelianized Galois groups: $G_{\mathbb{K}}^{ab} \cong G_{\mathbb{L}}^{ab}$ also not isomorphism; Onabe examples:

$$\mathbb{K} = \mathbb{Q}(\sqrt{-2}) \text{ and } \mathbb{L} = \mathbb{Q}(\sqrt{-3})$$

  not isomorphism $\mathbb{K} \neq \mathbb{L}$

- But ... absolute Galois groups $G_{\mathbb{K}} \cong G_{\mathbb{L}} \Rightarrow$ isomorphism $\mathbb{K} \cong \mathbb{L}$: Neukirch–Uchida theorem (Grothendieck's anabelian geometry)

Question: Can combine $\zeta_{\mathbb{K}}(s)$, $\mathbb{A}_{\mathbb{K}}$ and $G^{ab}_{\mathbb{K}}$ to something as strong as $G_{\mathbb{K}}$ that determines isomorphism class of $\mathbb{K}$?

Answer: Yes! Combine as a Quantum Statistical Mechanical system algebra and time evolution (A, σ)

$$A_{\mathbb{K}} := C(X_{\mathbb{K}}) \rtimes J^+_{\mathbb{K}}, \quad \text{with} \quad X_{\mathbb{K}} := G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \hat{\mathcal{O}}_{\mathbb{K}},$$

$\hat{\mathcal{O}}_{\mathbb{K}} =$ ring of finite integral adeles, $J^+_{\mathbb{K}} =$ is the semigroup of ideals, acting on $X_{\mathbb{K}}$ by Artin reciprocity

Time evolution $\sigma_{\mathbb{K}}$ acts on $J^+_{\mathbb{K}}$ as a phase factor $N(\mathfrak{n})^{it}$

QSM systems introduced by Ha–Paugam to generalize Bost–Connes system, also recently studied by Laca–Larsen–Neshveyev [LLN]

The setting of Quantum Statistical Mechanics: Data

- $\mathscr{A}$ unital C^*-algebra of observables
- σ_t time evolution, $\sigma : \mathbb{R} \to \mathrm{Aut}(\mathscr{A})$
- states $\omega : \mathscr{A} \to \mathbb{C}$ continuous, normalized $\omega(1) = 1$, positive

$$\omega(a^* a) \geq 0$$

- equilibrium states $\omega(\sigma_t(a)) = \omega(a)$ all $t \in \mathbb{R}$
- representation $\pi : \mathscr{A} \to \mathscr{B}(\mathscr{H})$, Hamiltonian H

$$\pi(\sigma_t(a)) = e^{itH} \pi(a) e^{-itH}$$

- partition function $Z(\beta) = \mathrm{Tr}(e^{-\beta H})$
- Gibbs states (equilibrium, inverse temperature β):

$$\omega_\beta(a) = \frac{\mathrm{Tr}(\pi(a) e^{-\beta H})}{\mathrm{Tr}(e^{-\beta H})}$$

- Generalization of Gibbs states: KMS states (Kubo–Martin–Schwinger) $\forall a, b \in A$, $\exists$ holomorphic $F_{a,b}$ on strip $I_\beta = \{0 < \operatorname{Im} z < \beta\}$, bounded continuous on ∂I_β,

$$F_{a,b}(t) = \omega(a\sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \omega(\sigma_t(b)a)$$

- Fixed $\beta > 0$: KMS_β state convex simplex: extremal states (like points in NCG)
- Isomorphism of QSM: $\varphi : (\mathscr{A}, \sigma) \to (\mathscr{B}, \tau)$

$$\varphi : \mathscr{A} \stackrel{\simeq}{\to} \mathscr{B}, \quad \varphi \circ \sigma = \tau \circ \varphi$$

C^*-algebra isomorphism intertwining time evolution

- Pullback of a state: $\varphi^*\omega(a) = \omega(\varphi(a))$

Theorem The following are equivalent:

1. $\mathbb{K} \cong \mathbb{L}$ are isomorphic number fields
2. Quantum Statistical Mechanical systems are isomorphic

$$(A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \simeq (A_{\mathbb{L}}, \sigma_{\mathbb{L}})$$

   C^*-algebra isomorphism $\varphi : A_{\mathbb{K}} \to A_{\mathbb{L}}$ compatible with time evolution, $\sigma_{\mathbb{L}} \circ \varphi = \varphi \circ \sigma_{\mathbb{K}}$
3. There is a group isomorphism $\psi : \hat{G}^{ab}_{\mathbb{K}} \to \hat{G}^{ab}_{\mathbb{L}}$ of Pontrjagin duals of abelianized Galois groups with

$$L_{\mathbb{K}}(\chi, s) = L_{\mathbb{L}}(\psi(\chi), s)$$

   identity of all L-functions with Großencharakter

   Note: Generalization of arithmetic equivalence:
   $\chi = 1$ gives $\zeta_{\mathbb{K}}(s) = \zeta_{\mathbb{L}}(s)$
   (now also purely number theoretic proof of (3) $\Rightarrow$ (1)
   by Hendrik Lenstra and Bart de Smit)

Setting and notation

• Artin reciprocity map

$$\vartheta_{\mathbb{K}} : \mathbb{A}^*_{\mathbb{K}} \to G^{ab}_{\mathbb{K}}.$$

$\vartheta_{\mathbb{K}}(\mathfrak{n})$ for ideal $\mathfrak{n}$ seen as idele by non-canonical section s of

$$\mathbb{A}^*_{\mathbb{K},f} \twoheadrightarrow J_{\mathbb{K}} \quad (s : J_{\mathbb{K}} \to \mathbb{A}^*_{\mathbb{K},f}) \qquad : \qquad (x_{\mathfrak{p}})_{\mathfrak{p}} \mapsto \prod_{\mathfrak{p} \text{ finite}} \mathfrak{p}^{v_{\mathfrak{p}}(x_{\mathfrak{p}})}$$

