Quantum statistical mechanics, $L$-series, Anabelian Geometry

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joint work with Gunther Cornelissen

Recovering a Number Field from invariants

- Dedekind zeta function $\zeta_K(s) = \zeta_L(s)$ arithmetic equivalence
  - Gaßmann examples:
    $$K = \mathbb{Q}(\sqrt[8]{3}) \text{ and } L = \mathbb{Q}(\sqrt[8]{3 \cdot 2^4})$$
    not isomorphism $K \neq L$

- Adeles rings $A_K \cong A_L$ adelic equivalence $\Rightarrow$ arithmetic equivalence
  - Komatsu examples:
    $$K = \mathbb{Q}(\sqrt[8]{2 \cdot 9}) \text{ and } L = \mathbb{Q}(\sqrt[8]{2^5 \cdot 9})$$
    not isomorphism $K \neq L$
Abelianized Galois groups: $G_{ab}^K \cong G_{ab}^L$ also not isomorphism;
Onabe examples:

\[ K = \mathbb{Q}(\sqrt{-2}) \text{ and } L = \mathbb{Q}(\sqrt{-3}) \]

not isomorphism $K \neq L$

But ... absolute Galois groups $G_K \cong G_L \Rightarrow$ isomorphism
$K \cong L$: Neukirch–Uchida theorem
(Grothendieck’s anabelian geometry)
Question: Can combine $\zeta_K(s)$, $A_K$ and $G_{ab}^K$ to something as strong as $G_K$ that determines isomorphism class of $K$?

Answer: Yes! Combine as a Quantum Statistical Mechanical system algebra and time evolution $(A, \sigma)$

$$A_K := C(X_K) \rtimes J_K^+, \quad \text{with} \quad X_K := G_{ab}^K \times \hat{\mathcal{O}}_K \ast \hat{\mathcal{O}}_K, \quad \hat{\mathcal{O}}_K = \text{ring of finite integral adeles}, \quad J_K^+ = \text{is the semigroup of ideals,}$$
acting on $X_K$ by Artin reciprocity

Time evolution $\sigma_K$ acts on $J_K^+$ as a phase factor $N(n)^{it}$

QSM systems introduced by Ha–Paugam to generalize Bost–Connes system, also recently studied by Laca–Larsen–Neshveyev [LLN]
The setting of Quantum Statistical Mechanics: Data

- $\mathcal{A}$ unital $C^*$-algebra of observables
- $\sigma_t$ time evolution, $\sigma : \mathbb{R} \to \text{Aut}(\mathcal{A})$
- states $\omega : \mathcal{A} \to \mathbb{C}$ continuous, normalized $\omega(1) = 1$, positive
  
  $$\omega(a^*a) \geq 0$$

- equilibrium states $\omega(\sigma_t(a)) = \omega(a)$ all $t \in \mathbb{R}$
- representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$, Hamiltonian $H$
  
  $$\pi(\sigma_t(a)) = e^{itH} \pi(a) e^{-itH}$$

- partition function $Z(\beta) = \text{Tr}(e^{-\beta H})$
- Gibbs states (equilibrium, inverse temperature $\beta$):
  
  $$\omega_\beta(a) = \frac{\text{Tr}(\pi(a)e^{-\beta H})}{\text{Tr}(e^{-\beta H})}$$
Generalization of Gibbs states: KMS states (Kubo–Martin–Schwinger) $\forall a, b \in A, \exists$ holomorphic $F_{a,b}$ on strip $I_\beta = \{ 0 < \text{Im} \, z < \beta \}$, bounded continuous on $\partial I_\beta$,

$$F_{a,b}(t) = \omega(\alpha \sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + i\beta) = \omega(\sigma_t(b)a)$$

Fixed $\beta > 0$: KMS$_\beta$ state convex simplex: extremal states (like points in NCG)

Isomorphism of QSM: $\varphi : (\mathcal{A}, \sigma) \rightarrow (\mathcal{B}, \tau)$

$$\varphi : \mathcal{A} \xrightarrow{\sim} \mathcal{B}, \quad \varphi \circ \sigma = \tau \circ \varphi$$

$C^*$-algebra isomorphism intertwining time evolution

Pullback of a state: $\varphi^* \omega(a) = \omega(\varphi(a))$
Theorem The following are equivalent:

