

- X smooth compact mfd $\dim = n$
 local coordinates $(x_1, \dots, x_n)$
 meaning: open sets $U \subset X$ s.t.
 $U \cong$ open set in $\mathbb{R}^n$ (coordinates $x_1, \dots, x_n$)
 homeom
 and changes of coord's on $U \cap V$ are
 diffeomorphisms

- de Rham cohomology of X $H_{dR}^k(X, \mathbb{R})$
 $\Omega^k(X)$ k -differential forms

$$dx_I = dx_{i_1} \wedge \dots \wedge dx_{i_k} \quad i_1 < \dots < i_k$$

(wedge prod $dx \wedge dy = -dy \wedge dx$)

$$\omega = \sum_I f_I dx_I \quad f_I \in \mathcal{C}^\infty(M)$$

de Rham differential $d: \Omega^k(M) \rightarrow \Omega^{k+1}(M)$

$$d\omega = \sum_I \sum_{i=1}^n \frac{\partial f_I}{\partial x_i} dx_i \wedge dx_I$$

main property: $d^2 = 0$

so consider closed forms: $\text{Ker}(d) \subset \Omega^k(M)$
 (cycles)

modulo coboundaries (exact forms)

$$\text{Ker}(d) / \text{Image}(d) =: H_{dR}^k(M, \mathbb{R})$$

- Singular homology

n -simplex : $\begin{array}{c} 0 \\ \hline 1 \end{array} \quad \triangle_2 \quad \triangle_3 \quad \dots \text{ etc}$

(complete graph on $n+1$ vertices as 1-skeleton, fill all triangles, all tetrahedra, etc. inductively)

$$\Delta_n = \{ (t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_i \geq 0 \forall i=0, \dots, n \}$$

$[v_0, \dots, v_n] := n$ -simplex on vertices $\{v_i\}_{i=0}^n$

boundary $\partial [v_0, \dots, v_n] := \sum_{k=0}^n (-1)^k [v_0, \dots, \underbrace{v_{i-1}, v_{i+1}, \dots, v_n}_{(n-1)\text{-faces of simplex}}]$

remove v_i
↓

main property: $\partial^2 = 0$

X topological space (or smooth mfd)

continuous functions $\sigma: \Delta_n \rightarrow X$ n -chains

$C_n(X) =$ abelian group generated by all $\{\sigma: \Delta_n \rightarrow X\}$

$$\partial: C_n(X) \rightarrow C_{n-1}(X)$$

$$\begin{aligned} \sigma(\Delta_n) &= [v_0, \dots, v_n] \\ \partial\sigma &= \sum_{k=0}^n (-1)^k \sigma|_{\substack{i=k \\ (n-1)\text{ face} \\ \text{of } \Delta_n}} \end{aligned}$$

n -cycles: $\ker(\partial) \subset C_n(X)$

modulo $(n+1)$ -boundaries

$$\ker(\partial) / \text{Image}(\partial) = H_k(X, \mathbb{Z})$$

get $H_k(X, \mathbb{Q}) = H_k(X, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}$ $H_k(X, \mathbb{R})$, $H_k(X, \mathbb{C})$, ...

- homology & cohomology pairing X smooth compact mfd
by integration

$$\langle \sigma, \omega \rangle = \int_{\sigma} \omega$$

$\uparrow$ $\uparrow$
 $\sigma \in C_k(X)$ $\omega \in \Omega^k(X)$

invariant under $\sigma \mapsto \sigma + \partial\tau$
 $\omega \mapsto \omega + d\eta$

also integration linear in both
 integrand and integration domain
 so pairing of vector spaces

$H_k(X, \mathbb{R}) \times \int_{dR}^k(X, \mathbb{R}) \rightarrow \mathbb{R}$
 is a perfect pairing (de Rham theorem)

- complex manifold X w/ complex coordinates
 $(z_1, \dots, z_n)$ $\dim_{\mathbb{C}} X = n$ $\dim_{\mathbb{R}} X = 2n$