• Crossed product algebra

$$A_{\mathbb{K}} := C(X_{\mathbb{K}}) \rtimes J^+_{\mathbb{K}} = C(G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \hat{\mathcal{O}}_{\mathbb{K}}) \rtimes J^+_{\mathbb{K}}$$

• semigroup crossed product: $\mathfrak{n} \in J_{\mathbb{K}}^{+}$ acting on $f \in C(X_{\mathbb{K}})$ as

$$\rho_{\mathfrak{n}}(f)(\gamma, \rho) = f(\vartheta_{\mathbb{K}}(\mathfrak{n})\gamma, s(\mathfrak{n})^{-1}\rho)e_{\mathfrak{n}},$$

$e_{\mathfrak{n}} = \mu_{\mathfrak{n}}^{*}\mu_{\mathfrak{n}}$ projector onto $[(\gamma, \rho)]$ with $s(\mathfrak{n})^{-1}\rho \in \hat{\mathcal{O}}_{\mathbb{K}}$

• partial inverse of semigroup action

$$\sigma_{\mathfrak{n}}(f)(x) = f(\mathfrak{n} * x) \quad \text{with} \quad \mathfrak{n} * [(\gamma, \rho)] = [(\vartheta_{\mathbb{K}}(\mathfrak{n})^{-1}\gamma, \mathfrak{n}\,\rho)]$$

Generators and Relations: $f \in C(X_{\mathbb{K}})$ and $\mu_{\mathfrak{n}}$, $\mathfrak{n} \in J_{\mathbb{K}}^{+}$

$$\mu_{\mathfrak{n}}\mu_{\mathfrak{n}}^* = e_{\mathfrak{n}};\ \mu_{\mathfrak{n}}^*\mu_{\mathfrak{n}} = 1;\ \rho_{\mathfrak{n}}(f) = \mu_{\mathfrak{n}} f \mu_{\mathfrak{n}}^*;$$

$$\sigma_{\mathfrak{n}}(f) e_{\mathfrak{n}} = \mu_{\mathfrak{n}}^* f \mu_{\mathfrak{n}};\ \ \sigma_{\mathfrak{n}}(\rho_{\mathfrak{n}}(f)) = f;\ \rho_{\mathfrak{n}}(\sigma_{\mathfrak{n}}(f)) = f e_{\mathfrak{n}}$$

Time evolution:

$$\sigma_{\mathbb{K},t}(f) = f \quad \text{and} \quad \sigma_{\mathbb{K},t}(\mu_{\mathfrak{n}}) = N(\mathfrak{n})^{it}\, \mu_{\mathfrak{n}}$$

for $f \in C(G_{\mathbb{K}}^{ab} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^*} \hat{\mathcal{O}}_{\mathbb{K}})$ and for $\mathfrak{n} \in J_{\mathbb{K}}^{+}$

Stratification of $X_{\mathbb{K}}$

- $\hat{\mathcal{O}}_{\mathbb{K},n} := \prod_{\mathfrak{p}|n} \hat{\mathcal{O}}_{\mathbb{K},\mathfrak{p}}$ and

$$X_{\mathbb{K},n} := G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \hat{\mathcal{O}}_{\mathbb{K},n} \quad \text{with} \quad X_{\mathbb{K}} = \varinjlim_n X_{\mathbb{K},n}$$

- Topological groups

$$G^{ab}_{\mathbb{K}} \times_{\hat{\mathcal{O}}^*_{\mathbb{K}}} \hat{\mathcal{O}}^*_{\mathbb{K},n} \simeq G^{ab}_{\mathbb{K}} / \vartheta_{\mathbb{K}}(\hat{\mathcal{O}}^*_{\mathbb{K},n}) = G^{ab}_{\mathbb{K},n}$$

  Gal of max ab ext *unramified* at primes dividing n
- $J^+_{\mathbb{K},n} \subset J^+_{\mathbb{K}}$ subsemigroup gen by prime ideals dividing n
- Decompose $X_{\mathbb{K},n} = X^1_{\mathbb{K},n} \coprod X^2_{\mathbb{K},n}$

$$X^1_{\mathbb{K},n} := \bigcup_{\mathfrak{n} \in J^+_{\mathbb{K},n}} \vartheta_{\mathbb{K}}(\mathfrak{n}) G^{ab}_{\mathbb{K},n} \quad \text{and} \quad X^2_{\mathbb{K},n} := \bigcup_{\mathfrak{p}|n} Y_{\mathbb{K},\mathfrak{p}}$$

  where $Y_{\mathbb{K},\mathfrak{p}} = \{(\gamma, \rho) \in X_{\mathbb{K},n} \,:\, \rho_{\mathfrak{p}} = 0\}$
- $X^1_{\mathbb{K},n}$ dense in $X_{\mathbb{K},n}$ and $X^2_{\mathbb{K},n}$ has $\mu_{\mathbb{K}}$-measure zero
- Algebra $C(X_{\mathbb{K},n})$ is generated by functions

$$f_{\chi,\mathfrak{n}} \,:\, \gamma \mapsto \chi(\vartheta_{\mathbb{K}}(\mathfrak{n}))\chi(\gamma), \quad \chi \in \widehat{G}^{ab}_{\mathbb{K},n}, \;\; \mathfrak{n} \in J^+_{\mathbb{K},n}$$

