

QFT 10 Parametric Feynman integrals and determinant hypersurfaces ①

$$\int_{\sigma_n} \frac{P_F(t, p)^{-n+D/2}}{\psi_F(t)^{-n+(l+1)D/2}} \quad P = (p_e)_{e \in E_{\text{ext}}(\Gamma)} \neq 0 \quad m=0$$

$$\text{or} \int_{\sigma_n} \frac{\omega_F}{\psi_F(t)^{D/2}} \quad m = -2n+D \quad P=0 \text{ \& } m \neq 0$$

$$m(\mathbb{P}^{n-1} \setminus X_F, \Sigma_n \setminus (\Sigma_n \cap X_F))$$

$$\partial \sigma_n \subset \Sigma_n = \{ \prod_e t_e = 0 \} \text{ divisor}$$

if convergent $\Rightarrow$ period of
~~the~~ motive
 (otherwise of suitable
 blowup)

$$\psi_F(t) = \det(M_F(t))$$

$$(M_F(t)) = \sum_e t_e \eta_{e,k} \eta_{e,r}$$

↑
circuit matrix

$$\tau = \tau_F : \mathbb{A}^n \rightarrow \mathbb{A}^{l^2}$$

$$l = b_1(\Gamma)$$

$$\tau(t_1, \dots, t_n) = \sum_e t_e \eta_{e,k} \eta_{e,r}$$

$$\hat{X}_F = \tau^{-1}(\hat{D}_l)$$

$$\hat{D}_l \subset \mathbb{A}^{l^2} \text{ determinant hypersurface}$$

$$\hat{D}_l = \{ x = (x_{ij})_{i,j=1,\dots,l} \mid \det(x) = 0 \}$$

$$\mathbb{A}^{l^2} \setminus \hat{D}_l = \text{GL}_l$$

(2) Some properties of the map $\tau: \mathbb{A}^n \rightarrow \mathbb{A}^{l^2}$

1) for $i \neq j$ entry (i, j) is sum of $\pm t_k$

for t_k edges common to i th & j th loop

w/ $+1$ if ~~both~~ ^{edge} orientation agrees or
disagrees w/ both loop orient.

-1 if agrees w/ one; disagrees w/ other

2) $i=j$ entry (i, i) is sum of t_k of
edges in k -th loop all w/ $+$ sign

3) for a fixed edge e

t_e appears in $M(\tau)$ iff e is part of
at least one loop in
chosen basis of $H_1(\Gamma)$

4) if e belongs to only one loop l_i

t_e only appears in diagonal entry (i, i)
(added to variables of other edges of same loop)

5) if two loops l_i, l_j contain e and have no other
edge in common

then $\pm t_e$ appears alone in (i, j) entry and
 (j, i) entry

If the map $\tau: \mathbb{A}^n \rightarrow \mathbb{A}^{l^2}$ is injective
 then the Feynman integral can be pushed forward
 and rewritten as

$$\int_{\tau(\Omega_n)} \frac{\tilde{P}_r(p, x)^{-n+D/2} \omega_r(x)}{\det(x)^{-n+(l+1)D/2}}$$

where $\tilde{P}_r(p, x)$ is a homog. polynomial on $\mathbb{A}^{l^2}$
 for fixed external momenta p

that agrees w/ $P_r(p, t)$ on image of τ

$\omega_r(x)$ form s.t. $\omega_r(x) \wedge \langle \xi_r, dx \rangle = \omega_{l^2}$ ← volume form in $\mathbb{A}^{l^2}$

$\xi_r = (l^2 - n)$ -frame assoc. to linear subspace
 $\tau(\mathbb{A}^n) \subset \mathbb{A}^{l^2}$

$$\langle \xi_r, dx \rangle = \langle \xi_1, dx \rangle \wedge \dots \wedge \langle \xi_{l^2-n}, dx \rangle$$

Note: if $\tau: \mathbb{A}^n \rightarrow \mathbb{A}^{l^2}$ injective then also have
 well defined map $\tau: \mathbb{P}^{n-1} \rightarrow \mathbb{P}^{l^2-1}$ (otherwise not defined)

Then can replace motive

$$m(\mathbb{P}^{n-1} \setminus X_r, \Sigma_n \setminus (\Sigma_n \cap X_r))$$

with motive

$$m(\mathbb{P}^{l^2-1} \setminus D_\ell, \Sigma \setminus (\Sigma \cap D_\ell))$$

where $D_\ell \subset \mathbb{P}^{l^2-1}$
 def hypersurf.

