1) Hopf algebra of Feynman graphs (Connes-Kreimer)

$\mathcal{H}$ as algebra: polynomial alg. on (1PI) Feyman graphs of given scalar QFT
(prod = disjoint union of graphs)

coalgebra

$$\Delta(X) = X\otimes 1 + 1\otimes X + \sum X'\otimes X''$$ X', X'' lower degrees

w/ $\mathcal{H} = \bigoplus_{n\geq 0} \mathcal{H}_n$ $\mathcal{H}_0 = \mathbb{C}$ $n = \#\text{loops}$ (or # internal edges)

graded

$$\Delta(\Gamma) = \Gamma\otimes 1 + 1\otimes\Gamma + \sum_{\gamma} \gamma\otimes \Gamma/\gamma$$ is coassociative

"connected"

for graded connected Hopf alg: antipode defined inductively

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X\otimes 1 + 1\otimes X + \sum X'\otimes X''$

Observations: graded connected commutative Hopf $\mathcal{H}/_K$

$\exists\ \mathcal{H}_i \subset \mathcal{H}$ $i\in I$ part. ordered set s.t.

$\mathcal{H}_i$ fin. gen. alg's$/_K$ s.t.

- $\Delta(\mathcal{H}_i) \subset \mathcal{H}_i\otimes\mathcal{H}_i$ $\forall i\in I$
- $S(\mathcal{H}_i) \subset \mathcal{H}_i$ $\forall i\in I$
- $\forall i,j\in I$ $\exists k\in I$ s.t. $\mathcal{H}_i \cup \mathcal{H}_j \subset \mathcal{H}_k$ and $\mathcal{H} = \bigcup_i \mathcal{H}_i$

and affine group scheme dual to $\mathcal{H}$ is proj. lim

$$G = \varprojlim_{i\in I} G_i$$

G_i aff grp scheme dual to $\mathcal{H}_i$
(lin alg. groups)

2) Lie algebra of an affine grp. scheme G

$\mathfrak{g} = \mathrm{Lie}\, G$ functor from cat $\mathcal{A}$ commutative k-algebras to cat. of Lie algebras$/_k$

$A \mapsto \mathfrak{g}(A)$

Lie alg. of linear maps $L : \mathcal{H} \to A$ satisfying

$$L(XY) = L(X)\,\varepsilon(Y) + \varepsilon(X)\,L(Y)$$

w/ $\varepsilon : \mathcal{H} \to k$ the counit of $\mathcal{H}$

& Lie bracket

$$[L_1, L_2](X) := \langle L_1 \otimes L_2 - L_2 \otimes L_1, \Delta(X) \rangle$$

Note: in general knowing $\mathrm{Lie}(G) = \mathfrak{g}$ not enough to reconstruct G

e.g. affine grp. schemes $\mathbb{G}_m$ and $\mathbb{G}_a$ have same Lie algebra

but: (Milnor–Moore theorem)

if $\mathcal{H}$ commutative Hopf alg.$/_k$ $\mathrm{char}(k) = 0$ graded & connected w/ graded pieces fin dim k-vect. spaces

then
- take dual Hopf algebra $\mathcal{H}^\vee$ (inverting all arrows, exchanging multipl & comultipl.)
- take primitive elements in $\mathcal{H}^\vee$ (i.e. elements w/ $\Delta(X) = X \otimes 1 + 1 \otimes X$)
- prim elements form a Lie algebra $\mathcal{L}$
- take universal enveloping algebra $U(\mathcal{L})$

$\mathcal{H} = U(\mathcal{L})^\vee \Leftarrow$

3) Note: $\mathcal{H} = \bigoplus_n \mathcal{H}_n$ has action γ of multiplicative group $\mathbb{G}_m$ by

$$u \in \mathbb{G}_m \qquad \gamma(u)(X) = u^n X \qquad X \in \mathcal{H}_n$$

if G affine grp. scheme dual to commutative graded connected $\mathcal{H}$ then have

$$G^* = G \rtimes \mathbb{G}_m$$

when $\mathrm{Lie}(G^*)$ has extra generator Z

$$[Z, X] = \gamma(X)$$

In the case of the Hopf algebra of Feynman graphs: additional structure if not just combinatorics of graphs but want to take care of assignments of external momenta to graphs "external structure" given by distributions

