

Q7/6

Bloch's motives of sums of graphs

(1)

$$\psi_{\Gamma}(t) = \det(M_{\Gamma}(t)) = \det\left(\sum_e t_e M_e\right)$$

$$\psi_{\Gamma}^{\vee}(t) = \det(N_{\Gamma}(t)) = \det\left(\sum_e t_e N_e\right)$$

$$\psi_{\Gamma}(t) = \left(\prod_{e \in E(\Gamma)} t_e\right) \psi_{\Gamma}^{\vee}\left(\frac{1}{t}\right)$$

$$\psi_{\Gamma/e} = \psi_{\Gamma} \big|_{t_e=0}$$

$$\psi_{\Gamma/e} = \frac{\partial}{\partial t_e} \psi_{\Gamma}$$

e neither
bridge nor
looping edge

$$\psi_{\Gamma/e}^{\vee} = \frac{\partial}{\partial t_e} \psi_{\Gamma}^{\vee}$$

$$\psi_{\Gamma/e}^{\vee} = \psi_{\Gamma}^{\vee} \big|_{t_e=0}$$

$e_1, \dots, e_p$ sting of edges (neither bridges nor looping)

$$\psi_{\Gamma \setminus (e_1, \dots, e_p)}^{\vee} = \psi_{\Gamma}^{\vee} \big|_{t_{e_1} = \dots = t_{e_p} = 0}$$

$$\mathbb{P}_{\Gamma} = \mathbb{P}^{n-1} \quad n = \#E(\Gamma)$$

$\Delta \subset \mathbb{P}_{\Gamma}$ divisor $\Delta = \left\{ \prod_{e \in E(\Gamma)} t_e = 0 \right\}$ of coord. hyperplanes.

$$\Pi_{\Gamma} = \mathbb{P}_{\Gamma} \setminus \Delta \quad \text{max. torus}$$

$X_{\Gamma} \subset \mathbb{P}_{\Gamma}$ graph hypersurface $X_{\Gamma} = \{ \psi_{\Gamma} = 0 \}$

$$X_{\Gamma}^{\circ} = X_{\Gamma} \cap \Pi_{\Gamma}$$

$$X_{\Gamma}^{\vee, \circ} = X_{\Gamma}^{\vee} \cap \Pi_{\Gamma}$$

$X_{\Gamma}^{\vee} = \{ \psi_{\Gamma}^{\vee} = 0 \}$ (if Γ planar = hypersurf of dual graph)

- by Gauss transformation

$$K_p \equiv K_p^*$$

- the distribution of K_p

$\Rightarrow$ distribution of K_p^*

there is
open lines
/ but open it

$$I(K_p^*) = \sum_{(q_1, q_2) \in E(K_p^*)} [(K_p^* \cap (q_1 - q_2 - 1))^*] (q_1, q_2)$$

there are all subsets of $E(K_p^*)$

- a subset (q_1, q_2) is disjoint if the linear space $(q_1 - q_2 - 1)$ is contained in K_p^*

- compute $I(K_p^*)$ as

$$I(K_p^*) = \sum_{(q_1, q_2) \in E(K_p^*)} [(K_p^* \cap (q_1 - q_2 - 1))^*] + t$$

with $t = \sum_{(q_1, q_2) \in E(K_p^*)} 1$

where $t = \sum_{(q_1, q_2) \in E(K_p^*)} 1$ because in this there
there is a line hypothesis

- K_p - complete graph on n vertices

$$n = 1, 2, 3, \dots \quad n = 1, 2, 3, \dots$$

- by Cremona transformation

$$X_\Gamma^0 \cong X_\Gamma^{V,0}$$

- toric stratification of Π_Γ

$\Rightarrow$ stratification of X_Γ^V

inters. w/
open torus:
 $t_e \neq 0 \quad e \notin \Delta_{q-p}$

- $[X_\Gamma^V] = \sum_{\{e_1, \dots, e_p\} \subset E(\Gamma)} [(X_\Gamma^V \cap \{t_{e_1} = \dots = t_{e_p} = 0\})^0] = \in K_0(\mathbb{Z})$