$\Sigma = \text{divisor } \partial(\Omega_n) \subset \Sigma$

④ Conditions for injectivity of the map τ

$$\tau_i = \pi_i \circ \tau \quad \pi_i = \text{projection onto } i\text{th row of matrix } M_r(t)$$

map of only the variables t_e of edges e that belong to i th loop

Lemma: If τ_i is injective for i ranging over a set of loops s.t. every $e \in E(T)$ is part of a loop in that set

$\Rightarrow \tau$ is also injective

Suppose a point $(t_1, \dots, t_n) = (c_1, \dots, c_n)$ is in Kernel of τ

each (i, j) entry in $M_r(t)$ = combination of variables of edges in i -th loop

$\Rightarrow \tau_i$ maps to zero the tuple of c_j 's that corresp. to edges in i th loop

if τ_i injective $\Rightarrow$ that tuple is $= 0$

since the τ_i correspond to loops that cover all $e \in E(T)$
then all $c_j = 0 \Rightarrow \tau$ also injective

Then using explicit form of entries of $M_r(t)$:

Lemma: τ_i injective if

1) $\forall e$ edge in i -th loop $\exists$ other loop having only e in common with i th loop

2) i th loop has at most one edge not shared w/ any other loop

(5)

Pf: if these conditions satisfied then

- all but one edge variable appear by themselves as an entry in i th row
- and possible last remaining variable appears summed together w/ ^{the} other variables

$t_{i_1}, \dots, t_{i_k}$ variables of edges in l_i loop

up to rearranging entries (order of variables)

$\tau_i : (t_{i_1}, \dots, t_{i_k}) \mapsto (t_{i_1} + \dots + t_{i_k}, \pm t_{i_1}, \dots, \pm t_{i_k})$
 if l_i has no edge not shared

$\tau_i : (t_{i_1}, \dots, t_{i_k}) \mapsto (t_{i_1} + \dots + t_{i_k}, \pm t_{i_1}, \dots, \pm t_{i_{k-1}})$
 if a single edge e_{i_k} not shared w/ any other loop

injective in both cases

What natural combinatorial condition on the graph ensures the existence of a basis of loops w/ properties as above?

Recall every finite graph can be embedded in a compact orientable surface S of some finite genus $g = g(S)$

- planar iff minimal genus is $g = 0$

⑥. An embedding $\Gamma \hookrightarrow S$ is a 2-cell embedding if complement $S \setminus \Gamma \cong \bigcup_i D_i^\circ$ homeomorphic union of open 2-cells

(D_i° = "faces" or "regions" of the embedding)

• embedding $\Gamma \hookrightarrow S$ is closed 2-cell embedding if the closure $\overline{D_i}$ of every face in S is a disk

Known: $\Gamma \hookrightarrow S$ is minimal genus iff it is 2-cell embedding

but conditions for closed 2-cell embedding are more subtle (see later)

Why closed 2-cell embeddings? Because given $\Gamma \hookrightarrow S$ closed 2-cell

- each face determines a loop (the union of edges in $\partial \overline{D_i}$)
- together w/ basis of homology of Riemann surface S also given by collections of edges

More precisely —

Lemma: $\Gamma \hookrightarrow S$ closed 2-cell

(7)

choose a face and a basis of $H_1(S, \mathbb{Z})$

$\Rightarrow$ get basis of $H_1(\Gamma, \mathbb{Z})$ by $2g + f - 1$ loops

$f = \# \text{faces of embedding}$

Pf: orient arbitrarily edges of Γ & faces

edges around each D_i w/ ∂ -orientation

$\Rightarrow f$ loops (all null-homotopic in S)

"internal faces"

not all independent: sum of $f-1$ of them gives f^{th} or

So pick one face & remove the corr. loop from counting
"external"

left w/ loops made of edges of Γ that span $H_1(\Gamma)$

(there are enough otherwise would not be min. genus)

$H_1(\Gamma)$ has rk $2g + f - 1$ because

$$b_0(S) - b_1(S) + b_2(S) = 2 - 2g = v - e + f$$

$$= b_0(\Gamma) - b_1(\Gamma) + f = 1 - l + f$$

so no other relations between these loops

Use this basis of $H_1(\Gamma)$ to compute maps τ & τ_v

(8) Lemma: $\Gamma \hookrightarrow S$ closed 2-cell embedding

(if) (i) any two faces have at most one edge in common

$\Rightarrow$ the $f-1$ maps τ_i are injective

(ii) if also every edge is in boundary of two faces

$\Rightarrow \tau$ is injective

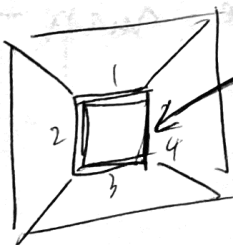
Pf: if ℓ loop of an internal face then variables in common w/ another face loop will appear as an isolated $\pm t$ in (i) entry

all but at most one edge (in common w/ external face)