$$\Delta(\Gamma, \sigma) = (\Gamma, \sigma) \otimes 1 + 1 \otimes (\Gamma, \sigma) + \sum_\gamma (\gamma, \sigma_{\pi(\gamma)}) \otimes (\Gamma/\gamma, \sigma)$$

main point here is assignment of external structure to extracted subgraphs

4) Birkhoff factorization of loops

$C \subset \mathbb{P}^1(\mathbb{C})$ $\quad C = \partial\Delta$ $\quad \Delta$ = small disk centered at $z=0$ in $\mathbb{P}^1(\mathbb{C})$

$G(\mathbb{C})$ connected complex Lie group

$\gamma : C \to G(\mathbb{C})$ a loop (smooth)

a <u>Birkhoff factorization</u> for γ is a pair γ_+, γ_- of holomorphic functions

$\gamma_\pm : C_\pm \longrightarrow G(\mathbb{C})$ $\quad C_\pm$ = two components of $\mathbb{P}^1(\mathbb{C}) \setminus C$

with $\quad \gamma(z) = \gamma_-(z)^{-1} \cdot \gamma_+(z) \quad \forall z \in C$

Not all loops can be factorized this way

<u>example</u> : for $G = GL_n$

$\gamma : C \to GL_n(\mathbb{C})$ in general only has a factorization of the form

$$\gamma(z) = \gamma_-(z)^{-1} \, \lambda(z) \, \gamma_+(z)$$

w/ holomorphic $\gamma_\pm : C_\pm \to GL_n(\mathbb{C})$ and

5)
$$\lambda(z) = \begin{pmatrix} z^{k_1} & & 0 \\ & \ddots & \\ 0 & & z^{k_n} \end{pmatrix}$$ integers $k_i \in \mathbb{Z}$

<u>geometrically</u>: $\gamma_\pm$ are trivializations on open sets $C_\pm$ of a holom. vector bundle E over $\mathbb{P}^1(\mathbb{C})$ while λ is the nontrivial transition function

$$E = L_1 \oplus \cdots \oplus L_n \qquad c_1(L_i) = k_i$$

When G is affine group scheme of a commutative Hopf algebra $/\mathbb{C}$ (or $\mathbb{Q}$)

so $G(\mathbb{C}) = \mathrm{Hom}_{cAlg_{\mathbb{C}}}(\mathcal{H}, \mathbb{C})$

take $K = \mathbb{C}(\{z\}) = \mathbb{C}\{z\}[z^{-1}]$
field of convergent Laurent series
= germs of meromorphic functions at the origin

K commutative $\mathbb{C}$ algebra so
$$G(K) = \mathrm{Hom}_{cAlg_{\mathbb{C}}}(\mathcal{H}, K)$$

6)

$G(K)$ describes infinitesimal loops

i.e. loops $\gamma : \Delta^* \longrightarrow G(\mathbb{C})$

w/ Δ^* an infinitesimal punctured disk at $z=0$

because think of K as field of funct's of Δ^*

so $\mathrm{Hom}_{\mathrm{cAlg}_{\mathbb{C}}}(\mathcal{H}, K)$ is dually

maps $\Delta^* \longrightarrow G(\mathbb{C})$

those functions that extend holomorphically to zero (center of Δ^*)

$\mathcal{O} = \mathbb{C}\{z\}$ functions on Δ

$G(\mathcal{O}) = \mathrm{Hom}_{\mathrm{cAlg}_{\mathbb{C}}}(\mathcal{H}, \mathcal{O})$

also those that extend to ∞

$\mathcal{Q} = z^{-1}\mathbb{C}[z^{-1}]$ and unitization $\tilde{\mathcal{Q}} = \mathbb{C}[z^{-1}]$

$G(\tilde{\mathcal{Q}}) = \mathrm{Hom}_{\mathrm{cAlg}_{\mathbb{C}}}(\mathcal{H}, \tilde{\mathcal{Q}})$ normalized by $\gamma_-(\infty) = 1$

7)

