

QFT II Determinant hypersurfaces (part II)

①

Motivation: in $K_0(V_{\mathbb{Z}})$

$$[A^{l^2}, \hat{S}_l] = l^{\binom{l}{2}} \prod_{i=1}^l (l^i - 1)$$

$$[IP^{l^2-1}, D_l] = l^{\binom{l}{2}} \prod_{i=2}^l (l^i - 1)$$

Parameterize locus of matrices of $rk = l$:

$E = l$ -dim vector space

- $W_1 = E \times \{0\}$ • first row of M must be non-zero vector

- $E_1 = E \times W_1$ trivial vector bundle $S_1 := L_1 \subseteq E_1$ line bundle
 $\downarrow$
 W_1 fiber over $v \in W_1$ = line spanned by v_1

- $W_2 \subseteq E_1$ given by complement $E_1 \setminus L_1$

• second row of M must be vector v_2 that is non-zero mod v_1

- W_2 w/ trivial vector bundle $E_2 = E \times W_2$ and two line bundles
 $\downarrow$
 W_2 • pullback of L_1

- L_1 & L_2 span rank 2 subbundle S_2 of E_2 • L_2 w/ fiber over v_2 = line spanned by v_2

- $W_3 \subseteq E_2$ complement $E_2 \setminus S_2$

etc.

• third row of M must be vector non-zero mod v_1 and mod v_2

So inductively construct W_k with k line bundles $L_1, \dots, L_k$ spanning rk bundle $S_k \subseteq E_k = E \times W_k$

$$W_{k+1} = E_k \setminus S_k$$

$E_{k+1} = E \times W_{k+1}$ $L_1, \dots, L_k$ pullbacks; L_{k+1} fiber over v_{k+1} is line spanned by v_{k+1}

$L_1, \dots, L_{k+1}$ span S_{k+1} rk $k+1$ in E_{k+1} ; stop at l -th step: S_l has rk l
 $E_l \setminus S_l = \emptyset$

(2) W_k constructed this way is $W_k \cong A^{l^2} \hat{\Delta}_k$

• The bundle S_k over W_k is trivial for all $1 \leq k \leq l$

because pts of $W_k \Leftrightarrow k$ -tuples $v_1, \dots, v_k$
spanning $S_k \subseteq E_k = K^l \times W_k$

hence map $K^k \times W_k \xrightarrow{\alpha} S_k$

$((c_1, \dots, c_k), (v_1, \dots, v_k)) \mapsto c_1 v_1 + \dots + c_k v_k$ is isom.

First look at motives in triang. category of mixed motives

— Recall $\mathcal{D}$ triang. cat. $\mathcal{D}' \subset \mathcal{D}$ full subcat.

— $\mathcal{D}'$ triang. subcat. iff 1) invariant under shift T
2) for any distinguished triangle

$$A \rightarrow B \rightarrow C \rightarrow A[1] \text{ of } \mathcal{D}$$

if A & B are in $\mathcal{D}'$ then $C \cong C'$ with $C' \in \mathcal{D}'$
isom.

— $\mathcal{D}'$ ^{full} triang. subcat. is thick if closed under direct sums

$DM_K =$ Verdsky category of mixed motives over K (triang. category)

$DMT_K =$ triang. cat of mixed Tate motives / K

full triangulated thick subcat. of DM_K gen. by Tate objects $\mathbb{Q}(n)$

[If $K = \#$ field $\ni$ t-str. & heart = abelian cat. of mixed Tate motives MT_K]

(3)

Determinant hypers. complement
 $A^{l^2} \hat{\mathcal{Q}}_l$ defines object in DMT_K (in partic. $DMT_{\mathbb{Q}}$)
 cat. of mixed Tate motives

Known: over $\mathbb{K}$ w/ $\text{char}(\mathbb{K}) = 0$ $Y \subset X$ closed embedding
 gives distinguished triangle in DM_K

$$m(Y) \rightarrow m(X) \rightarrow m(X \setminus Y) \rightarrow m(Y)[1]$$

also $m(X \times \mathbb{A}^1) = m(X) (-1)[2]$ in DM_K

then use previous construction of $A^{l^2} \hat{\mathcal{Q}}_l$
 and check at each step W_k of construction get
 motive in $DMT_{\mathbb{Q}}$