$\Rightarrow$ enough for τ_i injective

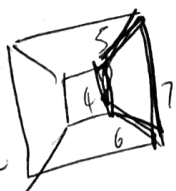
to get τ injective check that every edge of Γ is in one of the $f-1$ face loops which guaranteed by condition (ii)

Example with these conditions



for this loop row of $M_\Gamma(t)$ is

$$(t_1 + t_2 + t_3 + t_4, \pm t_4, \pm t_3, \pm t_2, \pm t_1)$$



this loop

$$(t_4 + t_5 + t_6 + t_7, \pm t_4, \pm t_5, \pm t_6)$$

etc. all τ_i injective

Connectivity of graphs

9

k -edge-connected : cannot be disconnected by removal of any set of $\leq k-1$ edges

2 -vertex-connected , no looping edges , ≥ 3 vertices

& cannot be disconnected by removal of any single vertex

($k \geq 2$) k -vertex-connected ; no looping ; $\geq k+1$ vertices

& cannot be disconnected by removal of any $\leq k-1$ vertices

1-vertex & 1-edge conn = connected

2-edge-conn. = 1PI

2-vertex-conn $\Rightarrow$ 1PI to see this

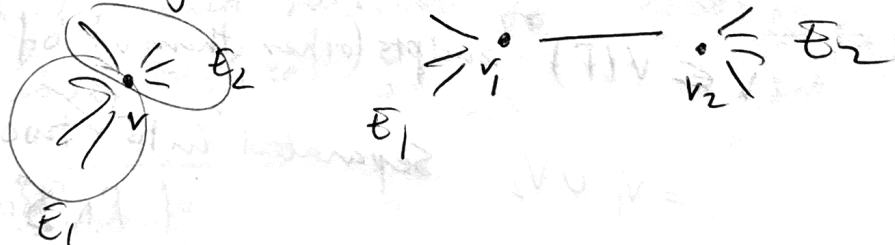
splitting at a vertex :

Γ' obtained by taking $v \in V(\Gamma)$

- partitioning set of edges of Γ adjacent to v

into $E_1 \cup E_2$

- inserting new edge with ;



Lemma:
(10) Γ $\#V(\Gamma) \geq 3$ no looping

(1) if Γ is 2-vertex-connected $\Rightarrow$ also Γ is 2-edge-conn.
(1PI)

(2) Γ is 2-vertex-conn. iff all splittings Γ'
at any $v \in V(\Gamma)$ are 1PI

Pf: (1) if Γ were not 1PI $\exists e \in E(\Gamma)$ s.t.
 $\Gamma - e = \Gamma_1 \cup \Gamma_2$ disjoint components

Γ no looping so

e has different endpts $v_1 \neq v_2$ $v_1 \in \Gamma_1, v_2 \in \Gamma_2$

$\#V(\Gamma) \geq 3$ so $\exists$ another vertex in one of the components
say $v \neq v_1$ in Γ_1

Remove vertex v_1 from $\Gamma \Rightarrow v$ and v_2 belong to
different components
(because in particular removed also e)

$\Rightarrow \Gamma$ not 2-vertex-conn.

Pf (2) if Γ not 2-vertex-conn.

$\exists$ vertex $v \in V(\Gamma)$ whose removal disconnects
 $v \in V(\Gamma)$ endpts (other than v) of edges adjacent to v

$V = V_1 \cup V_2$ Separated into ^{the} two components
of $\Gamma - \{v\}$
starting

$\Rightarrow$ gives the existence of a splitting Γ' of Γ at v
that is not 1PI

Conversely Suppose $\exists \Gamma'$ splitting Γ at v
that is not LPI

(11)

$\exists e \in E(\Gamma')$ that disconnects

If e already in $E(\Gamma) \Rightarrow \Gamma$ not 1-PI so Γ not 2-vertex-con

If $e = \text{additional edge}$: $v_1 \neq v_2$ endpts in Γ'

no looping $\Rightarrow \exists$ ~~edge~~ other vertex $v'_1 \neq v_2$ adjacent to v_1
 $\neq v_1$

v'_1 & v'_2 in different
components of $\Gamma' \setminus e$

$v'_2 \neq v_2$ adjacent to v_2
 $\neq v_1$

but v'_1 & v'_2 are also in $\Gamma \setminus S(v) \subset \Gamma' \setminus e$

$\Rightarrow$ removing $S(v)$ would also disconnect Γ

Lemma : Γ with $\#V(\Gamma) \geq 4$; no looping
no multiple edges } simple

$\Rightarrow (3\text{-vertex-connected} \Rightarrow 3\text{-edge-connected})$

Similar proof by contradiction

2PI

Connectivity properties and relation to
embedding into surfaces S

Face width of an embedding : $\nu: \Gamma \hookrightarrow S$

for $S = S^2$
set
 $fw(\Gamma, \nu) = \infty$

$fw(\Gamma, \nu) = \text{largest } k \in \mathbb{N} \text{ s.t. every non-contractible}$
 $\text{simple closed curve in } S \text{ intersects}$
 $\Gamma \text{ at least } k \text{ times}$