Birkhoff factorization for infinitesimal loops $\phi \in G(K)$

$$\phi = (\phi_- \circ S) * \phi_+$$ with

$$\phi_+ \in G(\mathcal{O}) \qquad \phi_- \in G(\tilde{\mathcal{Q}})$$

and $\phi_1 * \phi_2 (X) = \langle \phi_1 \otimes \phi_2 , \Delta(X) \rangle$
product in $G(K)$

Operator T on K Laurent series
= projection onto polar part

$$T : K \to \mathcal{Q}$$

is a Rota-Baxter operator of weight -1

(K, T) Rota-Baxter algebra of weight -1

$\Rightarrow$ Birkhoff factorization always exists and can be constructed inductively

8) $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

$$\phi = (\phi_- \circ S) * \phi_+ \quad \text{with}$$

$$(*) \begin{cases} \phi_-(X) = -T\left(\phi(X) + \sum \phi_-(X')\,\phi(X'')\right) \\ \phi_+(X) = (1+T)\left(\phi(X) + \sum \phi_-(X')\,\phi(X'')\right) \end{cases}$$

BPHZ formula for renormalization in physics

$\phi(X) + \sum \phi_-(X')\,\phi(X'')$

is the Bogoliubov-Parashchuk preparation

$(*)$ inductively defined: using ϕ_- on lower degree terms

normalization $\gamma_-(\infty) = 1$

$\underline{\varepsilon} \circ \phi_- = \varepsilon \leftarrow$ counit of $\mathcal{H}$

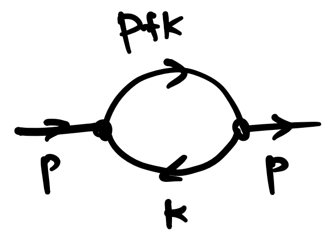
$\leftarrow$ augmentation of ring $\widehat{Q}$

fixes solution of this recursion uniquely

9) <u>Note</u>: using fact that G is dual to graded connected commut. Hopf (hence prounipotent group) so don't have problem like for Birkhoff factor. in $GL_n(\mathbb{C})$

<u>Dimensional regularization</u> of Feynman integrals and loops:

simplest example:



graph of ϕ^3 theory (w/ mass)

integral in momentum variables

$$\int \frac{1}{k^2+m^2} \frac{1}{(p+k)^2+m^2} \, d^D k$$

Schwinger parameters:

$$\frac{1}{k^2+m^2} \frac{1}{(p+k)^2+m^2} = \int_{s>0,\, t>0} e^{-s(k^2+m^2)-t((p+k)^2+m^2)} \, ds\, dt$$

quadratic form in exponential

$$-Q(k) = -\lambda \left((k+xp)^2 + ((x-x^2)p^2+m^2) \right)$$

$$s=(1-x)\lambda \qquad t = x\lambda$$

10) formally change order of integration

$\Rightarrow$ Gaussian int in dim D
in variable $q = k + xp$

$$\int e^{-\lambda q^2} d^D q = \pi^{D/2} \lambda^{-D/2}$$

<u>Note</u>: right hand side continues to make sense when D not an integer: use to define left-hand-side for non-integer D

So resulting Feynman integral is

$$\int_0^1 \int_0^\infty e^{-\lambda((x-x^2)p^2+m^2)} \int e^{-\lambda q^2} d^D q \, \lambda \, d\lambda \, dx$$

$$= \pi^{D/2} \int_0^1 \int_0^\infty e^{-\lambda((x-x^2)p^2+m^2)} \lambda^{-D/2} \lambda \, d\lambda \, dx$$

$$= \pi^{D/2} \, \Gamma(2 - D/2) \int_0^1 ((x-x^2)p^2+m^2)^{D/2-2} dx$$

here have pole when $D \in 4 + 2\mathbb{N}$

but well defined as function of

$D - z \in \mathbb{C}$ $\qquad z \in \Delta$ small disk centered at $z = 0$

Regularization: div. integral $\Rightarrow$ function of reg. param. z

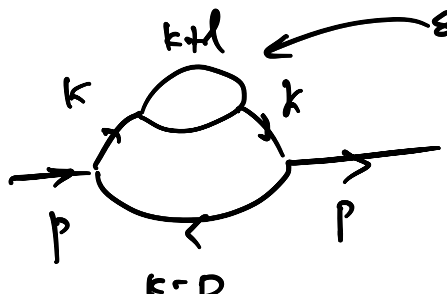
11)

in this case can eliminate the divergent part by

- projection onto polar part of Laurent series expansion in z
- subtraction of this projection
- evaluation at $z=0$ of remaining part