First Step of (2) $\Rightarrow$ (1): $(A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \simeq (A_{\mathbb{L}}, \sigma_{\mathbb{L}}) \Rightarrow \zeta_{\mathbb{K}}(s) = \zeta_{\mathbb{L}}(s)$

- QSM (A, σ) and representation $\pi : A \to B(\mathscr{H})$ gives Hamiltonian

$$\pi(\sigma_t(a)) = e^{itH}\pi(a)e^{-itH}$$

$$H_{\sigma_{\mathbb{K}}}\varepsilon_{\mathfrak{n}} = \log N(\mathfrak{n})\ \varepsilon_{\mathfrak{n}}$$

Partition function $\mathscr{H} = \ell^2(J^+_{\mathbb{K}})$

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \zeta_{\mathbb{K}}(\beta)$$

- Isomorphism $\varphi : (A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \simeq (A_{\mathbb{L}}, \sigma_{\mathbb{L}}) \Rightarrow$ homeomorphism of sets of extremal KMS$_\beta$ states by pullback $\omega \mapsto \varphi^*(\omega)$

- KMS$_\beta$ states for $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$ classified [LLN]: $\beta > 1$

$$\omega_{\gamma,\beta}(f) = \frac{1}{\zeta_{\mathbb{K}}(\beta)} \sum_{\mathfrak{m}\in J^+_{\mathbb{K}}} \frac{f(\vartheta_{\mathbb{K}}(\mathfrak{m})\gamma)}{N_{\mathbb{K}}(\mathfrak{m})^\beta}$$

parameterized by $\gamma \in G^{ab}_{\mathbb{K}}/\vartheta_{\mathbb{K}}(\hat{\mathscr{O}}^*_{\mathbb{K}})$

- Comparing GNS representations of $\omega \in \mathrm{KMS}_\beta(A_\mathbb{L}, \sigma_\mathbb{L})$ and $\varphi^*(\omega) \in \mathrm{KMS}_\beta(A_\mathbb{K}, \sigma_\mathbb{K})$ find Hamiltonians

$$H_\mathbb{K} = U\, H_\mathbb{L}\, U^* + \log \lambda$$

for some U unitary and $\lambda \in \mathbb{R}_+^*$

- Then partition functions give

$$\zeta_\mathbb{L}(\beta) = \lambda^{-\beta} \zeta_\mathbb{K}(\beta)$$

identity of Dirichlet series

$$\sum_{n \geq 1} \frac{a_n}{n^\beta} \quad \text{and} \quad \sum_{n \geq 1} \frac{b_n}{(\lambda n)^\beta}$$

with $a_1 = b_1 = 1$, taking limit as $\beta \to \infty$

$$a_1 = \lim_{\beta \to \infty} b_1 \lambda^{-\beta} \quad \Rightarrow \lambda = 1$$

Conclusion of first step: arithmetic equivalence $\zeta_{\mathbb{L}}(\beta) = \zeta_{\mathbb{K}}(\beta)$

Consequences:
From arithmetic equivalence already know $\mathbb{K}$ and $\mathbb{L}$ have same degree over $\mathbb{Q}$, discriminant, normal closure, unit groups, archimedean places.

But... not class group (or class number)

Intermezzo: a useful property (characterizing isometries)

Element u in $A_{\mathbb{K}}$:

• isometry: $u^* u = 1$

• eigenvector of time evolution: $\sigma_t(u) = q^{it} u$, for $q = n/m$

Then

$$u = \sum_{\mathfrak{n}} \mu_{\mathfrak{n}} f_{\mathfrak{n}}$$

with $f_{\mathfrak{n}} \in C(X_{\mathbb{K}})$ and $\mathfrak{n} \in J_{\mathbb{K}}^{+}$ with $N_{\mathbb{K}}(\mathfrak{n}) = n$ and $\sum_{\mathfrak{n}} |f_{\mathfrak{n}}|^2 = 1$

inner endomorphisms: $a \mapsto u\, a\, u^*$

Second Step of (2) $\Rightarrow$ (1): unraveling the crossed product

$$\varphi : C(X_{\mathbb{K}}) \rtimes J_{\mathbb{K}}^{+} \stackrel{\simeq}{\to} C(X_{\mathbb{L}}) \rtimes J_{\mathbb{L}}^{+} \quad \text{with} \quad \sigma_{\mathbb{L}} \circ \varphi = \varphi \circ \sigma_{\mathbb{K}}$$

Then is gives separately:

- A homeomorphism $X_{\mathbb{K}} \cong X_{\mathbb{L}}$
- A semigroup isomorphism $J_{\mathbb{K}}^{+} \cong J_{\mathbb{L}}^{+}$
- compatible with the crossed product action ρ

Test case: a single isometry

- If single isometry: continuous injective self-map γ of space X then semigroup crossed product $C(X) \rtimes_\rho \mathbb{Z}_+$ with $\mu f \mu^*(x) = \rho(f)(x) = \chi(x) f(\gamma^{-1}(x))$, with $\chi =$ characteristic function of range of γ; time evolution: $\sigma_t(\mu) = \lambda^{it}\mu$
- Then isomorphism $\varphi : (C(X) \rtimes_\rho \mathbb{Z}_+, \sigma) \simeq (C(X') \rtimes_{\rho'} \mathbb{Z}_+, \sigma')$ gives homeomorphism $\Phi : X \simeq X'$ with $\gamma' \circ \Phi = \Phi \circ \gamma$
- Basic step: write commutator ideals $\mathscr{C}_0$ in terms of Fourier modes $a = f_0 + \sum_{k>0}(\mu^k f_k + f_{-k}(\mu^*)^k)$ and get matching of maximal ideals $\varphi(\widetilde{I}_{\gamma(x),0}\mathscr{C}_0 + \mathscr{C}_0^2) = \widetilde{I}_{\Phi(\gamma(x)),0}\mathscr{C}_0' + (\mathscr{C}_0')^2$ where $\widetilde{I}_{y,0}\mathscr{C}_0 + \mathscr{C}_0^2 = \mathscr{C}_0 \widetilde{I}_{x,0} + \mathscr{C}_0^2$.