Sum over all subsets of $E(\Gamma)$

- a subset $\{e_1, \dots, e_p\}$ is degenerate if the linear space $\{t_{e_1} = \dots = t_{e_p} = 0\}$ is contained in X_Γ^V

- rewrite $[X_\Gamma^V]$ as

$$[X_\Gamma^V] = \sum_{\substack{S = \{e_1, \dots, e_p\} \subset E(\Gamma) \\ \text{nondegen}}} [(X_\Gamma^V \cap \{t_{e_1} = \dots = t_{e_p} = 0\})^0] + t$$

Where term $t \in \mathbb{Z}[\mathbb{Z}]$ because in those terms inters is a lin. subspace

- K_m = complete graph on m vertices

vertices v_i edges e_{ij}

$$\partial e_{ij} = v_i - v_j$$

Claim: $[X_{K_m}^V] \in \mathbb{Z}[L]$ for all m

(3)

Let $Q^m/Q =: Q^{m,0}$ vectors w/ entries adding up to zero

$$Q^m/Q \rightarrow Q^{E(K_m)}$$

$$v_k \mapsto \sum_{\mu > k} -e_{k\mu} + \sum_{r < k} e_{rk} \quad 1 \leq k \leq m-1$$

$\Rightarrow N_e$ in $\psi_P^V = \det(\sum_e t_e N_e)$ are given, for $P = K_m$ by a basis of space of all $(m-1) \times (m-1)$ symm. matrices

$$X_{K_m}^V \cong \mathbb{P} \left(\begin{array}{l} \text{space of all } (m-1) \times (m-1) \text{ symmetric} \\ \text{matrices of rank } \leq m-2 \end{array} \right)$$

• Let $S\text{Hom}(V, V^V)$ be set of $\phi \in \text{Hom}(V, V^V)$ s.t. corresp. quadr. form on V is symmetric

• stratification $\mathbb{P}(\text{Hom}(V, W)) = \bigsqcup_P \mathbb{P}(\text{Hom}_P(V, W))$ by rank

• set $c(m, p) = \left[\mathbb{P}(S\text{Hom}_P(K^m, K^m)) \right]$ $K = \text{field}$

$$c(m, 1) = [\mathbb{P}^{m-1}] \quad \sum_{p=1}^m c(m, p) = \left[\mathbb{P}^{\binom{m+1}{2}-1} \right]$$

$$c(m, p) = [Gr(m-p, m)] \cdot c(p, p)$$

$$\text{and } [X_{K_m}^V] = \sum_{p=1}^{m-2} c(m-1, p)$$

$$\Rightarrow [X_{K_m}^V] \in \mathbb{Z}[\mathbb{L}]$$

Note: if add any # of edges (no new vertices)
to ~~K_m~~ still class $X_{K_m}^V$ in $\mathbb{Z}[\mathbb{L}]$

(because either parallel multiple edges or looping edges)

$$\bullet \text{ Let } \mathcal{I}_m := \sum_{\substack{S = \{e_1, \dots, e_p\} \subseteq E(K_m) \\ \text{nondegen.}}} [X_{K_m \setminus \{e_1, \dots, e_p\}}]$$

• Let Γ be a simple graph (no multiple or looping edges)

with $\#V(\Gamma) = m$

then $[X_\Gamma]$ appears in the sum $\mathcal{I}_m$

with multiplicity $\frac{m!}{\#G}$

$G \subseteq \text{Sym}(V(\Gamma)) = S_m$
subgr. of symm. group S_m
that acts on set of edges
(preserves)