$\Rightarrow$ renormalized finite value

<u>But</u> problem of <u>subdivergences</u>



subgraph γ already contributes a divergence

(take $m=0$)

$$\int \frac{1}{k^4} \frac{1}{(k-p)^2} \frac{1}{(k+l)^2} \frac{1}{l^2} \, d^D l \, d^D k$$

as before have from

$$\int \frac{1}{(k+l)^2 \, l^2} \, d^D l \, d^D k \quad \text{a term} \quad \pi^{D/2} \, \Gamma(2-D/2) \int_0^1 (x-x^2)^{D/2-2} \, dx$$

$$\frac{\Gamma(D/2-1)^2}{\Gamma(D-2)}$$

12) but also get $\int (k^2)^{D/2-4} \frac{1}{(k-p)^2} d^D k$

where can use

$$x^{D/2-4} = \Gamma(4-D/2)^{-1} \int_0^\infty e^{-tx} t^{3-D/2} dt$$

changing variables $t_1 = \lambda s \quad t_2 = \lambda(1-s)$

$$t_1 k^2 + t_2 (k-p)^2 = \lambda q^2 + \lambda(s-s^2)p^2$$

$$q = k-(1-s)p$$

Schwinger param:

$$\Gamma(4-D/2)^{-1} \int e^{-t_1 k^2 - t_2 (k-p)^2} t_1^{3-D/2} dt_1 dt_2 d^D k$$

becomes

$$\Gamma(4-D/2)^{-1} \pi^{D/2} \int e^{-\lambda(s-s^2)p^2} \lambda^{3-D} s^{3-D/2} \lambda d\lambda\, ds$$

integrating in λ gives

$\Gamma(5-D)\left((s-s^2)p^2\right)^{D-5}$ and other integral

gives $\Gamma(D-4)\Gamma(D/2-1)\Gamma(3D/2-5)^{-1}$

Now possible problem: if expand in z variable for subtraction of divergence get coeff of $\frac{1}{z}$ non-polynomial in p

$$\sim p^2 \log\left(\frac{p^2}{\mu^2}\right)$$

μ energy scale for unit-less p^2/μ^2

13)

however in the divergence of smaller subgraph γ

Problem: not possible to cancel w/ "local" counterterms

$$\int \frac{1}{(k+l)^2} \frac{1}{l^2} d^D l \sim -\frac{1}{3}\pi^3 k^2 \frac{1}{z}$$

this divergence (counterterm) is local

replace l.h.s. w/ r.h.s. in previous integral computation

$$\int \frac{1}{k^4 (k-p)^2} \left(-\frac{1}{3}\pi^3 k^2 \frac{1}{z}\right) d^D k$$

then get divergence w/ local counterterm

$$-\frac{1}{3}\pi^3 p^2 \frac{1}{z} + \cdots$$

So this substitution is the BP preparation

$$\phi(\Gamma) + \sum_{\gamma} \phi_-(\gamma)\, \phi(\Gamma/\gamma)$$

Dim Regularization $U_\mu^z(\Gamma)$ in "$D-z$"

$$d^D k \Longleftrightarrow \mu^z d^{D-z} k$$

formally

14) dependence on a "mass scale" (energy scale) μ

view Dim-regularized Feynman integral $U_\mu^z(\Gamma)$ as

$\phi_\mu : \mathcal{H} \longrightarrow \mathbb{C}\{z\}[z^{-1}]$

$\phi_\mu \in \mathrm{Hom}_{\mathcal{CAlg}_{\mathbb{C}}}(\mathcal{H}, \mathbb{C}\{z\}[z^{-1}])$ i.e. dually as loop