- W_1 = complement of a pt in affine space: $\mathbb{A}^n$
- complement of line bdl S_1 in trivial vector bdl E_1 over W_1
 $m(S_1)$ & $m(E_1)$ in $DMT_{\mathbb{Q}} \Rightarrow m(E_1 \setminus S_1)$ is
- same for other steps where $m(E_k \setminus S_k)$ is in $DMT_{\mathbb{Q}}$
 since $m(E_k)$ & $m(S_k)$ are (by $m(X \times \mathbb{A}^1)$ in $DMT_{\mathbb{Q}}$ if $m(X)$ is)
 by htpy invariance property

$$\Rightarrow m(A^{l^2} \hat{\mathcal{Q}}_l) \in DMT_{\mathbb{Q}}$$

Explicit expression for the class in the Grothendieck ring

(4) $[W_k] = (\mathbb{L}^2 - 1)(\mathbb{L}^1 - \mathbb{L})(\mathbb{L}^1 - \mathbb{L}^2) \dots (\mathbb{L}^1 - \mathbb{L}^{k-1})$

by the inductive construction of W_k

$$= \mathbb{L}^{\binom{k}{2}} (\mathbb{L}^1 - 1) \dots (\mathbb{L}^{l-k+1} - 1)$$

Then use rel. between class of complement of affine and of proj. hypersurface to get $[P^{l-1} \setminus D_l]$

Note: $[P^{l-1} \setminus D_l] = (\mathbb{L}[P^1] \cdot \pi) \cdot (\mathbb{L}^2[P^2] \cdot \pi) \dots (\mathbb{L}^{l-1}[P^{l-1}] \cdot \pi)$

$\mathbb{L} = [A^1] \quad \pi = [G_m] \quad \text{and}$

$\mathbb{L}^l [P^1] \dots [P^{l-1}]$ is class of "variety of frames"
(see below)

examples: $l=2 \quad [P^1 \setminus D_2] = \mathbb{L}^3 - \mathbb{L} \quad ; \quad [D_2] = (\mathbb{L} + 1)^2$
 $l=3 \quad = \mathbb{L}^8 - \mathbb{L}^5 - \mathbb{L}^6 + \mathbb{L}^3 \quad ; \quad [D_3] = (\mathbb{L}^3 - \mathbb{L} + 1)(\mathbb{L}^2 + \mathbb{L} + 1)^2$

Conclusion: ambient variety $A^{l^2} \hat{=} \hat{D}_l$ or $P^{l-1} \setminus D_l$ better behaved than ambient varieties $A^n \setminus X_P$ or $P^{n-1} \setminus X_P$

Relative motive: $\mathcal{M}(P^{n-1} \setminus X_P, \Sigma_n \setminus (Z_n \cap X_P))$

Now instead of $\Sigma_n = \{ \prod_e t_e = 0 \}$ will have another locus

Σ_P that pulls back under map τ to Σ_n

$\hat{\Sigma}_P =$ Union of n linear spaces of $\text{codim}=1$ in A^{l^2} that meet image of A^n under τ along divisors w/ normal crossings

• If Γ graph such that the τ_i and τ are injective

(5)

then τ_i of "internal edges" of Γ

are preimages under $\tau: \mathbb{A}^n \rightarrow \mathbb{A}^{l^2}$

of n linear spaces X_i of codim one in $\mathbb{A}^{l^2}$

the X_i form a divisor $\hat{\Sigma}_\Gamma$ with normal crossings in $\mathbb{A}^{l^2}$

Pf: — e common to l_i & l_j (with no other common edge)
 then $\pm t_e$ only (i,j) entry $\Rightarrow t_e =$ pull back
 to $\mathbb{A}^n$ of X_{ij} coord in $\mathbb{A}^{l^2}$

— e belongs to single loop l_i :

Since τ_i injective any lin. comb. of variables t_k
 of edges in l_i

= lin. comb. of coord. x_{ir} of i -th row

In fact can improve construction of $\hat{\Sigma}_\Gamma$ by
 taking a larger normal crossings divisor

that contains $\hat{\Sigma}_\Gamma$ and that same for various Γ

$\sum_{g,l} 1$ depending only on minimal genus g of
 embeddings $\Gamma \hookrightarrow S$
 and on # loops