1. \( K \cong L \) are isomorphic number fields
2. Quantum Statistical Mechanical systems are isomorphic
   \[ (A_K, \sigma_K) \cong (A_L, \sigma_L) \]
   C*-algebra isomorphism \( \varphi : A_K \rightarrow A_L \) compatible with time evolution, \( \sigma_L \circ \varphi = \varphi \circ \sigma_K \)
3. There is a group isomorphism \( \psi : \hat{G}^{ab}_K \rightarrow \hat{G}^{ab}_L \) of Pontrjagin duals of abelianized Galois groups with
   \[ L_K(\chi, s) = L_L(\psi(\chi), s) \]

identity of all \( L \)-functions with Großencharakter

Note: Generalization of arithmetic equivalence:
\( \chi = 1 \) gives \( \zeta_K(s) = \zeta_L(s) \)
(now also purely number theoretic proof of (3) \( \Rightarrow \) (1) by Hendrik Lenstra and Bart de Smit)
Setting and notation

- Artin reciprocity map

\[ \vartheta_K : A^*_K \to G^\text{ab}_K. \]

\( \vartheta_K(n) \) for ideal \( n \) seen as idele by non-canonical section \( s \) of

\[ A^*_{K,f} \to J_K : (x_p)_p \mapsto \prod_{p \text{ finite}} p^{v_p(x_p)} \]

- Crossed product algebra

\[ A_K := C(X_K) \rtimes J^+_K = C(G^\text{ab}_K \times \widehat{\mathbb{O}}^*_K \times \widehat{\mathbb{O}}_K) \rtimes J^+_K \]
• semigroup crossed product: $n \in J_+^K$ acting on $f \in C(X_K)$ as

$$\rho_n(f)(\gamma, \rho) = f(\vartheta_K(n)\gamma, s(n)^{-1}\rho)e_n,$$

$e_n = \mu_n^*\mu_n$ projector onto $[(\gamma, \rho)]$ with $s(n)^{-1}\rho \in \hat{O}_K$

• partial inverse of semigroup action

$$\sigma_n(f)(x) = f(n*x) \quad \text{with} \quad n*(\gamma, \rho) = [(\vartheta_K(n)^{-1}\gamma, n\rho)]$$
Generators and Relations: \( f \in C(X_K) \) and \( \mu_n, n \in J_K^+ \)

\[
\mu_n \mu_n^* = e_n; \quad \mu_n^* \mu_n = 1; \quad \rho_n(f) = \mu_n f \mu_n^*;
\]

\[
\sigma_n(f) e_n = \mu_n^* f \mu_n; \quad \sigma_n(\rho_n(f)) = f; \quad \rho_n(\sigma_n(f)) = f e_n
\]

Time evolution:

\[
\sigma_{K,t}(f) = f \quad \text{and} \quad \sigma_{K,t}(\mu_n) = N(n)^t \mu_n
\]

for \( f \in C(\hat{G}_K^{ab} \times \hat{\hat{O}}_K^* \hat{\hat{O}}_K) \) and for \( n \in J_K^+ \)
Stratification of $X_K$

- $\hat{\Omega}_{K,n} := \prod_{p \mid n} \hat{\Omega}_{K,p}$ and
  
  $$X_{K,n} := G_{K}^{ab} \times \hat{\Omega}_{K,n}^{*} \quad \text{with} \quad X_K = \lim_{n \to \infty} X_{K,n}$$

- Topological groups
  
  $$G_{K}^{ab} \times \hat{\Omega}_{K,n}^{*} \cong G_{K}^{ab} / \nu_{K}(\hat{\Omega}_{K,n}^{*}) = G_{K,n}^{ab}$$

Gal of max ab ext unramified at primes dividing $n$

- $J_{K,n}^{+} \subset J_{K}^{+}$ subsemigroup gen by prime ideals dividing $n$

- Decompose $X_{K,n} = X_{K,n}^{1} \coprod X_{K,n}^{2}$
  
  $$X_{K,n}^{1} := \bigcup_{n \in J_{K,n}^{+}} \nu_{K}(n) G_{K,n}^{ab} \quad \text{and} \quad X_{K,n}^{2} := \bigcup_{p \mid n} Y_{K,p}$$

where $Y_{K,p} = \{ (\gamma, \rho) \in X_{K,n} : \rho_{p} = 0 \}$

- $X_{K,n}^{1}$ dense in $X_{K,n}$ and $X_{K,n}^{2}$ has $\mu_{K}$-measure zero

- Algebra $C(X_{K,n})$ is generated by functions
  
  $$f_{\chi,n} : \gamma \mapsto \chi(\nu_{K}(n)) \chi(\gamma), \quad \chi \in \hat{G}_{K,n}^{ab}, \quad n \in J_{K,n}^{+}$$
First Step of (2) ⇒ (1): $(A_K, \sigma_K) \simeq (A_L, \sigma_L) \Rightarrow \zeta_K(s) = \zeta_L(s)$