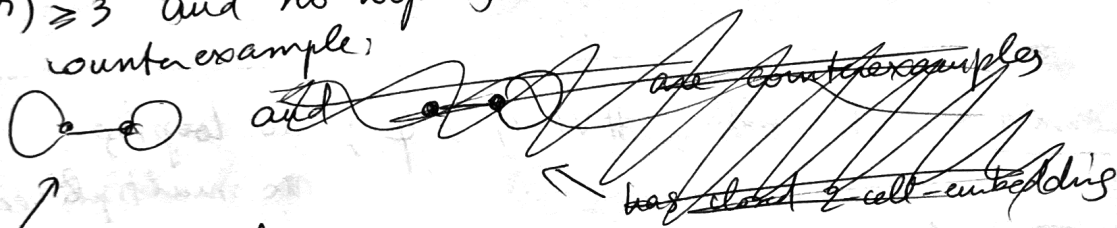
(12) for Γ with $\#V(\Gamma) \geq 3$, no looping

$\exists i: \Gamma \hookrightarrow S$ closed 2-cell-embedding
 $\Leftrightarrow \Gamma$ is 2-vertex-connected and
 $fw(\Gamma, v) \geq 2$

Known fact about embedded graphs

e.g. a planar graph w/ at least 3-vertices & no looping
has closed-2-cell embedding in S^2 iff 2-vertex-connected

$\#V(\Gamma) \geq 3$ and no looping are needed
counterexample;



2-vertex connected
no closed-2-cell embedding

Prop: Γ with $\#V(\Gamma) \geq 3$ no looping
& closed-2-cell embedding $i: \Gamma \hookrightarrow S$

then if any two faces have at most one edge
in common then τ is injective

Pf: enough to show every edge of Γ is in ∂ of two faces and
use previous

assume $\exists$ edge e not in ∂ of two faces

$\Rightarrow$ same face on both sides of e in S

but $\overline{D_i^0}$ is a disk; can take path in D_i^0

from one side of e to other;

γ splits cell into
two connected components

$D_i^0 \setminus \gamma$

but then this means removing e splits Γ into
two connected components (in two parts of cell $D_i^0 \setminus \gamma$)

$\Rightarrow \Gamma$ not 2-edge-connected

but conditions assume Γ would imply 2-vertex-connected
that stronger

Prop: Γ 3-edge-connected with $\#V(\Gamma) \geq 3$
no looping

with closed-2-cell embedding

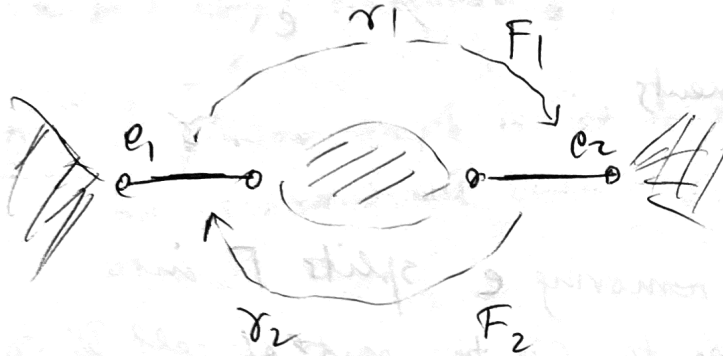
i.e. $\Gamma \hookrightarrow S$ with $fw(\Gamma, i) \geq 3$

then τ_i, τ are all injective

Proof: by previous know second condition of Lemma
(every edge in ∂ of two faces)
is satisfied

so need to check first condition of Lemma
(any two faces have at most one edge in common)

• Suppose faces F_1 & F_2 have more than one edge in common



can find paths γ_1, γ_2
in F_1, F_2
connecting pts on the
two edges

so that $\gamma_1 \cup \gamma_2$ closed paths
meeting Γ in two points

but $fw(\Gamma, v) \geq 3$ so $\gamma = \gamma_1 \cup \gamma_2$ has to be
null-homotopic in S

hence $S \setminus \gamma$ has two connected components

$\Rightarrow \Gamma$ splits into two components after
removing e_1 & $e_2 \Rightarrow$ not 3-edge-conn.

~~Planar surfaces~~ Corollary:

Known fact: if Γ 3-vertex-conn. : $fw(\Gamma, v) \geq 3 \Leftrightarrow$ every vertex has
a wheel neighb.

Coroll: Γ 3-vertex-conn w/ wheel neighb. at every vertex
 $\Rightarrow \tau_v, \tau$ are injective