⑥ Prop: $\hat{\Sigma}_{g,l} \subset A^{\ell^2}$

$$\hat{\Sigma}_{g,l} = X_1 \cup \dots \cup X_N \quad N = \binom{f}{2}$$

union of linear spaces of codim one in A^{ℓ^2}

s.t. if Γ has l loops & a genus g closed 2-cell embedding then $\hat{\Sigma}_{\Gamma} \subset \hat{\Sigma}_{g,l}$ a union of components with

$$\tau_{\Gamma}^{-1}(\hat{\Sigma}_{\Gamma}) = \Sigma_n \text{ in } A^n$$

Pf: $\hat{\Sigma}_{g,l}$ given explicitly by X_i $i=1, \dots, \binom{f}{2}$

with equations

$$x_{ij} = 0 \quad \text{for } 1 \leq i < j \leq f-1$$

$$x_{i1} + \dots + x_{i,f-1} = 0 \quad \text{for } 1 \leq i \leq f-1$$

with $f = l - 2g + 1$

- Recall that injectivity of map τ ensured by inj. of an $(l-2g) \times (l-2g)$ minor of $M(t)$
assume upper-left minor up to remaining variables

$\Rightarrow$ to hyperplanes by pulling back lin. spaces along this minor

(7)
 this $(f+1) \times (f+1)$ matrix has on the diagonal
 edges that make up each internal face $\partial \overline{D_i^0}$
 (w/ pos. sign)

$\Rightarrow$ pullbacks of equations given
 gives all edge variables to (w/ redundancies)

- among the components of $\sum_{g,l}^1$ select those of $\sum_p^1$
 by eliminating those equations in list
 that corresp. to zero entries of $M_p(t)$

Now want to understand the motive
 of $\sum_{g,l}^1 \setminus (\sum_{g,l}^1 \cap \hat{D}_l)$

need to construct a good stratification of
 this locus

Inclusion-exclusion:

reformulate: complement in a union of linear spaces
 as intersections

$\{Z_i\}_{i \in I}$ subcollection of varieties Z_i

set $Z_I^0 := \left(\bigcap_{i \in I} Z_i \right) \setminus \left(\bigcup_{j \notin I} Z_j \right)$

⑧ then notice that for all I

$$\bigcap_{i \in I} Z_i = \bigsqcup_{J \supseteq I} Z_J^{\circ} \quad \text{disjoint union}$$

hence get following:

Lemma:

$Z_1, \dots, Z_m$ collection of varieties

suppose know that $\bigcap_{i \in I} Z_i$ are mixed Tate

for all non-empty $I \subseteq \{1, \dots, m\}$

then $Z_1 \cup \dots \cup Z_m$ is mixed Tate

Pf show that the Z_I° are mixed Tate for all non-empty $I \subseteq \{1, \dots, m\}$

- true for $I = \{1, \dots, m\}$ by hypothesis: in this case $Z_I^{\circ} = \bigcap_i Z_i$
- suppose true for all $|I| > k$ then show true for $|I| = k$:
using distinguished triangles in Voevodsky's category

$$\text{and } Z_I^{\circ} = \left(\bigcap_{i \in I} Z_i \right) \setminus \left(\bigsqcup_{J \not\supseteq I} Z_J^{\circ} \right)$$

- then in particular get $Z_1 \cup \dots \cup Z_m = \bigsqcup_{I \neq \emptyset} Z_I^{\circ}$
is also mixed Tate

Then in case of $\hat{\Sigma}_{g,l} = X_1 \cup \dots \cup X_N$
with $\hat{\Sigma}_\Gamma \subset \hat{\Sigma}_{g,l}$ union of subcomponents as above

$\hat{\Sigma}_\Gamma \setminus (\hat{\Sigma}_l \cap \hat{\Sigma}_\Gamma)$ is mixed Tate for all Γ w/ l loops & genus g
if $(\bigcap_{i \in I} X_i) \setminus \hat{\Sigma}_l$ is mixed Tate
for all $I \subseteq \{1, \dots, N\}$
 $I \neq \emptyset$

Question becomes whether these loci are mixed Tate

Note: $\bigcap_{i \in I} X_i$ is a linear subspace of codim $|I|$ in A^{p^2}

in general intersection of linear spaces w/ det
hypersurface $\hat{\Sigma}_l$ need not be mixed Tate

e.g. $A^3 \cap \hat{\Sigma}_3 =$ cone over genus 1 curve
general not mixed Tate

So the intersections $\bigcap_{i \in I} X_i$ must have some special
property to ensure inters w/ $\hat{\Sigma}_l$ is mixed Tate