$\gamma_\mu : \partial\Delta \longrightarrow G(\mathbb{C})$ G affine grp scheme dual to $\mathcal{H}$

infinitesimal disk Δ

Scaling of mass parameter μ

$$\theta_t(X) = e^{nt} X \quad X \in \mathcal{H}_n \quad t \in \mathbb{C}$$

extended to $\mathcal{H}^\vee$ by $\langle \theta_t(u), X\rangle = \langle u, \theta_t(X)\rangle$

then

$$\gamma_{e^t\mu}(z) = \theta_{tz}(\gamma_\mu(z))$$

because if Γ loop number $b_1(\Gamma) = L$ then in dim. reg. acquire factor μ^{zL}

15)

when performing Birkhoff factorization

$$\gamma_\mu(z) = \gamma_{\mu -}(z)^{-1}\,\gamma_{\mu +}(z)$$

have $\dfrac{\partial}{\partial \mu}\,\gamma_{\mu^-}(z) = 0$ (*)

μ-dependence for counterterms in a renormalizable theory

independence because show that

- counterterms depend polynomially on momenta p & mass param m
- only occurrences of μ dependent terms would be powers of $\log\mu$ (as in example)
- dim analysis then gives no dependence on μ

expansion in z w/ μ-dependence in the form μ^{zL}

Residue: given loop $\gamma_\mu(z) \in \overbrace{L(G(\mathbb{C}),\mu)}^{\text{space of } G(\mathbb{C}) \text{ valued loops w/ (*)}}$

$$\operatorname{Res}_{z=0}\gamma_\mu := -\left(\frac{\partial}{\partial u}\,\gamma_-\left(\frac{1}{u}\right)\right)_{u=0}$$

16)

residue indep of μ since γ_- indep

$\beta := \underbrace{gr}_{\text{grading operator}} \operatorname{Res} \gamma_\mu$ β-function

grading operator $gr(X) = nX$ $X \in \mathcal{H}_n$

defines an element in Lie G

$\gamma_\mu + (z)$ pos. part of Birkhoff

$$\gamma_{e^t\mu} + (0) = F_t \, \gamma_{\mu +}(0)$$

with $F_t = \lim_{z \to 0} \gamma_-(z) \, \theta_{tz}(\gamma_-(z)^{-1})$

$$\frac{d}{dt} F_t \Big|_{t=0} = \beta$$

F_t = renormalization group flow

Structure of $\left\{ \begin{array}{l} \text{renormalization: counterterms } \gamma_- \\ \text{\& renormalized values } \gamma_{\mu+}(0) \end{array} \right.$

17)

- counterterms as iterated integrals
  (Gross-'t Hooft relations)
- geometric encoding of these: flat equisingular connections
- differential Galois groups and Tannakian formalism
- "universal symmetries" underlying renormalization

<u>Dyson's time ordered exponential</u>

notation $T e^{\int_a^b \alpha(t) dt}$

Lie grp H; trivial principal bundle $[a,b] \times H$ w/ left action of H on itself

connection defined by 1-form $\alpha(t) dt$ valued in Lie algebra $\mathrm{Lie}(H) = \mathfrak{g}(\mathbb{C})$

then $h(u) = T e^{\int_a^u \alpha(t) dt}$ is solution of diff. equation

$$dh(u) = h(u)\, \alpha(u)\, du \qquad h(a) = 1$$

18) these solutions can be described as iterated integrals

$\leftarrow$ topology: Chen's it. int.
op. alg.: Araki expansional

$\mathcal{L} = \mathfrak{g}(\mathbb{C})$-valued smooth function $\alpha(t)$
for $t \in [a,b]$

$$T e^{\int_a^b \alpha(t)\, dt} := 1 + \sum_{n=1}^{\infty} \int_{a \le s_1 \le \cdots \le s_n \le b} \alpha(s_1) \cdots \alpha(s_n)\, ds_1 \cdots ds_n \qquad (*)$$

where the product $\alpha(s_1) \cdots \alpha(s_n)$ is in
$\mathcal{H}^\vee = U(\mathcal{L})$ univ. env. alg. of Lie alg
$1 \in \mathcal{H}^\vee$ is unit (i.e. counit ε of $\mathcal{H}$)