From one to N isometries

Difficulty: no longer know just from time evolution that image of $C(X)$ does not involve terms like $\mu_i \mu_j^*$ but only $C(X')$
But ... Still works!

Result: for N commuting isometries and crossed products

$$\varphi : (\mathscr{A} = C(X) \rtimes_\rho \mathbb{Z}_+^N, \sigma) \stackrel{\simeq}{\to} (\mathscr{A}' = C(X') \rtimes_{\rho'} \mathbb{Z}_+^N, \sigma')$$

with $\sigma_t(f) = f$ and $\sigma_t(\mu_j) = \lambda^{it} \mu_j$ (both sides)
with density hypothesis: any multi-indices $\alpha, \beta \in \mathbb{Z}_+^N$ with $\gamma_\alpha \neq \gamma_\beta$

$$\{x \in X : \gamma_\alpha(x) \neq \gamma_\beta(x)\} \quad \text{dense in } X$$

(and same for $\mathscr{A}'$)

Then isomorphism of QSM system gives:
- homeomorphism $\Phi : X \simeq X'$
- and compatible isomorphism $\alpha_x : \mathbb{Z}_+^N \to \mathbb{Z}_+^N$

locally constant in $x \in X$ (permutations of the generators)

In fact:

- μ_j go to isometries u_j eigenvectors of time evolution $\Rightarrow$

$$\varphi(\mu_j) = \sum_k \mu'_k \, f_{jk}$$

- functions $f(x) = e^{ih(x)}$ (local phase) go to local phase in $C(X')$
- for all functions $\varphi(f_1)\varphi(f_2) = \varphi(f_2)\varphi(f_1)$
- applied to a local phase $h_2 = \varphi(f_2)$ and an arbitrary function f_1:

$$f_{\alpha,\beta} \cdot (h_2 \circ \gamma'_\beta) = f_{\alpha,\beta} \cdot (h_2 \circ \gamma'_\alpha).$$

where

$$\varphi(f_1) = \sum_{\alpha,\beta:|\alpha|=|\beta|} \mu'_\alpha \; f_{\alpha,\beta} \; \mu'^*_\beta$$

Conclusion: $C(X)$ goes to $C(X')$ and μ_i go to something with no μ'^*_j then same argument as for one isometry

Applied to QSM system $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$:
from finitely many to infinitely many isometries

- By separating eigenspaces of the time evolution by $N(\wp) = p$, apply case of N isoetries
- Density hypothesis: for any $\mathfrak{m} \neq \mathfrak{n}$ dense set of $x \in X_{\mathbb{K}}$ with $\mathfrak{m} * x \neq \mathfrak{n} * x$. In fact, check that set E of $\mathfrak{m} * x = \mathfrak{n} * x$ means exists $u \in \hat{\mathcal{O}}_{\mathbb{K}}^*$

$$\begin{cases} \vartheta_{\mathbb{K}}(\mathfrak{m}) = \vartheta_{\mathbb{K}}(u\, \mathfrak{n}) \\ s(\mathfrak{m})\rho = us(\mathfrak{n})\rho \end{cases}$$

$s \colon J_{\mathbb{K}}^{+} \to \mathbb{A}_{\mathbb{K},f}^{*}$ section (defined up to units)

$$E = \begin{cases} \emptyset & \text{if } \mathfrak{m} \not\sim \mathfrak{n} \in \mathrm{Cl}^{+}(\mathbb{K}); \\ G_{\mathbb{K}}^{\text{ab}} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^{*}} \{0\} \cong \mathrm{Cl}^{+}(\mathbb{K}) & \text{if } \mathfrak{m} \sim \mathfrak{n} \in \mathrm{Cl}^{+}(\mathbb{K}). \end{cases}$$

finite or empty: complement dense

Third step of (2): group isomorphism $G^{ab}_{\mathbb{K}} \simeq G^{ab}_{\mathbb{L}}$

- $\gamma \mapsto \epsilon_\gamma$ (faithful) action of $G^{ab}_{\mathbb{K}}$ as symmetries of $A_{\mathbb{K}}$
- $G^{ab}_{\mathbb{K}}$ acts freely transitively on extremal KMS

$$\omega_{\beta,\gamma_1} \circ \epsilon_{\gamma_2} = \omega_{\beta,\gamma_1\gamma_2}$$

- $\widetilde{\Phi}(\gamma) = \Phi(\gamma)\Phi(1)^{-1}$ group isomorphism, from

$$\varphi^*(\epsilon_\gamma)(\omega^{\mathbb{L}}_{\beta,\gamma'}) = \omega^{\mathbb{L}}_{\beta,\Phi(\Phi^{-1}(\gamma')\gamma)}$$

$$\varphi^*(\epsilon_{\gamma_2}) = \epsilon_{\Phi(\gamma_1)^{-1}\Phi(\gamma_1\gamma_2)} = \epsilon_{\Phi(1)\Phi(\gamma_2)}$$

$$\Phi(\gamma_1\gamma_2) = \Phi(1)\Phi(\gamma_1)\Phi(\gamma_2)$$

Back to step 2: Got homeomorphism $\Phi : X_{\mathbb{K}} \simeq X_{\mathbb{L}}$ and locally constant $\alpha_x : J_{\mathbb{K}}^{+} \simeq J_{\mathbb{L}}^{+}$