- QSM $(A, \sigma)$ and representation $\pi : A \to B(\mathcal{H})$ gives Hamiltonian

\[
\pi(\sigma_t(a)) = e^{itH} \pi(a) e^{-itH}
\]

\[
H_{\sigma_K} \varepsilon_n = \log N(n) \varepsilon_n
\]

Partition function $\mathcal{H} = \ell^2(J_+^K)$

\[
Z(\beta) = \text{Tr}(e^{-\beta H}) = \zeta_K(\beta)
\]

- Isomorphism $\varphi : (A_K, \sigma_K) \simeq (A_L, \sigma_L) \Rightarrow$ homeomorphism of sets of extremal KMS$_{\beta}$ states by pullback $\omega \mapsto \varphi^*(\omega)$

- KMS$_{\beta}$ states for $(A_K, \sigma_K)$ classified [LLN]: $\beta > 1$

\[
\omega_{\gamma, \beta}(f) = \frac{1}{\zeta_K(\beta)} \sum_{m \in J_+^K} \frac{f(\vartheta_K(m)\gamma)}{N_K(m)^\beta}
\]

parameterized by $\gamma \in G_K^{ab}/\vartheta_K(\hat{O}_K^*)$
Comparing GNS representations of $\omega \in \text{KMS}_\beta(A_L, \sigma_L)$ and $\varphi^*(\omega) \in \text{KMS}_\beta(A_K, \sigma_K)$ find Hamiltonians

$$H_K = U H_L U^* + \log \lambda$$

for some $U$ unitary and $\lambda \in \mathbb{R}^*_+$

Then partition functions give

$$\zeta_L(\beta) = \lambda^{-\beta} \zeta_K(\beta)$$

identity of Dirichlet series

$$\sum_{n \geq 1} \frac{a_n}{n^\beta} \quad \text{and} \quad \sum_{n \geq 1} \frac{b_n}{(\lambda n)^\beta}$$

with $a_1 = b_1 = 1$, taking limit as $\beta \to \infty$

$$a_1 = \lim_{\beta \to \infty} b_1 \lambda^{-\beta} \quad \Rightarrow \quad \lambda = 1$$
Conclusion of first step: arithmetic equivalence $\zeta_L(\beta) = \zeta_K(\beta)$

Consequences:
From arithmetic equivalence already know $K$ and $L$ have same degree over $\mathbb{Q}$, discriminant, normal closure, unit groups, archimedean places.

But... not class group (or class number)
Intermezzo: a useful property (characterizing isometries)

Element $u$ in $A_K^*$:
- isometry: $u^* u = 1$
- eigenvector of time evolution: $\sigma_t(u) = q^{it} u$, for $q = n/m$

Then

$$u = \sum_n \mu_n f_n$$

with $f_n \in C(X_K)$ and $n \in J_K^+$ with $N_K(n) = n$ and $\sum_n |f_n|^2 = 1$

inner endomorphisms: $a \mapsto u a u^*$
Second Step of (2) \( \Rightarrow \) (1): unraveling the crossed product

\[ \varphi : C(X_K) \rtimes J_K^+ \xrightarrow{\sim} C(X_L) \rtimes J_L^+ \quad \text{with} \quad \sigma_L \circ \varphi = \varphi \circ \sigma_K \]

Then is gives separately:

- A homeomorphism \( X_K \cong X_L \)
- A semigroup isomorphism \( J_K^+ \cong J_L^+ \)
- Compatible with the crossed product action \( \rho \)
Test case: a single isometry

If single isometry: continuous injective self-map $\gamma$ of space $X$ then semigroup crossed product $C(X) \rtimes_\rho \mathbb{Z}_+$ with $\mu f \mu^*(x) = \rho(f)(x) = \chi(x)f(\gamma^{-1}(x))$, with $\chi = \text{characteristic function of range of } \gamma$; time evolution: $\sigma_t(\mu) = \lambda^t \mu$

Then isomorphism $\varphi : (C(X) \rtimes_\rho \mathbb{Z}_+, \sigma) \simeq (C(X') \rtimes_{\rho'} \mathbb{Z}_+, \sigma')$ gives homeomorphism $\Phi : X \simeq X'$ with $\gamma' \circ \Phi = \Phi \circ \gamma$

