

Graph complete interactions Λ_r and Feynman integral (1)

$$\gamma_{k,j} = \begin{cases} \pm 1 & e_k \in l_j \text{ w/ } \pm \text{ orientation} \\ 0 & e_k \notin l_j \end{cases}$$

circuit matrix of the graph

$$Y_{t,m} = \left\{ x = (x_{ij}) \mid \sum_{i=1}^n x_{ij} = b_j(t) \mid x^T Q_t(x) + m^2 = 0 \right\}$$

$$\Lambda_r = \{ (t, \beta) \in \mathbb{A}^n \times \mathbb{A}^l \mid Q_t(\beta) = 0 \}$$

$$R_r = \{ (t, \beta, x) \mid \underbrace{(t, \beta) \in \Lambda_r}_{t \in \Sigma_n \text{ simplex}}, x \in Y_{t,m}^c \}$$

$\leftarrow$ complement

where:

$$(Q_t(\beta))_j = \sum_{e,r} t_e \gamma_{e,r} \gamma_{e,j} \quad \beta_r = M_r(t) \cdot \beta$$

$$U(r) = \int \frac{\delta(1 - \sum_e t_e) \delta(Q_t(\beta))}{(m^2 + \sum_x x_{ij} Q_{ij}(x))^n} dt_1 \dots dt_n \omega(x, \beta)$$

Volume form of
fiber in R_r over
point $t \in \Sigma_n$

to see this start again from

$$U(r) = \int \frac{\delta(\sum_{e \in E_{int}} \varepsilon_{e,v} k_e + \sum_{e' \in E_{ext}} \varepsilon_{e',v} p_{e'})}{q_1(k) \dots q_n(k)} d^D k_1 \dots d^D k_n$$

(2) Same change of variables

$$k_e = u_e + \sum_{j=1}^l \eta_{e,j} x_j$$

with Kirchhoff constraint $\sum_e t_e u_e \eta_{e,j} = 0$ (*)

$\forall j=1, \dots, l$

and $\sum_{e \in E_{\text{int}}} \epsilon_{e,r} u_e + \sum_{e' \in E_{\text{ext}}} \epsilon_{e',r} p_{e'} = 0$

Feynman trick & change of variables

$$\int \frac{\delta(1 - \sum_e t_e) \delta(\sum_e \epsilon_{e,r} u_e + \sum_{e'} \epsilon_{e',r} p_{e'})}{(\sum_e t_e (u_e^2 + m^2) + \frac{t}{x} Q_t(x))^n} dt_1 \dots dt_n dx_1 \dots dx_l$$

no mixed terms
in denominator

But now instead of assuming $(p_{e'})_{e' \in E_{\text{ext}}}$ nonzero
and $m=0$

do opposite keep $m \neq 0$ and set $p_{e'} = 0$ trivial external momenta

The u_e solutions of $\sum_{e \in E_{\text{int}}} \epsilon_{e,r} u_e = 0$

in kernel of incidence matrix

$\Rightarrow$ by orthogonality of incidence & circuit matrices

$$u_e = \sum_{r=1}^l \eta_{e,r} \beta_r \quad \text{for some } \beta = (\beta_r)_{r=1, \dots, l}$$

and the other Kirchhoff constraint (*) becomes

$$\sum_{e,r} t_e \eta_{e,j} \eta_{e,r} \beta_r = Q_t(\beta) = 0 \quad (= M_T(t) \cdot \beta)$$

Consider all possible solutions
(not only $\beta \equiv 0$) ~~with region~~
and integrate over space of solutions get **

Set of solutions $\{ \beta \mid Q_t(\beta) = 0 \}$
is a
 $(t, \beta) \in \mathbb{A}^n \times \mathbb{A}^l$
complete intersection Λ_Γ in $\mathbb{A}^n \times \mathbb{A}^l$ (or in $\mathbb{P}^{n-1} \times \mathbb{P}^{l-1}$)

Equivalently describe it in projective environment

$$\Lambda_\Gamma \subset X_\Gamma \times \mathbb{P}^{b_1(\Gamma)-1} \quad \text{with projection } \Lambda_\Gamma \longrightarrow X_\Gamma$$

$$(t, \beta) \longmapsto t$$

$\uparrow$
 sol'n of
 $\det(Q_t) = 0$
 $\parallel$
 $\det(d\Lambda_\Gamma(t)) = 0$

Bloch proved that

$$[\Lambda_\Gamma] \in \mathbb{Z}[\mathbb{L}] \quad \text{for all graphs } \Gamma$$

sketch of argument; consider second proj. $\Lambda_\Gamma \longrightarrow \mathbb{P}^{b_1(\Gamma)-1}$

Λ_Γ has a stratification defined by sets

$$S_\Gamma^{(m)} \subset \mathbb{P}^{b_1(\Gamma)-1} \quad S_\Gamma^{(m)} = T_\Gamma^{(m)} \setminus T_\Gamma^{(m+1)} \quad \text{where each}$$

$$T_\Gamma^{(m)} \text{ is a union of linear subspaces}$$

$$\text{" } \{ \beta \in \mathbb{P}^{b_1(\Gamma)-1} \mid \varepsilon(\beta) \geq m \}$$

in $Q^{b_1(\Gamma)}$
 with $\varepsilon(\beta) = \text{codim span}(\gamma_{e_i})$
 with $\sum_i \gamma_{e_i} \beta_i \neq 0$