<u>Properties</u> :

- when paired w/ elements $X \in \mathcal{H}$ the sum in $(*)$ is finite
  (i.e. if $X \in \mathcal{H}_n$ all terms $\langle \alpha(s_1) \cdots \alpha(s_m), X \rangle$ $m > n$ vanish)
- $(*)$ defines an element in $G(\mathbb{C}) = \mathrm{Hom}_{\text{Alg}}(\mathcal{H}, \mathbb{C})$
- $(*)$ is $g(b)$ value of unique solution $g(t) \in G(\mathbb{C})$ of $dg(t) = g(t)\, \alpha(t)\, dt$ w/ $g(a) = 1$
- iterated integral is multipl. under concatenation of paths

19) $$T e^{\int_a^c \alpha(t)\,dt} = T e^{\int_a^b \alpha(t)\,dt} \, T e^{\int_b^c \alpha(t)\,dt}$$
for $a < b < c$

- under group homomorphisms
$$\rho : G(\mathbb{C}) \longrightarrow H$$
$$\rho\left(T e^{\int_a^b \alpha(t)\,dt}\right)$$ is sol'n for connection $\rho(\alpha(t))\,dt$

- inverse in grp $G(\mathbb{C})$:
$$\left(T e^{\int_a^b \alpha(t)\,dt}\right)^{-1} = T' e^{-\int_a^b \alpha(t)\,dt}$$
where time reversal
$$T' e^{\int_a^b \alpha(t)\,dt} = 1 + \sum_n \int_{a \le s_1 \le \cdots \le s_n \le b} \alpha(s_n) \cdots \alpha(s_1)\, ds_1 \cdots ds_n$$

20) $\Omega \subset \mathbb{R}^2$ open domain

(s,t) coord's

$$\omega = \alpha(s,t)\,ds + \eta(s,t)\,dt \qquad \mathfrak{g}(\mathbb{C})\text{-valued connection}$$

flat if $\partial_s \eta - \partial_t \alpha + [\alpha, \eta] = 0$

$\gamma : [0,1] \to \Omega$ path (contin)

- then time ordered exponential

$T e^{\int_0^1 \gamma^* \omega}$ only depends on homotopy class $[\gamma]$ of path

(using $G(\mathbb{C})$ projective limit

$G(\mathbb{C}) = \varprojlim_i G_i(\mathbb{C})$ of Lie groups

and for given Lie grp $G_i(\mathbb{C})$ additivity on paths & inverse properties)

<u>Differential</u> field: (K, δ) w/ field of constants $= \mathbb{C}$ $\operatorname{Ker}(\delta)$

$\delta(f+g) = \delta(f) + \delta(g) \qquad \delta(fg) = \delta(f)g + f\delta(g)$

21) If (K, δ) differential field

$G(K) = \mathrm{Hom}_{\mathrm{eAlg}}(\mathcal{H}, K)$ has a

logarithmic derivative

$$D(g) := g^{-1}\,\delta(g) \quad \forall g \in G(K)$$

$g' := \delta(g) : \mathcal{H} \to K$ linear map

$$g'(X) = \delta(g(X))$$

---

as an element in $\mathfrak{g}(K)$

$$D(g) : \mathcal{H} \to K$$

$$\langle D(g), X \rangle = g^{-1} * g'(X) =$$

$$\langle g^{-1} \otimes g', \Delta X \rangle \qquad \text{vista}$$

$$\langle D(g), XY \rangle = \langle D(g), X \rangle\, \varepsilon(Y) + \varepsilon(X) \langle D(g), Y \rangle$$