So can rewrite

$$J_m = \sum_{\Gamma: \#V(\Gamma)=m} \frac{[X_\Gamma]}{\#Aut(\Gamma)}$$

Thm: (Bloch)

$$[J_m] \in \mathbb{Z}[\mathbb{L}]$$

Pr: from $[X_{K_m}^V] = \sum_{\substack{S=\{e_1 \dots e_p\} \subset E(K_m) \\ \text{nondeg}}} [X_{K_m \setminus \{e_1 \dots e_p\}}^0] + t$ $\in \mathbb{Z}[\mathbb{L}]$

$\uparrow$
and previous result that this is in $\mathbb{Z}[\mathbb{L}]$

- then need to pass from $X_{K_m \setminus \{e_1 \dots e_p\}}^0$ to full $X_{K_m \setminus \{e_1 \dots e_p\}}$

- $[X_{K_m \setminus \{e_1 \dots e_p\}}] = \sum_{\substack{\{f_1 \dots f_k\} \subset E(K_m \setminus \{e_1 \dots e_p\}) \\ \{e_1 \dots e_p, f_1 \dots f_k\} \text{ nondeg.}}} [X_{(K_m \setminus \{e_1 \dots e_p\}) / \{f_1 \dots f_k\}}^0] + t$ $\in \mathbb{Z}[\mathbb{L}]$

So want to check this term is in $\mathbb{Z}[\mathbb{L}]$

- but this can rewrite as

$$\sum_{\{f_1 \dots f_k\}} \sum_{\substack{\{e_1 \dots e_p\} \in E(K_m / \{f_1 \dots f_k\}) \\ \text{nondeg}}} [X_{(K_m / \{f_1 \dots f_k\}) \setminus \{e_1 \dots e_p\}}^0]$$

all degen. terms where get (in dual picture) a lin space

then note that

$K_m / \{f_1, \dots, f_\ell\}$ obtained from $K_{m-\ell}$
by adding looping edges and multiple edges

$\Rightarrow [X_{K_m / \{f_1, \dots, f_\ell\}}^V] \in \mathbb{Z}[\mathbb{L}]$ because $[X_{K_{m-\ell}}^V] \in \mathbb{Z}[\mathbb{L}]$
as shown before

but then $\sum_{\substack{\{e_1, \dots, e_p\} \subset E(K_m / \{f_1, \dots, f_\ell\}) \\ \text{nondeg}}} [X_{(K_m / \{f_1, \dots, f_\ell\}) \setminus \{e_1, \dots, e_p\}}^0] \in \mathbb{Z}[\mathbb{L}]$

$\Downarrow$

$\mathcal{L}_m \in \mathbb{Z}[\mathbb{L}]$

Note: cannot draw directly conclusion about
periods because periods don't factor through
Grothendieck ring

Dichotomy in the Grothendieck ring (Aluffi-M.)

(7)

- Seen that (Belkale-Burnham) $[X_r]$ span ~~$S^{-1}K_0(V_{\mathbb{Z}})$~~ $S^{-1}K_0(V_{\mathbb{Z}})$ where S multipl. system gen. by $\mathbb{L}^n - \mathbb{L}$ (in particular $\mathbb{L}$ invertible)

- if set $\mathbb{L} = 0$ $K_0(V_{\mathbb{Z}}) / (\mathbb{L}) \cong \mathbb{Z}[SB]$ (Larsen-Hunts)
 $SB =$ stable birational equivalence classes;

$$X \sim_{sb} Y \quad \text{if} \quad \exists P^m, P^m \quad X \times P^m \sim_{\text{birat.}} Y \times P^m$$

- What about X_P 's in $K_0(V_{\mathbb{Z}}) / (\mathbb{L}) = \mathbb{Z}[SB]$?