Basic step: write commutator ideals $\mathcal{C}_0$ in terms of Fourier modes $a = f_0 + \sum_{k>0}(\mu^k f_k + f_{-k}(\mu^*)^k)$ and get matching of maximal ideals $\varphi(\widetilde{I}_{\gamma(x),0} + \mathcal{C}_0^2) = \widetilde{I}_{\Phi(\gamma(x)),0} \mathcal{C}_0' + (\mathcal{C}_0')^2$ where $\widetilde{I}_{y,0} \mathcal{C}_0 + \mathcal{C}_0^2 = \mathcal{C}_0 \widetilde{I}_{x,0} + \mathcal{C}_0^2$. 
From one to $N$ isometries

Difficulty: no longer know just from time evolution that image of $C(X)$ does not involve terms like $\mu_i \mu_j^*$ but only $C(X')$

But ... Still works!

Result: for $N$ commuting isometries and crossed products

$$\varphi : (\mathcal{A} = C(X) \rtimes \rho \mathbb{Z}_N^+, \sigma) \cong (\mathcal{A}' = C(X') \rtimes \rho' \mathbb{Z}_N^+, \sigma')$$

with $\sigma_t(f) = f$ and $\sigma_t(\mu_j) = \lambda^{it} \mu_j$ (both sides)

with density hypothesis: any multi-indices $\alpha, \beta \in \mathbb{Z}_N^+$ with $\gamma_{\alpha} \neq \gamma_{\beta}$

$$\{ x \in X : \gamma_{\alpha}(x) \neq \gamma_{\beta}(x) \} \text{ dense in } X$$

(and same for $\mathcal{A}'$)

Then isomorphism of QSM system gives:

- homeomorphism $\Phi : X \cong X'$
- and compatible isomorphism $\alpha_x : \mathbb{Z}_N^+ \to \mathbb{Z}_N^+$

locally constant in $x \in X$ (permutations of the generators)
In fact:

- $\mu_j$ go to isometries $u_j$ eigenvectors of time evolution $\Rightarrow$
  \[ \varphi(\mu_j) = \sum_k \mu'_k f_{jk} \]

- Functions $f(x) = e^{ih(x)}$ (local phase) go to local phase in $C(X')$

- For all functions $\varphi(f_1)\varphi(f_2) = \varphi(f_2)\varphi(f_1)$

- Applied to a local phase $h_2 = \varphi(f_2)$ and an arbitrary function $f_1$:
  \[ f_{\alpha,\beta} \cdot (h_2 \circ \gamma'_\beta) = f_{\alpha,\beta} \cdot (h_2 \circ \gamma'_\alpha). \]

where
  \[ \varphi(f_1) = \sum_{\alpha,\beta: |\alpha|=|\beta|} \mu'_\alpha f_{\alpha,\beta} \mu'_\beta. \]

**Conclusion:** $C(X)$ goes to $C(X')$ and $\mu_i$ go to something with no $\mu'_j$ then same argument as for one isometry
Applied to QSM system \((A_K, \sigma_K)\):
from finitely many to infinitely many isometries

- By separating eigenspaces of the time evolution by \(N(\varphi) = \rho\), apply case of \(N\) isoetries
- Density hypothesis: for any \(m \neq n\) dense set of \(x \in X_K\) with \(m \ast x \neq n \ast x\). In fact, check that set \(E\) of \(m \ast x = n \ast x\) means exists \(u \in \hat{O}_K^*\)

\[
\begin{align*}
\vartheta_K(m) &= \vartheta_K(u \cdot n) \\
\rho(s(m)) &= \rho(u \cdot s(n))
\end{align*}
\]

\(s: J^+_K \rightarrow A^*_K, f\) section (defined up to units)

\[
E = \begin{cases} \\
\emptyset & \text{if } m \not\sim n \in \text{Cl}^+(K) \\
G^a_K \times \hat{O}_K^* \{0\} \cong \text{Cl}^+(K) & \text{if } m \sim n \in \text{Cl}^+(K).
\end{cases}
\]

finite or empty: complement dense
Third step of (2): group isomorphism $G_{K}^{ab} \cong G_{L}^{ab}$

- $\gamma \mapsto \epsilon_{\gamma}$ (faithful) action of $G_{K}^{ab}$ as symmetries of $A_{K}$
- $G_{K}^{ab}$ acts freely transitively on extremal KMS

$$\omega_{\beta,\gamma_{1}} \circ \epsilon_{\gamma_{2}} = \omega_{\beta,\gamma_{1}\gamma_{2}}$$