- The locally constant $\alpha_x : J_{\mathbb{K}}^{+} \simeq J_{\mathbb{L}}^{+}$ is constant on $x \in G_{\mathbb{K}}^{ab}$. Use symmetries action of $G_{\mathbb{K}}^{ab}$ on $(A_{\mathbb{K}}, \sigma_{\mathbb{K}})$. Isomorphism φ intertwines action of symmetries and get

$$\alpha_{\gamma x}(\mathfrak{n})\widetilde{\Phi}(\gamma x) = \widetilde{\Phi}(\theta_{\mathbb{K}}(\mathfrak{n})\gamma x) = \varphi(\gamma)\widetilde{\Phi}(\theta_{\mathbb{K}}(\mathfrak{n})x) = \alpha_x(\mathfrak{n})\widetilde{\Phi}(\gamma x)$$

Though don't know if constant on all of $X_{\mathbb{K}}$

Note: Isomorphism type of $G_{\mathbb{K}}^{ab}$: Ulm invariants

Fourth step of (2): Preserving ramification

Result: $N \subset G^{ab}_{\mathbb{K}}$ subgroup, $G^{ab}_{\mathbb{K}}/N \stackrel{\sim}{\to} G^{ab}_{\mathbb{L}}/\Phi(N)$

$$\mathfrak{p} \text{ ramifies in } \mathbb{K}'/\mathbb{K} \iff \varphi(\mathfrak{p}) \text{ ramifies in } \mathbb{L}'/\mathbb{L}$$

$\mathbb{K}' = (\mathbb{K}^{ab})^N$ finite extension and $\mathbb{L}' := (\mathbb{L}^{ab})^{\Phi(N)}$

- Mapping projectors $\mu_{\mathfrak{n}}\mu_{\mathfrak{n}}^* = e_{\mathbb{K},\mathfrak{n}}$ (divisibility by $\mathfrak{n}$)

$$\varphi(e_{\mathbb{K},\mathfrak{n}}) = \varphi(\mu_{\mathfrak{n}}\mu_{\mathfrak{n}}^*) = \mu_{\varphi(\mathfrak{n})}\mu_{\varphi(\mathfrak{n})}^* = e_{\mathbb{L},\varphi(\mathfrak{n})}$$

- Use these to show matching of $H_{\mathbb{K}}$

$$H_{\mathbb{K}} \cong G^{ab}_{\mathbb{K}}/\vartheta_{\mathbb{K}}\left(\prod_{\mathfrak{q}\neq\mathfrak{p}} \hat{\mathcal{O}}^*_{\mathfrak{q}}\right) \cong \mathring{G}^{ab}_{\mathbb{K},\mathfrak{p}}, \quad \text{and} \quad \Phi(H_{\mathbb{K}}) \cong \mathring{G}^{ab}_{\mathbb{L},\varphi(\mathfrak{p})}$$

$\mathring{G}^{ab}_{\mathbb{K},\mathfrak{p}}$ Gal group of max ab extension unramified *outside* $\mathfrak{p}$

Intermezzo: ramification matching proves (2) $\Rightarrow$ (3)
isomorphism $\varphi : (A_{\mathbb{K}}, \sigma_{\mathbb{K}}) \to (A_{\mathbb{L}}, \sigma_{\mathbb{L}}) \Rightarrow$ matching of L-series

- isom of G^{ab} groups $\Rightarrow$ character groups

$$\psi \; : \; \widehat{G}^{ab}_{\mathbb{K}} \stackrel{\sim}{\to} \widehat{G}^{ab}_{\mathbb{L}}$$

- character $\chi \in \widehat{G}^{ab}_{\mathbb{K}}$ extends to function $f_\chi \in C(X_{\mathbb{K}})$
- check $\varphi(f_\chi) = f_{\psi(\chi)}$: need matching divisors of conductor
- $\mathfrak{p}$ is coprime to $\mathfrak{f}_\chi$ iff χ factors over $G^{ab}_{\mathbb{K},\mathfrak{p}}$
- seen by ramification result these match: $\psi(\chi) = \Phi^*(\chi)$ factoring over $\Phi(G^{ab}_{\mathbb{K},\mathfrak{p}}) = G^{ab}_{\mathbb{L},\varphi(\mathfrak{p})}$
- then $\chi(\vartheta_{\mathbb{K}}(\mathfrak{n})) = \psi(\chi)(\vartheta_{\mathbb{L}}(\varphi(\mathfrak{n})))$
- then matching KMS$_\beta$ states on $f = f_\chi$

$$\omega^{\mathbb{L}}_{\gamma,\beta}(\varphi(f)) = \omega^{\mathbb{K}}_{\tilde{\gamma},\beta}(f)$$

and using arithmetic equivalence

Conclusion: $L_{\mathbb{K}}(\chi, s) = L_{\mathbb{L}}(\psi(\chi), s)$

Fifth Step of (2) $\Rightarrow$ (1): from QSM isomorphism get also

- Isomorphism of local units

$$\varphi : \hat{\mathcal{O}}_{\mathfrak{p}}^{*} \xrightarrow{\sim} \hat{\mathcal{O}}_{\varphi(\mathfrak{p})}^{*}$$

max ab ext where $\mathfrak{p}$ unramified = fixed field of inertia group $I_{\mathfrak{p}}^{ab}$, by ramification preserving

$$\Phi(I_{\mathfrak{p}}^{ab}) = I_{\varphi(\mathfrak{p})}^{ab}$$

and by local class field theory $I_{\mathfrak{p}}^{ab} \simeq \hat{\mathcal{O}}_{\mathfrak{p}}^{*}$

