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full $\otimes$ -functor

①

$$\text{Chow}(k)_F = \text{Mot}_{\text{rat}}(k)_F \longrightarrow \text{Mot}_{\sim}(k)_F$$

for another good equiv. rel $\sim$

Kernel $I_{\sim}$ = morphisms in $\text{Chow}(k)_F$ that become trivial in $\text{Mot}_{\sim}(k)_F$

is a $\otimes$ -ideal of $\text{Chow}(k)_F$

a bilateral ideal I of an F -linear category $\mathcal{C}$ is the datum for all pairs of objects A, B in $\mathcal{C}$ of a sub- F -module of $\text{Hom}_{\mathcal{C}}(A, B)$ s.t. the collection of these sub- F -modules is stable under composition of morphisms (on the right and on the left) by any morphism in $\mathcal{C}$

$\Rightarrow$ one can form a quotient category $\mathcal{C}/I$ same objects as $\mathcal{C}$ and morphisms are quotients by I

If $\mathcal{C}$ is also a $\otimes$ -category then have a notion of $\otimes$ -ideal

I $\otimes$ -ideal if also stable under $\otimes$ product

~~with~~ $[$ ~~also~~ $\text{id}_C \otimes f$ and $f \otimes \text{id}_C$ for all objects C $]$
 with arbitrary morphisms in $\mathcal{C}$

$\Rightarrow \mathcal{C}/I$ still has $\otimes$ -structure

(2)

$I_{\sim}$ is a $\otimes$ -ideal of $\text{Chow}(k)_F$

this follows from stability of a good equiv. rel. under composition, linear combinations, and $\otimes$ -prod of correspondences

Lemma: all $\otimes$ -ideals I in $\text{Chow}(k)_F$ are of the form $I_{\sim}$ for some good equiv. rel. on alg. cycles

$\Rightarrow$ resulting category $\text{Mot}_{\sim}(k)_F^{\otimes} = (\text{Chow}(k)_F / I_{\sim})^{\otimes} \leftarrow \text{pseudo-ab envelope}$

Pf: Now can write Chow groups motivically as

$$\text{CH}^r(X)_F = \text{Hom}_{\text{Chow}(k)_F}(\mathbb{1}, h(X)(r))$$

set $\alpha \sim 0$ iff $\alpha \in I(\mathbb{1}, h(X)(r))$

This is an F -linear equiv. relation

to show it is a good equiv. rel. check that

$\exists$ representatives that intersect properly

and that if $\alpha \sim 0$ $\gamma_X(\alpha) \sim 0$ under a corresp.

this is OK because start with already taking $\otimes$ -equiv. rel.

need to check:

$$\gamma \in \text{CH}^s(X \times Y) = \text{Hom}(\mathbb{1}, h(X \times Y)(s))$$

$$\Delta_{X \times Y} : X \times Y \rightarrow (X \times Y)^2 \text{ diagonal}$$

$$\gamma_Y : \mathbb{1} \rightarrow h(Y) \text{ given by } [Y]$$

class of Y in Chow

$$\gamma_*(\alpha): \mathbb{1} \longrightarrow h(Y)(r+s-\dim X)$$

$$\parallel$$

$$(\text{pr}_Y^{X \times Y})_* \circ \Delta_{X \times Y}^* \circ (\gamma \otimes \alpha \otimes \gamma_Y)$$

since I is a $\otimes$ -ideal if $\alpha \sim 0$ then $\gamma \otimes \alpha \otimes \gamma_Y \sim 0$
and compositions w/ other morphisms also
 $\Rightarrow \gamma_*(\alpha) \sim 0$

Note: the Lemma above does not hold for effective motives!
need the Tate twists (i.e. not just fact that
 $\otimes$ -category but rigid $\otimes$ -cat.)

Example: $\otimes$ -ideal of morphisms
that factorize through a $\otimes$ -multiple
of the Lefschetz motive
(Kahn, Sujatha "birational motives")

Examples of $I_{\sim}$ ideals

$$1) \quad I_{\sim}^{(M,N)} = \{ f \in \text{Hom}_{\text{Chow}(k)_F}(M, N) \mid \bigcap_{i=0}^{\otimes n} f \in \text{Hom}(M^{\otimes n}, N^{\otimes n}) \}$$

is a $\otimes$ -ideal denoted $\bigcap_{i=0}^{\otimes n}$

induces the Voevodsky smash-nilpotence equivalence

$$2) \quad I_{\sim}(M, N) = \{ f \in \text{Hom}_{\text{Chow}(k)_F}(M, N) \mid \forall g \in \text{Hom}(N, M) \left. \begin{array}{l} \text{Tr}(g \circ f) = 0 \end{array} \right\}$$

$\Rightarrow$ induces numerical equiv. rel.

for $f \in \text{CH}^r(X)_F$ $g \in \text{CH}^{d-r}(X)_F$ $\text{Tr}(g \circ f) \in \text{End}(\mathbb{1}) = F$ is intersection number $\langle f, g \rangle$

(4)