- $\tilde{\Phi}(\gamma) = \Phi(\gamma)\Phi(1)^{-1}$ group isomorphism, from

$$\varphi^{*}(\epsilon_{\gamma})(\omega_{\beta,\gamma'}) = \omega_{\beta,\Phi(\Phi^{-1}(\gamma')\gamma)}$$

$$\varphi^{*}(\epsilon_{\gamma_{2}}) = \epsilon\Phi(\gamma_{1})^{-1}\Phi(\gamma_{1}\gamma_{2}) = \epsilon\Phi(1)\Phi(\gamma_{2})$$

$$\Phi(\gamma_{1}\gamma_{2}) = \Phi(1)\Phi(\gamma_{1})\Phi(\gamma_{2})$$
Back to step 2: Got homeomorphism $\Phi : X_K \simeq X_L$ and locally constant $\alpha_x : J^+_K \simeq J^+_L$

- The locally constant $\alpha_x : J^+_K \simeq J^+_L$ is constant on $x \in G^{ab}_K$. Use symmetries action of $G^{ab}_K$ on $(A_K, \sigma_K)$. Isomorphism $\varphi$ intertwines action of symmetries and get

$$\alpha_{\gamma x}(n)\tilde{\Phi}(\gamma x) = \tilde{\Phi}(\theta_K(n)\gamma x) = \varphi(\gamma)\tilde{\Phi}(\theta_K(n)x) = \alpha_x(n)\tilde{\Phi}(\gamma x)$$

Though don’t know if constant on all of $X_K$

Note: Isomorphism type of $G^{ab}_K$: Ulm invariants
Fourth step of (2): Preserving ramification

Result: \( N \subset G^\text{ab}_K \) subgroup, \( G^\text{ab}_K / N \cong G^\text{ab}_L / \Phi(N) \)

\( \mathfrak{p} \) ramifies in \( K'/K \iff \varphi(\mathfrak{p}) \) ramifies in \( L'/L \)

\( K' = (K^\text{ab})^N \) finite extension and \( L' := (L^\text{ab})^{\Phi(N)} \)

- Mapping projectors \( \mu_n \mu_n^* = e_{K,n} \) (divisibility by \( n \))

\[ \varphi(e_{K,n}) = \varphi(\mu_n \mu_n^*) = \mu_{\varphi(n)} \mu_n = e_{L,\varphi(n)} \]

- Use these to show matching of \( H_K \)

\[ H_K \cong G^\text{ab}_K / \vartheta_K \left( \prod_{q \neq \mathfrak{p}} \hat{\vartheta}_q^* \right) \cong \hat{G}^\text{ab}_{K,p}, \text{ and } \Phi(H_K) \cong \hat{G}^\text{ab}_{L,\varphi(\mathfrak{p})} \]

\( \hat{G}^\text{ab}_{K,p} \) Gal group of max ab extension unramified outside \( \mathfrak{p} \)

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Quantum statistical mechanics, \( L \)-series, Anabelian Geometry
Intermezzo: ramification matching proves \((2) \Rightarrow (3)\) isomorphism \(\varphi : (A_K, \sigma_K) \rightarrow (A_L, \sigma_L) \Rightarrow\) matching of \(L\)-series

- isom of \(G^{ab}\) groups \(\Rightarrow\) character groups

\[
\psi : \hat{G}_{K}^{ab} \sim \hat{G}_{L}^{ab}
\]

- character \(\chi \in \hat{G}_{K}^{ab}\) extends to function \(f_\chi \in C(X_K)\)
- check \(\varphi(f_\chi) = f_{\psi(\chi)}\): need matching divisors of conductor
- \(p\) is coprime to \(\int_\chi\) iff \(\chi\) factors over \(G_{K,p}^{ab}\)
- seen by ramification result these match: \(\psi(\chi) = \Phi^{*}(\chi)\) factoring over \(\Phi(G_{K,p}^{ab}) = G_{L,\varphi(p)}^{ab}\)

- then \(\chi(\varphi_K(n)) = \psi(\chi)(\varphi_L(\varphi(n)))\)
- then matching KMS_\beta states on \(f = f_\chi\)

\[
\omega_{\gamma,\beta}(\varphi(f)) = \omega_{\tilde{\gamma},\beta}(f)
\]

and using arithmetic equivalence

Conclusion: \(L_K(\chi, s) = L_L(\psi(\chi), s)\)
Fifth Step of (2) $\Rightarrow$ (1): from QSM isomorphism get also