- by product of the local units: isomorphism

$$\varphi : \hat{\mathcal{O}}_{\mathbb{K}}^{*} \xrightarrow{\sim} \hat{\mathcal{O}}_{\mathbb{L}}^{*}$$

- Semigroup isomorphism

$$\varphi : (\mathbb{A}_{\mathbb{K},f}^{*} \cap \hat{\mathcal{O}}_{\mathbb{K}}, \times) \xrightarrow{\sim} (\mathbb{A}_{\mathbb{L},f}^{*} \cap \hat{\mathcal{O}}_{\mathbb{L}}, \times)$$

by exact sequence

$$0 \to \hat{\mathcal{O}}_{\mathbb{K}}^{*} \to \mathbb{A}_{\mathbb{K},f}^{*} \cap \hat{\mathcal{O}}_{\mathbb{K}} \to J_{\mathbb{K}}^{+} \to 0$$

(non-canonically) split by choice of uniformizer $\pi_{\mathfrak{p}}$ at every place

Recover multiplicative structure of the field

- Endomorphism action of $\mathbb{A}^*_{\mathbb{K},f} \cap \hat{\mathcal{O}}_{\mathbb{K}}$

$$\epsilon_s(f)(\gamma,\rho) = f(\gamma, s^{-1}\rho)e_\tau, \quad \epsilon_s(\mu_{\mathfrak{n}}) = \mu_{\mathfrak{n}}\, e_\tau$$

  e_τ char function of set $s^{-1}\rho \in \hat{\mathcal{O}}_{\mathbb{K}}$
- $\hat{\mathcal{O}}^*_{\mathbb{K}} =$ part acting by automorphisms
- $\overline{\mathcal{O}^*_{\mathbb{K},+}}$ (closure of tot pos units): trivial endomorphisms
- $\mathcal{O}^\times_{\mathbb{K},+} = \mathcal{O}_{\mathbb{K},+} - \{0\}$ (non-zero tot pos elements of ring of integers): *inner endomorphisms* (isometries eigenv of time evolution)
- $\varphi(\varepsilon_s) = \varepsilon_{\varphi(s)}$ for all $s \in \mathbb{A}^*_{\mathbb{K},f} \cap \hat{\mathcal{O}}_{\mathbb{K}}$

Conclusion: isom of multiplicative semigroups of tot pos non-zero elements of rings of integers

$$\varphi : (\mathcal{O}^\times_{\mathbb{K},+}, \times) \xrightarrow{\sim} (\mathcal{O}^\times_{\mathbb{L},+}, \times)$$

Last Step of (2) $\Rightarrow$ (1): Recover additive structure of the field

Extend by $\varphi(0) = 0$ the map

$$\varphi : (\mathcal{O}^{\times}_{\mathbb{K},+}, \times) \xrightarrow{\sim} (\mathcal{O}^{\times}_{\mathbb{L},+}, \times)$$

Claim: it is additive

Start from induced multipl map of local units $\varphi : \hat{\mathcal{O}}^*_{\mathfrak{p}} \xrightarrow{\sim} \hat{\mathcal{O}}^*_{\varphi(\mathfrak{p})}$

- Fix rational prime p totally split in $\mathbb{K}$
- Teichmüller lift $\tau_{\mathbb{K},p} : \overline{\mathbb{K}}^*_p \hookrightarrow \hat{\mathcal{O}}^*_{\mathbb{K},p}$ gives multiplicative map of residue fields

$$\begin{array}{ccc} \hat{\mathcal{O}}^*_{\mathbb{K},p} & \xrightarrow{\varphi} & \hat{\mathcal{O}}^*_{\mathbb{L},p} \\ {\scriptstyle \tau_{\mathbb{K},p}} \uparrow & & \downarrow {\scriptstyle \mathrm{mod}\, p} \\ \overline{\mathbb{K}}^*_p & \overset{\tilde{\varphi}}{\dashrightarrow} & \overline{\mathbb{L}}^*_p \end{array}$$

- To show additive (hence identity) on residue field, extend Teichmüller lift to

$$\tau_{\mathbb{K},p} : \hat{\mathcal{O}}^*_{\mathbb{K},p} \to \hat{\mathcal{O}}^*_{\mathbb{K},p} : x \mapsto \lim_{n \to +\infty} x^{p^n}$$

Show then $\varphi : \hat{\mathcal{O}}_{\mathfrak{p}}^* \xrightarrow{\sim} \hat{\mathcal{O}}_{\varphi(\mathfrak{p})}^*$ identity on $\hat{\mathcal{O}}_{\mathfrak{p}}^* \cap \mathbb{Z}$

- Set $\mathbb{Z}_{(p\Delta)}$ integers coprime to $p\Delta$ with $\Delta = \Delta_{\mathbb{K}} = \Delta_{\mathbb{L}}$ discriminant
- rational prime a coprime to $\Delta \Rightarrow$ ideal $(a) \mapsto \alpha_x(a)$ also in $\mathbb{Z}_{(p\Delta)}$, since $(a) = \mathfrak{p}_1 \dots \mathfrak{p}_r$ (distinct primes: totally spit) and α_x permutes primes above same rational prime
- φ fixes the element $[(1, 1_p)]$ (preserving ramification) $\Rightarrow$ $\varphi(a \cdot 1_p) = a \cdot 1_p$, for $a \in \mathbb{Z}_{(p\Delta)}$
- Injective map $\varpi_{\mathbb{K}} : \mathbb{Z}_{(p\Delta)} \to X_{\mathbb{K}} : a \mapsto [(1, a \cdot 1_p)]$
- Then $\varphi(\varpi_{\mathbb{K}}(a)) = \varphi((a) * [(1, 1_p)]) = (a) * [(1, 1_p)] = \varpi_{\mathbb{L}}(a)$