$z_i = x_i + iy_i$ $(x_1, \dots, x_n, y_1, \dots, y_n)$ real coord's

$$\Omega^{1,0}(X) = \left\{ \sum_i f_i dz_i \right\} \quad \Omega^{0,1} = \left\{ \sum_i f_i d\bar{z}_i \right\}$$

$$\Omega^{p,q} = \underbrace{\Omega^{1,0} \wedge \dots \wedge \Omega^{1,0}}_{p\text{-times}} \wedge \underbrace{\Omega^{0,1} \wedge \dots \wedge \Omega^{0,1}}_{q\text{-times}}$$

differential forms $\omega \in \Omega^k(X)$ (in real coord's)
 w/ complex coeff's

$$\Omega^k(X) = \bigoplus_{p+q=k} \Omega^{p,q}(X)$$

deRham differential $d: \Omega^k \rightarrow \Omega^{k+1}$

$\Rightarrow$ Dolbeault differentials

$$\partial = \pi^{p+1, q} \circ d: \Omega^{p, q} \rightarrow \Omega^{p+1, q}$$

$$d = \partial + \bar{\partial}$$

$$\bar{\partial} = \pi^{p, q+1} \circ d: \Omega^{p, q} \rightarrow \Omega^{p, q+1}$$

$$\partial^2 = \bar{\partial}^2 = \partial\bar{\partial} + \bar{\partial}\partial = 0$$

$$H^{p, q}(X, \mathbb{C}) = \frac{\text{Ker}(\bar{\partial}: \Omega^{p, q} \rightarrow \Omega^{p, q+1})}{\text{Im}(\bar{\partial}: \Omega^{p, q+1} \rightarrow \Omega^{p, q})}$$

(Note $\bar{\partial}w = 0$ generalizes Cauchy-Riemann
 $\frac{\partial}{\partial \bar{z}} f = 0$ for holom. functions)

holomorphic p -forms $w \in \Omega^{p, 0}(X)$ $\bar{\partial}w = 0$

- $X \subset \mathbb{CP}^N$ complex mfd & smooth proj. alg. var.
 round metric (Fubini-Study) on $\mathbb{CP}^N$
 induces Riemannian metric on X
 compatible w/ complex structure (Kähler)

$$H_{dR}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p, q}(X, \mathbb{C}) \quad \text{Hodge decomp. of deRham cohomology}$$

$$b_k(X) = \text{rk } H_k(X, \mathbb{Z}) = \dim_{\mathbb{R}} H_{dR}^k(X, \mathbb{R}) \quad \text{Betti numbers}$$

$$h^{p, q}(X) = \dim H^{p, q}(X) \quad \text{Hodge numbers}$$

symmetry: $h^{p, q} = h^{q, p} \quad h^{p, q} = h^{n-p, n-q}$

X n -dim
 $\mathbb{C}$ -variety
(smooth proj.)

$V \subset X$ closed subvariety
of codim p

Poincaré duality

$$[V] \in H_{2(n-p)}(X, \mathbb{Z}) \cong H^{2p}(X, \mathbb{Z}) \ni \gamma_V$$

$$\int_V \omega = \langle \gamma_V, \omega \rangle = \int_X \gamma_V \wedge \omega$$

viewed in $H^{2p}(X, \mathbb{C})$ the class
 γ_V is in the middle piece of Hodge
decomposition: $\gamma_V \in H^{p,p}(X, \mathbb{C})$

(this because in local coord's $(z_1, \dots, z_n)$
can describe embedding $V \hookrightarrow X$ if $\dim_{\mathbb{C}} V = k$
as $z_{k+1} = \dots = z_n = 0$
if $\deg \omega > k$ $\int_V \omega$ has terms in $d\bar{z}_i$ $i \geq k+1$
that vanish
also any terms w/ $d\bar{z}_i$ $i \geq k+1$
so only nonvanishing terms if $\omega \in \Omega^{k,k}$)

Hodge conjecture: all classes in $H^{2k}(X, \mathbb{Q}) \cap H^{k,k}(X, \mathbb{C})$
are $\mathbb{Q}$ -linear combinations of
classes γ_{V_i} of subvarieties $V_i \subset X$ codim k
i.e. alg. cycles