- Isomorphism of local units

$$\varphi : \hat{\mathcal{O}}_p^* \sim \hat{\mathcal{O}}^*_\varphi(p)$$

max ab ext where $p$ unramified = fixed field of inertia group $I_p^{\text{ab}}$,
by ramification preserving

$$\Phi(I_p^{\text{ab}}) = I_p^{\varphi(ab)}$$

and by local class field theory $I_p^{\text{ab}} \simeq \hat{\mathcal{O}}_p^*$

- by product of the local units: isomorphism

$$\varphi : \hat{\mathcal{O}}_K^* \sim \hat{\mathcal{O}}_L^*$$

- Semigroup isomorphism

$$\varphi : (A_{K,f}^*, \times) \sim (A_{L,f}^*, \times)$$

by exact sequence

$$0 \to \hat{\mathcal{O}}_K^* \to A_{K,f}^* \cap \hat{\mathcal{O}}_K \to J_K^+ \to 0$$

(non-canonically) split by choice of uniformizer $\pi_p$ at every place
Recover multiplicative structure of the field

- **Endomorphism action of** \( \mathbb{A}^{\times}_{K,f} \cap \hat{\mathcal{O}}_{K} \)

\[
\epsilon_s(f)(\gamma, \rho) = f(\gamma, s^{-1}\rho)e_\tau, \quad \epsilon_s(\mu) = \mu e_\tau
\]

\( e_\tau \) char function of set \( s^{-1}\rho \in \hat{\mathcal{O}}_{K} \)

- \( \hat{\mathcal{O}}_{K}^{\times} \) = part acting by automorphisms
- \( \overline{\mathcal{O}_{K,+}^{\times}} \) (closure of tot pos units): trivial endomorphisms
- \( \mathcal{O}_{K,+,+}^{\times} = \mathcal{O}_{K,+} - \{0\} \) (non-zero tot pos elements of ring of integers): *inner endomorphisms* (isometries eigenv of time evolution)

- \( \varphi(\epsilon_s) = \epsilon_{\varphi(s)} \) for all \( s \in \mathbb{A}^{\times}_{K,f} \cap \hat{\mathcal{O}}_{K} \)

**Conclusion:** isom of multiplicative semigroups of tot pos non-zero elements of rings of integers

\[
\varphi : (\overline{\mathcal{O}_{K,+,+}^{\times}}, \times) \xrightarrow{\sim} (\mathcal{O}_{L,+,+}^{\times}, \times)
\]
Last Step of (2) $\Rightarrow$ (1): Recover additive structure of the field

Extend by $\varphi(0) = 0$ the map

$$
\varphi : (\mathcal{O}_K^\times, +, \times) \xrightarrow{\sim} (\mathcal{O}_L^\times, +, \times)
$$

Claim: it is additive

Start from induced multipl map of local units $\varphi : \hat{\mathcal{O}}_p^* \xrightarrow{\sim} \hat{\mathcal{O}}_{\varphi(p)}^*$

- Fix rational prime $p$ totally split in $K$
- Teichmüller lift $\tau_{K,p} : \overline{K}_p^* \hookrightarrow \hat{\mathcal{O}}_{K,p}^*$ gives multiplicative map of residue fields

$$
\begin{array}{c}
\hat{\mathcal{O}}_{K,p}^* \xrightarrow{\varphi} \hat{\mathcal{O}}_{L,p}^* \\
\downarrow \tau_{K,p} \uparrow \varphi \uparrow \downarrow \text{mod } p
\end{array}
$$

- To show additive (hence identity) on residue field, extend Teichmüller lift to $\tau_{K,p} : \hat{\mathcal{O}}_{K,p}^* \rightarrow \hat{\mathcal{O}}_{K,p}^*$

$$
\tau_{K,p} : \hat{\mathcal{O}}_{K,p}^* \rightarrow \hat{\mathcal{O}}_{K,p}^* : x \mapsto \lim_{n \to +\infty} x^{p^n}
$$
Show then $\varphi : \hat{\mathcal{O}}^* \sim \hat{\mathcal{O}}^*_{\varphi(p)}$ identity on $\hat{\mathcal{O}}^* \cap \mathbb{Z}$

- Set $\mathbb{Z}_{(p\Delta)}$ integers coprime to $p\Delta$ with $\Delta = \Delta_K = \Delta_L$ discriminant

- rational prime $a$ coprime to $\Delta \implies$ ideal $(a) \mapsto \alpha(a)$ also in $\mathbb{Z}_{(p\Delta)}$, since $(a) = p_1 \ldots p_r$ (distinct primes: totally split) and $\alpha$ permutes primes above same rational prime