Start with residue class $\tilde{a}$ in $\overline{\mathbb{K}}_p^*$ and choose integer a congruent to $\tilde{a}$ mod p and coprime to $p\Delta$ (by Chinese remainder thm) $\Rightarrow$ $\tau_{\mathbb{K},p}(\tilde{a}) = \tau_{\mathbb{K},p}(a)$
Conclusion: Continuity $\Rightarrow$ φ identity map mod any totally split prime

$$\varphi(x+y) = \varphi(x) + \varphi(y) \mod p$$

tot split primes of arbitrary large norm $\Rightarrow$ φ additive

Then Conclusion of (2) $\Rightarrow$ (1):

- Have isomorphism of semigroups of totally positive integers (additive and multiplicative)
- $\mathcal{O}_{\mathbb{K}}$ has $\mathbb{Z}$-basis of totally positive elements
- Then obtain $\varphi : \mathcal{O}_{\mathbb{K}} \xrightarrow{\sim} \mathcal{O}_{\mathbb{L}} \Rightarrow$ $\mathbb{K} \simeq \mathbb{L}$ field isomorphism

□

First Step of (3) $\Rightarrow$ (2): identify $J_{\mathbb{K}}^{+}$ and $J_{\mathbb{L}}^{+}$ compatibly with Artin map
Method: Fourier analysis on Number Fields

- Observation: matching of zeta functions, so know same number of primes $\mathfrak{p}$ in $\mathbb{K}$ and $\mathfrak{q}$ in $\mathbb{L}$ over the same rational prime p with inertia degree f
- Need to find a way to match them compatible with the Artin map: $\mathfrak{p} \mapsto \mathfrak{q}$ so that $\psi(\chi)(\theta_{\mathbb{L}}(\mathfrak{q})) = \chi(\theta_{\mathbb{K}}(\mathfrak{p}))$ for all characters χ with conductor coprime to p
- Need to show this can be done with a bijection between primes of $\mathbb{K}$ and $\mathbb{L}$
- Idea: use a combination of L-series as counting function for number of such $\mathfrak{q}$

L-series and counting functions

- Fix a finite quotient $G_{\mathbb{K}}^{ab} \twoheadrightarrow G$
- Set $b_{\mathbb{K},G,n}(\gamma) := \# B_{\mathbb{K},G,n}(\gamma)$ cardinality of set

$$B_{\mathbb{K},G,n}(\gamma) = \{\mathfrak{n} \in J_{\mathbb{K}}^{+} \,:\, N_{\mathbb{K}}(\mathfrak{n}) = n \text{ and } \pi_G(\vartheta_{\mathbb{K}}(\mathfrak{n})) = \pi_G(\gamma)\}$$

- Then use known fact that

$$\sum_{\substack{\mathfrak{n}\in J_{\mathbb{K}}^{+} \\ N_{\mathbb{K}}(\mathfrak{n})}} \left(\sum_{\widehat{G}} \chi(\pi_G(\gamma)^{-1})\chi(\vartheta_{\mathbb{K}}(\mathfrak{n})) \right) = b_{\mathbb{K},G,n}(\gamma).$$

- Identity of L-functions gives, for fixed norm n,

$$\sum_{\mathfrak{n}\in J^+_{\mathbb{K}},\gamma\in\widehat{G}} \chi(\pi_G(\gamma)^{-1})\chi(\vartheta_{\mathbb{K}}(\mathfrak{n})) = \sum_{\mathfrak{m}\in J^+_{\mathbb{L}},\gamma\in\widehat{G}} \chi(\pi_G(\gamma)^{-1})\psi(\chi)(\vartheta_{\mathbb{L}}(\mathfrak{m}))$$

- Using isomorphism $\psi : G^{ab}_{\mathbb{K}} \to G^{ab}_{\mathbb{L}}$ preserving $G^{ab}_{\mathbb{K},n} =$ Gal of max abelian ext unramified above prime divisors of n, right-hand-side above gives, for $(\psi^{-1})^*(G) = G'$,

$$\sum_{\widehat{G'}} \psi^{-1}(\eta)(\pi_G(\gamma)^{-1})\eta(\pi_{G'}(\vartheta_{\mathbb{L}}(\mathfrak{m})))$$

- $\mathfrak{m}$ coprime to $\mathfrak{f}_\eta$: character on $\widehat{G'}$
$\Xi_{\mathfrak{m}} : \eta \mapsto \psi^{-1}(\eta)(\pi_G(\gamma)^{-1})\eta(\pi_{G'}(\vartheta_{\mathbb{L}}(\mathfrak{m})))$ so that

$$\sum_{\widehat{G'}} \psi^{-1}(\eta)(\pi_G(\gamma)^{-1})\eta(\pi_{G'}(\vartheta_{\mathbb{L}}(\mathfrak{m}))) = \begin{cases} |G'| & \text{if } \Xi_{\mathfrak{m}} \equiv 1; \\ 0 & \text{otherwise.} \end{cases}$$

- $\Xi_{\mathfrak{m}} \equiv 1$ gives

$$\eta(\pi_{G'}(\vartheta_{\mathbb{L}}(\mathfrak{m}))) = \psi^{-1}(\eta)(\pi_G(\gamma)) \text{ for all } \eta \in G'$$

  so that $\pi_{G'}(\vartheta_{\mathbb{L}}(\mathfrak{m})) = \pi_{G'}((\psi^{-1})^*(\gamma))$.
- So from identity of L-function get counting identity

$$b_{\mathbb{K},G,n}(\gamma) = b_{\mathbb{L},(\psi^{-1})^* G,n}((\psi^{-1})^*(\gamma))$$