22) in particular $K = \mathbb{C}((z)) = \mathbb{C}[[z]][z^{-1}]$

with $\delta(f) = \frac{d}{dz} f$ or $K = \mathbb{C}(\{z\})$
conv. Laurent

<u>Monodromy</u> $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ $\mathcal{H}_0 = \mathbb{C}$
commut.
G dual affine grp scheme graded conn. Hopf

given $\omega \in \mathfrak{g}(K)$ $K = \mathbb{C}(\{z\})$

$\exists$ monodromy representation
$M_i(\omega) : \mathbb{Z} \longrightarrow G_i(\mathbb{C})$

for $G = \varprojlim_i G_i$

and condition $M(\omega) = 1$
(trivial monodromy)
well defined in limit $G(\mathbb{C})$

in fact:
$\omega \in \mathfrak{g}(K)$ defines $\omega_i \in \mathfrak{g}_i(K) = \mathrm{Lie}\, G_i\,(K)$
by restriction $\omega_i = \omega|_{\mathcal{H}_i \subset \mathcal{H}}$

23)

each $\mathcal{H}_i$ fin. generated

so w_i determined by fin. many elements in K (images of gen. of $\mathcal{H}_i$)

$\Rightarrow$ radius $\rho_i > 0$ of convergence of all these in disk Δ_i^*

$\Rightarrow$ flat $\mathfrak{g}_i(\mathbb{C})$-valued connections $\omega_i := w_i(z)dz$ on Δ_i^*

and for path $\gamma : [0,1] \to \Delta_i^*$

$$M_i(\omega)(\gamma) := T e^{\int_0^1 \gamma^* \omega_i} \in G_i(\mathbb{C})$$

this depends on htopy class $[\gamma]$

$\gamma(0) = \gamma(1) = z_i \in \Delta_i^*$

$\Rightarrow M_i(\omega) : \pi_1(\Delta_i^*, z_i) \to G_i(\mathbb{C})$

when passing to limit: changes of basepoint $z_i \to 0$ as radii $\rho_i \searrow 0$

but change of basept is conjugation of $\pi_1(\Delta_i^*, z_i)$ by path

condition of $M_i(\omega) = 1$ invariant under conjug.

24)

Non-trivial monodromy is an <u>obstruction</u> to existence of solutions to equation

$$D(g) = \omega \qquad g \in \mathfrak{g}(K)$$

<u>Example</u>: $\mathbb{G}_a$ additive group

$\mathbb{G}_a(K) = K \qquad \delta(f) = \frac{d}{dz} f \quad K = \mathbb{C}(\{z\})$

$D(f) = \omega$ non-trivial residue of ω is obstruction

For $\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n \quad \mathcal{H}_0 = \mathbb{C}$ graded connected commut. Hopf w/ G affine grp scheme

if $\omega \in \mathfrak{g}(K)$ has $M(\omega) = 1$ trivial monodromy then $\exists$ solution to

$$D(g) = \omega$$

25) with $G = \varprojlim_i G_i$

$$g_i(z) = T e^{\int_{z_i}^{z} \omega_i}$$ indep of path as trivial monodr.

has property that $h_i(z) = \langle g_i(z), X \rangle$
for $X \in \mathcal{H}_i$
is convergent Laurent series in Δ_i^*
so $h_i \in K$
so $g_i \in G_i(K)$

and satisfies $D(g_i) = \omega_i$

two solutions have $D(gh^{-1}) = 0$
field of constants is $\mathbb{C}$
so scale factor $g = a \cdot h$

so system of $g_i \in G_i(K)$ w/ factors in $G_i(\mathbb{C})$ so that
compat. under projections

<u>Note</u> graded connected $\mathcal{H}$ needed:

for $G = \mathbb{G}_m$ $\quad \mathbb{G}_m(K) = K^*$

$D(f) = f^{-1}\dfrac{df}{dz}$ $\quad$ equation $D(f) = \omega$

w/ $\omega = \dfrac{1}{z^2} \in K$ $\quad$ has solution $f(z) = e^{-1/z}$
but not a solution in K^*