Note: it was proved by Y. Andre' & B. Kahn
(Nilpotence, radicaux et structures monoidales)
that this $I_{\sim}$ (simil (numerical)) is
= largest $\otimes$ -ideal $\mathcal{N}$ distinct from $\text{Chow}(\mathcal{M})_{\mathbb{F}}$ itself

Theorem (Tannaka 1991): F field
the category $\text{Num}(k)_{\mathbb{F}} = \text{Mot}_{\text{num}}(k)_{\mathbb{F}}$
is abelian and semi-simple and $\sim_{\text{num}}$ is the
only equiv. rel.
for which this is true

Semi-simplicity:

for all
objects X

$\text{End}(X) = \text{Hom}(X, X)$
is semi-simple
as K -algebra

$(\Rightarrow)$ an abelian category is semi-simple
if every object has a direct sum
decomposition into finitely many
simple objects

Prop 2.1.2
Andre-Kahn

(and $\forall V, W$ objects $\bigoplus_{i \in I} \text{Hom}(V, X_i) \otimes \text{Hom}(X_i, W)$
 $\downarrow$
isomorphism $\text{Hom}(V, W)$)

An F -linear pseudo-abelian semi-simple category is abelian

Thm
A.2.10
of Andre-Kahn

Abelian category: additive &
all morphisms have Ker & Coker and each
monomorphism = ker of some morphism; each epimorphism = coker of some
(or canonical morphism from $\text{coim}(f)$ to $\text{im}(f)$ is isomorphism)

A more abstract formulation of the theorem
(Andre - Kahn)

(5)

$\mathcal{C}$ a $\otimes$ -category, rigid, over F , with $\text{End}(1) = F$

suppose exists a $\otimes$ -functor, F -linear,

$$H: \mathcal{C} \longrightarrow \mathcal{C}'$$

with $\mathcal{C}'$ rigid $\otimes$ -cat over an extension K of F

s.t. over K Hom's are fin. dim and nilpotent endomorphisms have $\text{tr} = 0$ (eg. true if $\mathcal{C}'$ abelian)

then

(i) the category $(\mathcal{C}/\mathcal{N})^g \leftarrow$ pseudo-ab. envelope is abelian and semi-simple

$\mathcal{N} =$ largest $\otimes$ -ideal

(ii) the only $\otimes$ -ideal I for which $\mathcal{C}/I$ is semi-simple is $I = \mathcal{N}$

$\Rightarrow$ Jannsen's result then applying this to $\mathcal{C} = \text{Chow}(K)_F$ $H =$ Weil cohomology

The original proof of Jannsen of semisimplicity:

by showing that the endom. algebras $\text{End}(M)$ of all objects $M \in \text{Num}(K)_F$ are semisimple F -alg's

using fact that Ker ideal $C_{\text{hom}}^*(X)_F \rightarrow C_{\text{num}}^*(X)_F$ is nilpotent

$f \in \mathcal{N} = \text{Ker}(C_{\text{hom}}^{\dim X}(X^2)_F \rightarrow C_{\text{num}}^{\dim X}(X^2)_F)$ $\forall g$ fg^t nilpotent $\Rightarrow$ (left det formula)
 $0 = \text{Tr}(fg^t | H^+) - \text{Tr}(fg^t | H^-) = \langle f, g \rangle$

⑥ Proof of Andre-Kahn version of Theorem :

(for simplicity, assume $K = F$)

$\text{Ker}(H) \subset \mathcal{N} = \text{largest } \otimes\text{-ideal}$

$\Rightarrow$ can first pass to quotient $\mathcal{C}/\text{Ker}(H)$ (homological equivalence)

— then Hom-spaces are finite dimensional (because so are images under H) ~~that is~~ ← (note fin dim part of properties of Weil cohom)

$\Rightarrow$ the radical of the F -algebra $\mathcal{C}/\text{Ker}(H)$ (= ideal consisting of all elements that annihilate every simple left module (contains all nilpotent ideals and if alg. fin dim. is nilpotent)) is nilpotent

and the quotient by the radical is semi-simple

— then show that any nilpotent ideal I_M of $\text{End}(M)$ is contained in $\mathcal{N}(M, M)$

i.e. $\forall f \in I_M \quad \forall g \in \text{Hom}_{\mathcal{C}/\text{Ker}(H)}(M, M) = \text{End}(M)$

gf is nilpotent $\Rightarrow H(gf)$ also nilpotent

$\Rightarrow \text{tr}(H(gf)) = 0$ by hypothesis

hence also $\text{tr}(gf) = 0 \Rightarrow f \in \mathcal{N}(M, M)$

Using characterization of $\mathcal{N}$ max $\otimes$ -ideal in terms of traces

$\Rightarrow \text{Hom}_{\mathcal{C}/\mathcal{N}}(M, M)$ is semi-simple for all M

this implies $\mathcal{C}/\mathcal{N}$ is semi-simple

$\Rightarrow$ pseudo-abelian envelope $(\mathcal{C}/\mathcal{N})^\natural$ is abelian

to see $\mathcal{N}$ only ideal for which this holds:

(7)

suppose other (smaller) $\mathcal{I}$: take $\mathcal{C}/\mathcal{I}$ and ideal $\mathcal{N}$ then

given $f \in \mathcal{N}(M, N)$ suffices to show $f = 0$

because rigid $\otimes$ -cat can replace N with $M \otimes N$

and assume $M = \mathbb{1} \leadsto f \in \mathcal{N}(\mathbb{1}, N)$

take then $g \in \mathcal{C}/\mathcal{I}(N, \mathbb{1})$ $g \circ f = 0$ (same as tr) by hypothesis that $f \in \mathcal{N}$

but ~~Rare~~ assuming $\mathcal{C}/\mathcal{I}$ is

semi-simple $\leadsto$ then $g \circ f = 0 \quad \forall g$ implies $f = 0$

$\mathcal{V}$ subcategory of $\text{Smproj}(k)$ stable under disj. union and products

F field $\text{char}(F) = 0$

$\text{Mot}_{\sim}(V)_F$ ~~full rigid~~ $\otimes$ -subcategory of $\text{Mot}_{\sim}(k)_F$ formed by motives $M = (X, p, w)$ $X \in \text{Obj}(V)$

H = Weil cohomology coeffs in K extension of F

$\Rightarrow$ faithful $\otimes$ -functor $H^*: \text{Mot}_{\text{hom}}(V)_F \rightarrow \text{slVect}_K$
super vector spaces

If standard conj. $C(X)$ (sign conj.) holds for all X in V

then $M_{\text{hom}}(V)_F$ is $\mathbb{Z}/2\mathbb{Z}$ -graded and H^* compatible w/ $\mathbb{Z}/2\mathbb{Z}$ -grading also compat. w/ $\otimes$ -structure

(8) To apply Tannakian formalism to motives
want $\otimes$ -functor w/ values in Vect_K instead
of slVect_K

This requires modification of
commutativity constraint of $\otimes$ -structure
by Koszul rule

$$(-1)^{ij} c_{MN} : Gr_i(M) \otimes Gr_j(N) \xrightarrow{\sim} Gr_j(N) \otimes Gr_i(M)$$

$ij \in \mathbb{Z}/2\mathbb{Z}$

Denote by $\text{Mot}_{\sim}^+(V)_F$ for $\sim = \sim_{\text{hom}}$ or $\sim = \sim_{\text{num}}$
the category obtained by changing c_{MN} from this way

$$\Rightarrow \chi(M) = \text{Tr}(1_M) = \begin{cases} \sum_i (-1)^i b_i(M) = \text{Euler char} \in \mathbb{Z} \\ \text{in } \text{Mot}_{\text{hom}}^+(V)_F \\ \sum_i b_i(M) \geq 0 \text{ in } \text{Mot}_{\text{hom}}^+(V)_F \end{cases}$$

Then $\text{Mot}_{\text{num}}^+(V)_F$ is Tannakian semi-simple

if ~~dim~~'s of all objects are positive in $\text{Mot}_{\text{hom}}^+(V)_F$
then also in $\text{Mot}_{\text{num}}^+(V)_F$

know this is abelian & semi-simple

then by Deligne's characterization of Tannakian
for $\text{char}(F) = 0$ get Tannakian

Modification of commutativity constraint
via Künneth projectors:

(9)

$$C_{MN}^+ = C_{MN} \circ (e_M \otimes e_N)$$

$$e_M = 2\pi_M^+ - \text{id}_M$$

Next step: from Tannakian to neutral Tannakian
and motivic Galois groups

- Assume that the subcategory $\mathcal{V}$ is chosen so that both $C(X)$ and $D(X)$ hold for all objects of $\mathcal{V}$

then have a faithful fiber functor given by

$$H^*: M_{\text{num}}^+(\mathcal{V})_F = M_{\text{hom}}^+(\mathcal{V})_F \longrightarrow \text{Vect}_F$$

if H^* is Weil cohomology w/ values in F itself

Motivic Galois group $\text{Aut}^{\otimes} \left(H^* \Big|_{M_{\text{num}}^+(\mathcal{V})_F} \right) = \text{Gal}(M_{\text{num}}^+(\mathcal{V})_F)$

Remark:

The fact that category is semi-simple implies G is pro-reductive

E.g. if $K \subset \mathbb{C}$ and $F = \mathbb{Q}$ then
can take as $H^* = H_B^*$ Betti cohomology

(or can use deRham cohomology if $K = F$)

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Given subcategory of V generated by a single object X
(and all sums, products, etc.) V_X

get $\text{Gal}(V_X) = \text{"Mumford-Tate group of } X\text{"}$
(for elliptic curve is MT group)
reductive group

$$\text{Gal}(V) = \varprojlim_{X \in \text{Obj}(V)} \text{Gal}(V_X)$$

there are projection maps $\text{Gal}(V) \rightarrow \text{Gal}(V_X)$
corresponding to the inclusions of categories

$$\text{Mot}_{\text{num}}^+(V_X) \hookrightarrow \text{Mot}_{\text{num}}^+(V)$$