Note: any smooth projective X that is rational would ~~be~~ have

$$[X]_{SB} = [pt]_{SB} \text{ in } \mathbb{Z}[SB]$$

but if X rational but quasi-proj. or singular not in general true

• Example: $\exists$ rational surface X whose class $[X]_{SB}$ satisfies $[X]_{SB} = 2 - [C]_{SB}$ $C =$ elliptic curve

Known: $\exists$ proj. rational surfaces X w/ one isolated sing. pt p s.t. divisor of $\tilde{X}$ minimal resol. of sing.
 $\downarrow$
 X is $\cong C$ smooth ell. curve

• Bittner relations for blowups give for Grothendieck class

$$[X] = [\tilde{X}] - [C] + [p]$$

so for classes in quotient $\mathbb{Z}[SB]$

$$[X]_{SB} = [\tilde{X}]_{SB} - [C]_{SB} + [p]_{SB} \xrightarrow{\sim \text{rational \& smooth proj.}} \text{since } [\tilde{X}]_{SB} = [pt]_{SB} = 2 - [C]_{SB}$$

⑧ $\hat{Y}_\Gamma = A^n - \hat{X}_\Gamma$ graph hypersurface complement

remember from deletion-contraction formula:

- if e bridge: $[\hat{Y}_\Gamma] = \mathbb{L} [\hat{Y}_{\Gamma, e}]$
- if e looping: $[\hat{Y}_\Gamma] = (\mathbb{L} - 1) [\hat{Y}_{\Gamma, e}]$
- if neither: $[\hat{Y}_\Gamma] = \mathbb{L} [A^{n+1} - (\hat{Y}_{\Gamma, e} \wedge \hat{Y}_{\Gamma, e})] - [\hat{Y}_{\Gamma, e}]$

Claim: deletion-contr. formula implies

$$[\hat{Y}_\Gamma]_{SB} = \begin{cases} 0 & \text{if } \Gamma \text{ has edges that are not looping edges} \\ (-1)^n & \text{if } \Gamma \text{ has only } n \geq 0 \text{ looping edges} \end{cases}$$

take formulae above mod $\mathbb{L}$: if any bridge $[\hat{Y}_\Gamma]_{SB} = 0$
if any other kind of edge

$$[\hat{Y}_\Gamma]_{SB} = - [\hat{Y}_{\Gamma, e}]$$

Keep removing edges: if $\exists$ edge not looping
can remove all other edges so
left with a bridge $\Rightarrow [\hat{Y}_\Gamma]_{SB} = 0$

if only looping edges left with
only vertices but then $\hat{Y}_{\text{graph w/ only vertices}} = A^0$

$$\text{class} = [\text{pt}]_{SB}$$

with sign $(-1)^n$ (all edge removals)

So very far from "generating" $\mathbb{Z}[SB]$!!

Note $[\hat{Y}_p]$ also do not span $S'^{-1}K_0(V)$

(9)

where S' multipl. system. gen. by $L^n - 1$

because localization commutes with quotients

$$\text{so } (S'^{-1}K_0(V)) / (L) = S'^{-1}(K_0(V) / (L))$$

all elements of S' are invertible mod L ($L^n - 1 = -1 \pmod{L}$)

$$\text{so } S'^{-1}K_0(V) / (L) = K_0(V) / (L) \quad \text{localizing at invertibles does nothing}$$

but know $[\hat{Y}_p]$ don't span this quotient
so they cannot span $S'^{-1}K_0(V)$ either

Consequences on Hodge structures

Hodge-Deligne polynomial

$$P_X(u,v) = \sum_{p,q} e^{p,q}(X) u^p v^q \quad \text{where } e^{p,q}(X) = \sum_k (-1)^k h^{p,q}(CH_c^k(X), \mathbb{Q})$$

is a motivic measure $P: K_0(V_{\mathbb{C}}) \rightarrow \mathbb{Z}[u,v]$

that ~~passes through~~ ^{induces on} quotient: ~~P~~

$$\text{since } P_{\mathbb{A}^1}(u,v) = uv$$

$$\bar{P}: K_0(V_{\mathbb{C}}) / (L) \rightarrow \mathbb{Z}[u,v] / (uv)$$

$\Rightarrow$ all graph hypersurfaces have

$$P_{X_p}(u,v) = \underset{\substack{\uparrow \\ \text{constant}}}{c} + uv \underset{\substack{\uparrow \\ \text{another polyn.}}}{f(u,v)}$$