(4)

$$\Lambda_P \big|_{S_P^{(m)}}$$

i.e. preimage of $S_P^{(m)}$ under proj.
 $\Lambda_P \rightarrow \mathbb{P}^{n-b_1(U)-1}$

is a projective bundle over $S_P^{(m)}$ with fiber $\mathbb{P}^{n-b_1(U)+m-1}$

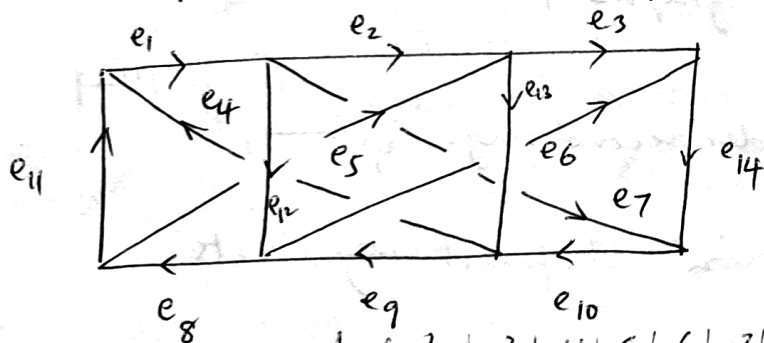
$$\text{So } [\Lambda_P] = \sum_m [S_P^{(m)}] [\mathbb{P}^{n-b_1(U)+m-1}]$$

and shows inductively that the

$$[S_P^{(m)}] = P_{P,m}(\pi) \text{ polynomials}$$

Dorj's counterexample to polynomial countability

$m=14$ edges $l=7$ loops



$\eta_{e,j} = 4$

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	1	0	0	0	0	0	0	1	0	0	1	1	0	0
2	0	0	0	1	0	0	0	1	1	0	1	0	0	0
3	0	0	0	0	1	0	0	1	1	0	0	0	1	0
4	0	1	0	0	0	0	0	0	1	0	0	-1	1	0
5	0	0	1	0	0	0	0	0	0	1	0	0	-1	1
6	0	0	0	0	0	1	0	0	1	1	0	0	0	1
7	0	0	0	0	0	0	1	0	1	1	0	-1	0	0

Call edge variables $t_0, \dots, t_6, s_0, \dots, s_6$
and get $M_P(t, s)$ matrix

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	s_0	t_0	t_2	t_1	0	0	t_1							
2	t_0	s_1	t_2+t_3	t_3	0	t_3	t_3							
3	t_2	t_2+t_3	s_2	t_3+t_4	t_4	t_3	t_3							
4	t_1	t_3	t_3+t_4	s_3	t_4	t_3	t_3+t_4							
5	0	0	t_4	t_4	s_4	t_5	t_6							
6	0	t_3	t_3	t_3	t_5	s_5	t_5+t_6							
7	t_1	t_3	t_3	t_3+t_4	t_6	t_5+t_6	s_6							

$$\hat{X}_P \subset A^{14}$$

Suppose that counting $N_q(\hat{X}_P)$ is polynomial $P(q)$ w/ $\mathbb{Z}$ -coeffs

Step 1: $\psi_P = s_0 \psi_{P|e_0} + \underbrace{\psi_P}_{\psi_P|e_0=0}$

look at image of $\hat{X}_P$ under projection from A^{14} to A^{13}
(projecting away s_0)

$$[\hat{X}_P] = [\underbrace{\hat{X}_P}_{\parallel} \cap \underbrace{Z(\psi_{P|e_0})}_{\parallel}] + [\underbrace{\hat{X}_P}_{\parallel} \cap \underbrace{(X_P \cap Z(\psi_{P|e_0}))}_{\parallel}]$$

$$\parallel \quad \parallel$$

$$\mathbb{L}^2 \cdot [\hat{X}_{P|e_0} \cap \hat{X}_{P|e_0}] \quad \mathbb{L}^{13} - \mathbb{L} [\hat{X}_{P|e_0}]$$

iterate one more time find

$$[\hat{X}_P] = \mathbb{L}^{13} \pm \mathbb{L}^2 \text{ (coeffs)} [\hat{X}_{P|e_0, e_1} \cap \hat{X}_{P|e_0, e_2}] + \mathbb{L}^2 [\hat{X}_{P|e_0, e_1} \cap \hat{X}_{P|e_0, e_2} \cap \hat{X}_{P|e_0}]$$

⑥ if counting polynomial would be

$$P(q) = q^{13} + q^2 F(q) \quad \text{with } F(q) \text{ of deg} \leq 10$$

then computer search for solutions: counting pts and fit polyn.

find $q = 3, 5, 7, 11, 13, 17, 19, 23$ compatible with q

$$\Rightarrow P(q) = q^{13} + q^{11} + 23q^{10} - 78q^9 + 90q^8 - 35q^7 + (q-2)q^6 - 34q^5 + 66q^4 - 32q^3 + (q-1)q^2$$

while $q=2$ and powers 4, 8, 16 const. with

$$P(q) = \text{same above} - (q-1)q^2$$

Not a single polynomial that interpolates

Can one obtain better examples where polynomiality fails in a more interesting way?

Several examples by Oliver Schreier