- $\varphi$ fixes the element $[(1, 1_p)]$ (preserving ramification) $\implies$
  $\varphi(a \cdot 1_p) = a \cdot 1_p$, for $a \in \mathbb{Z}_{(p\Delta)}$

- Injective map $\varpi_K : \mathbb{Z}_{(p\Delta)} \to X_K : a \mapsto [(1, a \cdot 1_p)]$

- Then $\varphi(\varpi_K(a)) = \varphi((a) * [(1, 1_p)]) = (a) * [(1, 1_p)] = \varpi_L(a)$
Start with residue class $\tilde{a}$ in $\overline{K}_p^*$ and choose integer $a$ congruent to $\tilde{a}$ mod $p$ and coprime to $p\Delta$ (by Chinese remainder thm) $\Rightarrow$

$\tau_{K,p}(\tilde{a}) = \tau_{K,p}(a)$

Conclusion: Continuity $\Rightarrow$ $\varphi$ identity map mod any totally split prime

$$\varphi(x + y) = \varphi(x) + \varphi(y) \mod p$$

tot split primes of arbitrary large norm $\Rightarrow$ $\varphi$ additive

Then Conclusion of (2) $\Rightarrow$ (1):

- Have isomorphism of semigroups of totally positive integers (additive and multiplicative)
- $\mathcal{O}_K$ has $\mathbb{Z}$-basis of totally positive elements
- Then obtain $\varphi : \mathcal{O}_K \sim \rightarrow \mathcal{O}_L \Rightarrow K \simeq L$ field isomorphism
First Step of (3) $\Rightarrow$ (2): identify $J^+_K$ and $J^+_L$ compatibly with Artin map.

Method: Fourier analysis on Number Fields

- Observation: matching of zeta functions, so know same number of primes $p$ in $K$ and $q$ in $L$ over the same rational prime $p$ with inertia degree $f$.

- Need to find a way to match them compatible with the Artin map: $p \mapsto q$ so that $\psi(\chi)(\theta_L(q)) = \chi(\theta_K(p))$ for all characters $\chi$ with conductor coprime to $p$.

- Need to show this can be done with a bijection between primes of $K$ and $L$.

- Idea: use a combination of $L$-series as counting function for number of such $q$. 
**L-series and counting functions**

- **Fix a finite quotient** $G^{ab}_{K} \rightarrow G$
- **Set** $b_{K,G,n}(\gamma) := \#B_{K,G,n}(\gamma)$ cardinality of set
  
  $B_{K,G,n}(\gamma) = \{n \in J_{K}^+: N_{K}(n) = n \text{ and } \pi_{G}(\vartheta_{K}(n)) = \pi_{G}(\gamma)\}$

- Then use known fact that
  
  $$
  \sum_{n \in J_{K}^+: N_{K}(n)} \left( \sum_{\hat{G}} \chi(\pi_{G}(\gamma)^{-1})\chi(\vartheta_{K}(n)) \right) = b_{K,G,n}(\gamma).
  $$
Identity of $L$-functions gives, for fixed norm $n$,

$$
\sum_{n \in J^+_K, \gamma \in \hat{G}} \chi(\pi_G(\gamma)^{-1}) \chi(\vartheta_K(n)) = \sum_{m \in J^+_L, \gamma \in \hat{G}} \chi(\pi_G(\gamma)^{-1}) \psi(\chi)(\vartheta_L(m))
$$

Using isomorphism $\psi : G_{ab}^K \to G_{ab}^L$ preserving $G_{ab}^K$, $n = \text{Gal of max abelian ext unramified above prime divisors of } n$, right-hand-side above gives, for $(\psi^{-1})^*(G) = G'$,

$$
\sum_{\hat{G}'} \psi^{-1}(\eta)(\pi_G(\gamma)^{-1}) \eta(\pi_{G'}(\vartheta_L(m)))
$$

$m$ coprime to $f_\eta$; character on $\hat{G}'$

$\Xi_m : \eta \mapsto \psi^{-1}(\eta)(\pi_G(\gamma)^{-1}) \eta(\pi_{G'}(\vartheta_L(m)))$ so that

$$
\sum_{\hat{G}'} \psi^{-1}(\eta)(\pi_G(\gamma)^{-1}) \eta(\pi_{G'}(\vartheta_L(m))) = \begin{cases} 
|G'| & \text{if } \Xi_m \equiv 1; \\
0 & \text{otherwise.}
\end{cases}
$$
• $\Xi_m \equiv 1$ gives

$$\eta(\pi_{G'}(\vartheta_{\mathbb{L}}(m))) = \psi^{-1}(\eta)(\pi_G(\gamma)) \text{ for all } \eta \in G'$$

so that $\pi_{G'}(\vartheta_{\mathbb{L}}(m)) = \pi_{G'}((\psi^{-1})^*(\gamma))$.