(10)

Lemma:

E fixed l -dim vector space
 $I \subseteq \{1, \dots, N\}$ gives choice of lin subspaces
 $V_1, \dots, V_\ell$ of E s.t.

$$\bigcap_{k \in I} X_k = \{ (v_1, \dots, v_\ell) \in \mathbb{A}^{l^2} \mid v_i \in V_i \}$$

with $\dim V_i \geq i-1$

and $\exists$ basis $\{e_1, \dots, e_\ell\}$ of E s.t. each V_i is
 span of subset of $\geq i-1$ vectors e_j

Pf components X_k of $\sum g_{i,l}$ are given by
 those matrices $X \in \mathbb{A}^{l^2}$ with $x_{ij} = 0$ for $1 \leq i < j \leq l-2g$
 or by $x_{i1} + \dots + x_{i, l-2g} = 0$ for $1 \leq i \leq l-2g$

so each $X_k = l$ -uples $(v_1, \dots, v_\ell)$ s.t. one v_i
 belongs to a fixed hyperplane in E

(one of hyperplanes $x_1 + \dots + x_{l-2g} = 0$; $x_2 = 0$; $\dots$; $x_{l-2g} = 0$)

Then take $V_i =$ intersection of hyperplanes corresp to X_k
 (in list) for i
 at most $l-2g-i+1$ of those

$$\Rightarrow \dim V_i \geq l - (l-2g-i+1) = 2g + i - 1 \geq i-1$$

$(e_1, \dots, e_\ell) =$ basis ^{vectors} dual to hyperplanes $(x_1 + \dots + x_{l-2g}, x_2, \dots, x_\ell)$

Varieties of frames:

(11)

$$\mathbb{F}(V_1, \dots, V_\ell) := \{ (v_1, \dots, v_\ell) \in \mathbb{A}^{l^2} \mid v_k \in V_k \} \setminus \hat{D}_\ell$$

Complement of det hypersurface in set of matrices determined by $V_1, \dots, V_\ell$

Question: ^{II} is $\mathbb{F}(V_1, \dots, V_\ell)$ mixed Tate?

This would imply $\hat{\Sigma}_p \setminus (\hat{\Sigma}_\ell \cap \hat{\Sigma}_p)$ mixed Tate for all p w/ ℓ loops & satisfying embedding condition

weaker question with same implication

Question: ^{III} $(e_1, \dots, e_\ell)$ basis of an ℓ -dim vect. space;
for $i=1, \dots, \ell$ V_i subsp. spanned by $\geq i-1$ vectors in this basis
is $\mathbb{F}(V_1, \dots, V_\ell)$ mixed Tate?

Question: ^{I',r} $\mathbb{F}_\ell(V_1, \dots, V_r) = \{ (v_1, \dots, v_r) \mid v_i \in V_i \text{ dim } \langle v_1, \dots, v_r \rangle = r \}$
is $\mathbb{F}_\ell(V_1, \dots, V_r)$ mixed Tate?

$$\mathbb{F}_\ell(V_1) = V_1 \setminus \{0\} \quad \text{ok}$$

Question $I_\ell = I'_{\ell,\ell}$

Question $I'_{\ell,r} = \text{Question } I_\ell \text{ w/ } V_{r+1} = \dots = V_\ell = E \quad \left. \vphantom{\text{Question } I_\ell} \right\} I_\ell \Leftrightarrow I'_{\ell,r}$

(12) Where difficulty? Suppose have constructed all $(V_1, \dots, V_k)$ controlling $\dim(V_{k+1} \cap \langle V_1, \dots, V_k \rangle)$ depends on positions of vectors v_i : many possibilities

Special cases:

- (1) all V_k hyperplanes $\Rightarrow F(V_1, \dots, V_k)$ mixed Tate
 because $V_{k+1} \cap \langle V_1, \dots, V_k \rangle$ only possibilities: $\begin{cases} \langle V_1, \dots, V_k \rangle \\ \text{a hyperplane in } \langle V_1, \dots, V_k \rangle \end{cases}$
- first case OK by induction (same as $E = V_k$) on $\dim E$
 - other case is complement of this: complement of MT inside MT $\Rightarrow$ also MT