Gross – 't Hooft relations

counterterms as iterated integrals

$$\gamma_\mu(z) \in L(G(\mathbb{C}), \mu)$$

$\gamma_-(z)$ = negative part of Birkhoff factorization (counterterms)

expansion as series

$$\gamma_-(z) = 1 + \sum_{n=1}^{\infty} \frac{d_n}{z^n} \quad \begin{cases} d_n \in \mathcal{H}^\vee \\ Y(d_{n+1}) = d_n \frac{d}{dt}F_t\Big|_{t=0} \\ Y(d_1) = \underbrace{\frac{d}{dt}F_t\Big|_{t=0}}_{\beta} \end{cases}$$

then get that

$$d_1 = \int_0^\infty \theta_{-s}(\beta)\, ds \quad \text{and}$$

$$d_n = \int_{s_1 \geq s_2 \geq \cdots \geq s_n \geq 0} \theta_{-s_1}(\beta)\, \theta_{-s_2}(\beta) \cdots \theta_{-s_n}(\beta)\, ds_1 \cdots ds_n$$

so that

$$\gamma_-(z) = T e^{-\frac{1}{z}\int_0^\infty \theta_{-t}(\beta)\, dt}$$

- writing as time ordered exponential means can describe as solutions of a diff. equation

- $L(G(\mathbb{C}), \mu)$ loops $\gamma_\mu(z)$ satisfying

$$(*) \begin{cases} \gamma_{e^t \mu}(z) = \theta_{tz}(\gamma_\mu(z)) \\ \frac{\partial}{\partial \mu} \gamma_-(z) = 0 \end{cases}$$

give a more geometric characterization of these data

- first observation: given a loop $\gamma_{reg}(z)$ regular at $z=0$ can form a $\gamma_\mu(z)$ satisfying $(*)$ by taking

$$\gamma_\mu(z) = \theta_{z \log \mu}(\gamma_{reg}(z))$$

remaining data: an element $\beta \in \mathfrak{g} = \mathrm{Lie}\, G$

in fact get characterization of $L(G(\mathbb{C}), \mu)$:

28) all loops $\gamma_\mu(z) \in L(G(\mathbb{C}), \mu)$ have the form

$$\gamma_\mu(z) = T e^{-\frac{1}{z}\int_0^{-z\log\mu} \underline{\theta}_t(\beta)\, dt} \cdot \theta_{z\log\mu}(\gamma_{reg}(z))$$

$\beta \in \mathfrak{g}(\mathbb{C})$ $\quad \gamma_{reg}$ regular at $z=0$

w/ Birkhoff factorization

$$\begin{cases} \gamma_{\mu+}(z) = T e^{-\frac{1}{z}\int_0^{-z\log\mu} \theta_{-t}(\beta)\, dt} \theta_{z\log\mu}(\gamma_{reg}(z)) \\ \gamma_-(z) = T e^{-\frac{1}{z}\int_0^{\infty} \theta_{-t}(\beta)\, dt} \end{cases}$$

---

in fact suppose $\gamma_\mu(z) \in L(G(\mathbb{C}), \mu)$

and $\gamma_\mu(z) = \gamma_-(z)^{-1}\gamma_{\mu+}(z)$

take $\quad \alpha_\mu(z) := \theta_{z\log\mu}(\gamma_-(z)^{-1})$

constructed so that scaling property satisfied

$$\alpha_{e^s\mu}(z) = \theta_{sz}(\alpha_\mu(z))$$

then take $\quad \alpha_\mu(z)^{-1}\gamma_\mu(z) \quad$ still scaling property

 also $\alpha_\mu(z)^{-1}\gamma_\mu(z)$ regular at $z=0$
(see for $\mu=1$ & then by scaling for all μ)

so $$\alpha_\mu(z)^{-1}\gamma_\mu(z) = \theta_{z\log\mu}(\gamma_{reg}(z))$$

then note that can write

$$\alpha_\mu(z)^{-1} = T e^{-\frac{1}{z}\int_{-z\log\mu}^{\infty} \theta_{-t}(\beta)\,dt}$$

using Gross-'t Hooft expr. for $\gamma_-(z)$

(shift in integral because

$$d_n = \int_{s_1 \geq s_2 \cdots \geq s_n \geq 0} \theta_{-s_1}(\beta)\cdots\theta_{-s_n}(\beta)\,ds_1\cdots ds_n$$

and $\theta_{z\log\mu}(d_n)$ shifts from 0 to $-z\log\mu$)

- conversely given choice of β, γ_{reg} get $\gamma_\mu(z) \in L(G(\mathbb{C}),\mu)$

Can then characterise $L(G(\mathbb{C}),\mu)$ in terms of a class of differential systems (of flat singular connections)