• So from identity of $L$-function get counting identity

$$b_{K,G,n}(\gamma) = b_{\mathbb{L},(\psi^{-1})^*G,n}((\psi^{-1})^*(\gamma))$$

• $G_{K,n}^{ab}$ as inverse limit over finite quotients: same cardinality of

$$S_1 = \{ n \in J_{K}^{+} : N_K(n) = n, \pi_{G_{K,n}^{ab}}(\vartheta_{K}(n)) = \pi_{G_{K,n}^{ab}}(\gamma) \}$$

$$S_2 = \{ m \in J_{\mathbb{L}}^{+} : N_{\mathbb{L}}(m) = n, \pi_{G_{L,n}^{ab}}(\vartheta_{\mathbb{L}}(m)) = \pi_{G_{L,n}^{ab}}((\psi^{-1})^*(\gamma)) \}$$
Artin map $\vartheta_K : J_K^+ \to G_{K,n}^{ab}$ injective on ideals dividing $n$:

- Get $\#S_1 = 1$.
- $\#S_2 = 1$ gives unique ideal $m \in J_L^+$ with $N_L(m) = N_K(n)$ and
  \[ \pi_{G_{K,n}^{ab}}(\vartheta_L(m)) = \pi_{G_{L,n}^{ab}}((\psi^{-1})^*(\vartheta_K(n))) \]

Get multiplicative map $\Psi(n) := m$, isomorphism of $J_K^+$ and $J_L^+$
compatible with Artin map.
Second Step of (3) ⇒ (2): matching $C(X_K)$ and $C(X_L)$ compatibly with $J^+_K$ and $J^+_L$ actions

- Idea: extend identification $\psi : C(G_{ab}^K) \sim \to C(G_{ab}^L)$ from $G_{ab}^K$ to $G_{ab}^K \times \hat{\mathcal{O}}_K^n \hat{\mathcal{O}}_K^n$.

- Using $X_{K,n} := G_{ab}^K \times \hat{\mathcal{O}}_K^n \hat{\mathcal{O}}_K^n$ and $J^+_K$ gen by prime ideals dividing $n$.

- Know algebra $C(X_{K,n})$ is generated by the functions

$$f_{\chi,n} : \gamma \mapsto \chi(\nu_K(n))\chi(\gamma), \quad \chi \in \hat{G}_{ab}^K, \quad n \in J^+_K,n$$

- Map $\psi_n : C(X_{K,n}) \rightarrow C(X_{L,n})$ by

$$f_{n,\chi} \mapsto f_{\psi(n),\psi(\chi)}$$

well defined by matching ramification and conductors.
• Direct limit $\psi = \lim_{n \to} \psi_n : C(X_K) \simto C(X_L)$

• Check algebra homomorphism: from compatibility with Artin map

$$\psi(f_{\chi,n})(\gamma') = f_{\psi(\chi),\psi(n)}(\gamma') = \psi(\chi)(\vartheta_L(\Psi(n))\psi(\chi)(\gamma')$$

$$\psi(f_{\chi,n})(\gamma') = \chi(\vartheta_K(n))\chi(\psi^*(\gamma')) = (\psi^{-1})^* f_{\chi,n}$$

$$\psi(f_n,\chi \cdot f_{n',\chi'}) = (\psi^{-1})^* (f_n,\chi \cdot f_{n',\chi'}) =$$

So get multiplicative map:

$$(\psi^{-1})^* (f_n,\chi) \cdot (\psi^{-1})^* (f_{n',\chi'}) = \psi(f_n,\chi) \cdot \psi(f_{n',\chi'})$$

• Compatibility with time evolution since $N_L(\Psi(n)) = N_K(n)$

This completes all implications of main Theorem $\square$
What then?

- Function fields $K = \mathbb{F}_{p^m}(C)$, curve $C$ over finite field
- Analogies between number fields and function fields
- Same type of QSM systems
- Sneak Preview: purely NT proof seems not to work for function fields ... but NCG proof does!

... coming soon to a lecture hall near you

Thank you!