(2) $V_1 \subseteq V_2 \subseteq \dots \subseteq V_r \Rightarrow F_r(V_1, \dots, V_r)$ mixed Tate

$\langle V_1, \dots, V_k \rangle \subseteq V_{k+1}$ always in this case

with $V_{k+1} \in V_{k+1} \setminus \langle V_1, \dots, V_k \rangle \Rightarrow$ MT again by complement

in this case

$$[F_r(V_1, \dots, V_r)] = (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1) \dots (\mathbb{L}^{d_r} - \mathbb{L}^{r-1})$$

$d_k = \dim V_k$

For more general cases reformulation

an inductive approach aims at constructing l -tuples $(V_1, \dots, V_l)$ with

(1) $v_i \in V_i$

(2) $\dim \langle V_1, \dots, V_r \rangle = r \quad \forall 1 \leq r \leq l$

$$\hookrightarrow 0 \subset \langle V_1 \rangle \subset \langle V_1, V_2 \rangle \subset \dots \subset \langle V_1, \dots, V_l \rangle = E$$

form a complete flag in E

$$E_r = \langle V_1, \dots, V_r \rangle$$

Question III_l:

(13)

$V_1, \dots, V_\ell$ subspaces of l -dim E
 $d_i, e_i \in \mathbb{N}$

$\text{Flag}_{l, \{d_i, e_i\}}(V_1, \dots, V_\ell) = \text{locus of complete flags}$

$$0 \subset E_1 \subset E_2 \subset \dots \subset E_\ell = E$$

with $\dim(E_i \cap V_i) = d_i$

$\dim(E_i \cap V_{i+1}) = e_i$

are these $\text{Flag}_{l, \{d_i, e_i\}}(V_1, \dots, V_\ell)$ mixed Tate?

If true for all $\underline{d}, \underline{e} \Rightarrow$ Question I_l also true

because

$$\mathbb{A}^1(V_1, \dots, V_\ell)$$

fibration



$$\text{Flag}_{l, \{d_i, e_i\}}(V_1, \dots, V_\ell)$$

over a flag $E_\bullet$ in here $\uparrow$ the ℓ -uples $(V_1, \dots, V_\ell)$ in $\mathbb{A}^1(V_1, \dots, V_\ell)$ are obtained by

- choosing $v_1 \in (E_1 \cap V_1) \setminus \{0\}$

- $v_2 \in (E_2 \cap V_2) \setminus (E_1 \cap V_2)$

- $v_3 \in (E_3 \cap V_3) \setminus (E_2 \cap V_3)$

etc.

$$\Rightarrow [\mathbb{A}^1(V_1, \dots, V_\ell)] = [\text{Flag}_{l, \{d_i, e_i\}}(V_1, \dots, V_\ell)] \cdot (\mathbb{L}^{d_1} - 1) (\mathbb{L}^{d_2} - \mathbb{L}^{e_1}) \cdot (\mathbb{L}^{d_3} - \mathbb{L}^{e_2}) \cdot \dots \cdot (\mathbb{L}^{d_\ell} - \mathbb{L}^{e_{\ell-1}})$$

Note (i) the V_i form a complete flag themselves

$$\Rightarrow \text{Flag}_{l, \{d, e\}}(\{V_i\}) = \text{disjoint union of cells}$$

$\Rightarrow$ mixed Tate

(already seen before)

- In general : set of flags E with $\dim(E_i \cap F)$ fixed is union of Schubert cells in the flag manifold of all subspaces of F

$\Rightarrow \text{Flag}_{l, \{d_i, e_i\}}(V_1, \dots, V_k)$ is intersection of unions of Schubert cells in the flag manifold

- This kind of loci widely studied in alg. geom.
mixed Tate question is hard

(counting functions related to Kac-Moody-Lusztig theory)

First cases :

2 subspaces V_1, V_2

$$F(V_1, V_2) = \{ (v_1, v_2) \mid v_1 \in V_1, v_2 \in V_2; \dim \langle v_1, v_2 \rangle = 2 \}$$

two pieces :

(1) $v_1 \in V_1 \setminus (V_1 \cap V_2)$ and $v_2 \in V_2 \setminus \{0\}$

(2) $v_1 \in (V_1 \cap V_2) \setminus \{0\}$ and $v_2 \in V_2 \setminus \{v_1\}$

first piece is $(V_1 \setminus (V_1 \cap V_2)) \times (V_2 \setminus \{0\})$

second piece is obtained by:

- take $\mathbb{P}(V_1 \cap V_2)$ and trivial bdl's $V_{1,2} \subseteq V_2$ fibers $V_1 \cap V_2$ and V_2
- $V_{1,2}$ contains tautological $\mathcal{O}_{1,2}(-1)$ over $\mathbb{P}(V_1 \cap V_2)$
- (v_1, v_2) obtained by choosing pt $p \in \mathbb{P}(V_1 \cap V_2)$ vector $v_1 \neq 0$ in fiber of $\mathcal{O}_{1,2}(-1)$ over p and v_2 in $V_2 \setminus \mathcal{O}_{1,2}(-1)$ fiber of

$\Rightarrow$ class

$$[\mathbb{P}(V_1, V_2)] = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{1,2}+1} + \mathbb{L}^{d_{1,2}} + \mathbb{L}$$

$$d_1 = \dim V_1 \quad d_{1,2} = \dim (V_1 \cap V_2)$$

because class of first piece is

$$(\mathbb{L}^{d_1} - \mathbb{L}^{d_{1,2}}) \cdot (\mathbb{L}^{d_2} - 1)$$

and class of second piece is

$$(\mathbb{L}^{d_{1,2}} - 1)(\mathbb{L}^{d_2} - \mathbb{L})$$

Note: classes of pieces (1) & (2) not symm. in d_1, d_2 but resulting class is

Case of 3 subspaces $\mathbb{P}(V_1, V_2, V_3)$

many more possibilities of obtaining (V_1, V_2, V_3) : ~~the pieces~~

Stratification of $V_3 \setminus \{0\}$
with five strata

$$S_{123} := (V_1 \cap V_2 \cap V_3) \setminus \{0\}$$

$$S_{13} := (V_1 \cap V_3) \setminus (V_1 \cap V_2 \cap V_3)$$

$$S_{23} := (V_2 \cap V_3) \setminus (V_1 \cap V_2 \cap V_3)$$

$$S_{(1,2)3} := ((V_1 + V_2) \cap V_3) \setminus ((V_1 \cup V_2) \cap V_3)$$

$$S_3 := V_3 \setminus ((V_1 + V_2) \cap V_3)$$

(16) Then use following general strategy:

- if $\{S_\alpha\}$ stratif. of V_r with
- finer than $V_1 \cap V_r, \dots, V_{r-1} \cap V_r$
 - $S_\alpha = \# V_i$ s.t. $V_r \in S_\alpha$ belongs to V_i indep of vector only dep. of α
 - for $V_r \in S_\alpha$ $\mathbb{F}_\alpha = [\mathbb{F}(\pi(V_1), \dots, \pi(V_{r-1}))]$
also depends only on α not on vector $V_r \in S_\alpha$
- where $\pi: V \rightarrow V/\langle V_r \rangle$ projection

$$\text{then } [\mathbb{F}(V_1, \dots, V_r)] = \sum_{\alpha} \mathbb{L}^{S_\alpha} \cdot [\mathbb{F}_\alpha] \cdot [S_\alpha]$$

For five listed strata have explicit $[\mathbb{F}_\alpha]$ and $[S_\alpha]$
assembling together gives

$$\begin{aligned} [\mathbb{F}(V_1, V_2, V_3)] &= (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1)(\mathbb{L}^{d_3} - 1) \\ &\quad - (\mathbb{L} - 1) \left((\mathbb{L}^{d_1} - \mathbb{L})(\mathbb{L}^{d_{23}} - 1) + (\mathbb{L}^{d_2} - \mathbb{L})(\mathbb{L}^{d_{13}} - 1) \right. \\ &\quad \left. + (\mathbb{L}^{d_3} - \mathbb{L})(\mathbb{L}^{d_{12}} - 1) \right) \\ &\quad + (\mathbb{L} - 1)^2 (\mathbb{L}^{d_1 + d_2 + d_3 - D} - \mathbb{L}^{d_{123} + 1}) + (\mathbb{L} - 1)^3 \end{aligned}$$

$$d_i = \dim V_i$$

$$d_{123} = \dim(V_1 \cap V_2 \cap V_3)$$

$$d_{ij} = \dim V_i \cap V_j$$

$$D = \dim(V_1 + V_2 + V_3)$$