- $G^{ab}_{\mathbb{K},n}$ as inverse limit over finite quotients: same cardinality of

$$S_1 = \{\mathfrak{n} \in J^+_{\mathbb{K}} : \, N_{\mathbb{K}}(\mathfrak{n}) = n, \, \pi_{G^{ab}_{\mathbb{K},n}}(\vartheta_{\mathbb{K}}(\mathfrak{n}))) = \pi_{G^{ab}_{\mathbb{K},n}}(\gamma)\}$$

$$S_2 = \{\mathfrak{m} \in J^+_{\mathbb{L}} : \, N_{\mathbb{L}}(\mathfrak{m}) = n, \, \pi_{G^{ab}_{\mathbb{L},n}}(\vartheta_{\mathbb{L}}(\mathfrak{m})) = \pi_{G^{ab}_{\mathbb{L},n}}((\psi^{-1})^*(\gamma))\}$$

- Artin map $\vartheta_{\mathbb{K}} : J_{\mathbb{K}}^{+} \to G_{\mathbb{K},n}^{ab}$ injective on ideals dividing n: get $\#S_1 = 1$
- $\#S_2 = 1$ gives unique ideal $\mathfrak{m} \in J_{\mathbb{L}}^{+}$ with $N_{\mathbb{L}}(\mathfrak{m}) = N_{\mathbb{K}}(\mathfrak{n})$ and with

$$\pi_{G_{\mathbb{K},n}^{ab}}(\vartheta_{\mathbb{L}}(\mathfrak{m})) = \pi_{G_{\mathbb{L},n}^{ab}}((\psi^{-1})^{*}(\vartheta_{\mathbb{K}}(\mathfrak{n})))$$

- Get multiplicative map $\Psi(\mathfrak{n}) := \mathfrak{m}$, isomorphism of $J_{\mathbb{K}}^{+}$ and $J_{\mathbb{L}}^{+}$ compatible with Artin map

Second Step of (3) $\Rightarrow$ (2): matching $C(X_{\mathbb{K}})$ and $C(X_{\mathbb{L}})$ compatibly with $J_{\mathbb{K}}^{+}$ and $J_{\mathbb{L}}^{+}$ actions

- Idea: extend identification $\psi : C(G_{\mathbb{K}}^{ab}) \xrightarrow{\sim} C(G_{\mathbb{L}}^{ab})$ from $G_{\mathbb{K}}^{ab}$ to $G_{\mathbb{K}}^{ab} \rtimes_{\hat{\mathcal{O}}_{\mathbb{K}}^{*}} \hat{\mathcal{O}}_{\mathbb{K}}$
- Using $X_{\mathbb{K},n} := G_{\mathbb{K}}^{ab} \times_{\hat{\mathcal{O}}_{\mathbb{K}}^{*}} \hat{\mathcal{O}}_{\mathbb{K},n}$ and $J_{\mathbb{K},n}^{+}$ gen by prime ideals dividing n
- know algebra $C(X_{\mathbb{K},n})$ is generated by the functions

$$f_{\chi,\mathfrak{n}} : \gamma \mapsto \chi(\vartheta_{\mathbb{K}}(\mathfrak{n}))\chi(\gamma), \quad \chi \in \widehat{G}_{\mathbb{K},n}^{ab}, \quad \mathfrak{n} \in J_{\mathbb{K},n}^{+}$$

- Map $\psi_n : C(X_{\mathbb{K},n}) \to C(X_{\mathbb{L},n})$ by

$$f_{\mathfrak{n},\chi} \mapsto f_{\Psi(\mathfrak{n}),\psi(\chi)}$$

well defined by matching ramification and conductors

- Direct limit $\psi = \lim_{\overrightarrow{n}} \psi_n \; : \; C(X_{\mathbb{K}}) \stackrel{\sim}{\to} C(X_{\mathbb{L}})$
- Check algebra homomorphism: from compatibility with Artin map

$$\psi(f_{\chi,\mathfrak{n}})(\gamma') = f_{\psi(\chi),\Psi(\mathfrak{n})}(\gamma') = \psi(\chi)(\vartheta_{\mathbb{L}}(\Psi(\mathfrak{n}))\psi(\chi)(\gamma')$$

$$\psi(f_{\chi,\mathfrak{n}})(\gamma') = \chi(\vartheta_{\mathbb{K}}(\mathfrak{n}))\chi(\psi^*(\gamma')) = (\psi^{-1})^* f_{\chi,\mathfrak{n}}$$

$$\psi(f_{\mathfrak{n},\chi} \cdot f_{\mathfrak{n}',\chi'}) = (\psi^{-1})^* \left(f_{\mathfrak{n},\chi} \cdot f_{\mathfrak{n}',\chi'}\right) =$$

So get multiplicative map:

$$(\psi^{-1})^* \left(f_{\mathfrak{n},\chi}\right) \cdot (\psi^{-1})^* \left(f_{\mathfrak{n}',\chi'}\right) = \psi(f_{\mathfrak{n},\chi}) \cdot \psi(f_{\mathfrak{n}',\chi'})$$

- Compatibility with time evolution since $N_{\mathbb{L}}(\Psi(\mathfrak{n})) = N_{\mathbb{K}}(\mathfrak{n})$

This completes all implications of main Theorem $\square$

What then?

- Function fields $\mathbb{K} = \mathbb{F}_{p^m}(C)$, curve C over finite field
- Analogies between number fields and function fields
- Same type of QSM systems
- Sneak Preview: purely NT proof seems not to work for function fields ... but NCG proof does!

... coming soon to a lecture hall near you

Thank you !