

Kontsevich conjecture: graph polynomials

$$\Psi_{\Gamma}(t) = \sum_{T \in \mathcal{T}} \prod_{e \in T} t_e$$

$$\hat{Y}_{\Gamma} = A^n \setminus X_{\Gamma} \quad n = \#E(\Gamma)$$

- Conjecture: $\hat{Y}_{\Gamma}$ polynomially countable for all Γ

i.e. $N_q(\hat{Y}_{\Gamma}) = \# \hat{Y}_{\Gamma}(\mathbb{F}_q) = P_{\hat{Y}_{\Gamma}}(q)$ polynomial in q $P_{\hat{Y}_{\Gamma}} \in \mathbb{Z}[q]$

- Let $S :=$ saturated multiplicative system in $\mathbb{Z}[q]$ generated by $q^n - q$ for $n > 1$

- $\text{Mot}^+ :=$ coarse version of $K_0(V)$: group generated by $N_q(X)$ $\forall X \in V_{\mathbb{Z}}$
w/ prod: $N_q(X)N_q(Y) = N_q(X \times Y)$

$N_q(A^1) = q$ so Mot^+ is a $\mathbb{Z}[q]$ module

- $\text{Mot} := S^{-1} \text{Mot}^+$ localization at S ; $R = S^{-1} \mathbb{Z}[q]$ known it is a principal ideal domain

- $\text{Graphs} :=$ R -module generated by all $N_q(\hat{Y}_{\Gamma})$ for all Γ

- Stanley's reformulation of the conjecture:

$$Q_{\Gamma} = \sum_{T \in \mathcal{T}} \prod_{e \in T} t_e \quad \text{homog. of deg } n - b_1(\Gamma)$$

$$Z_{\Gamma} := A^n - \{Q_{\Gamma} = 0\}$$

Stanley showed Kontsevich conj $\Leftrightarrow Z_{\Gamma}$ are polynomially countable

(the Z_{Γ} and the $\hat{Y}_{\Gamma}$ generate the same R -mod Graphs)

- advantage of this reformulation: Z_{Γ} 's are easier than $\hat{Y}_{\Gamma}$ when Γ has no looping edges & no multiple edges ^(simple) and Γ has an apex i.e. $v \in V(\Gamma)$ (st. $\forall w \in V(\Gamma)$ $w \neq v \exists e \in E(\Gamma)$ with $\partial(e) = \{v, w\}$)
(there is a vertex w/ an edge to each other vertex)

(2). Given Γ simple graph construct Γ^* by adding an apex v_0

$Z_{\Gamma^*} =$ set of $m \times m$ nondegenerate symmetric matrices with $m = \#V(\Gamma)$

$M_{ij} = 0$ if $i \neq j$ and there's no edge from v_i to v_j .

So Stanley's version of conjecture:

Conj 2: all Z_{Γ^*} are polynomially countable

(Would follow from Kontsevich conjecture)

• $\gamma \subset \Gamma$ subgraph $\Gamma \setminus \gamma =$ removal of all edges of γ keeping all vertices same

$$(\Gamma \setminus \gamma)^* = \Gamma^* \setminus \gamma$$

• each simple graph Γ with $m = \#V(\Gamma)$ contained in complete graph $\Gamma \subset K_m$

define $\Gamma^0 = K_m \setminus \Gamma$ complement

• $(\Gamma^0)^* = (D\Gamma)^0$ where $D\Gamma = \Gamma$ with added disjoint vertex

• set $\hat{Z}_{\Gamma} := Z_{(\Gamma^0)^*}$ is the scheme of $m \times m$ matrices M with $M_{ij} = 0$ if $\exists$ edge between v_i & v_j

• $\text{Graphs}_* = \mathbb{R}\text{-mod gen. by } N_q(\hat{Z}_{\Gamma})$ $\text{Graphs}_* \subset \text{Graphs}$

Theorem (Belkale-Brosnan): $\text{Graphs}_* = \text{Mot}$

this disproves Conj. 2 and disproves Kontsevich conj.

Steps: incidence schemes $A_{\Gamma}(s, r, k)$ with the property that

$$\text{if } m = \#V(\Gamma) \quad N_q(A_{\Gamma}(m, m, m)) = N_q(\hat{Z}_{\Gamma}) \cdot N_q(GL_m)$$

since $N_q(GL_m) \in \mathcal{R} \Rightarrow N_q(A_r(m, m, m)) \in \text{Graphs}_*$

then using relations between the $A_r(s, r, k)$ for varying s, r, k

- show that $\mathcal{R}$ -mod generated by $N_q(A_r(s, r, k)) = \text{Graphs}_*$
- show that for $J_r(s) = \bigcup_k A_r(s, s, k)$ $\mathcal{R}$ -mod gen. by $N_q(J_r(s))$ is also Graphs_*

- then compare the schemes $J_r(s)$ to representation spaces of matroids

M matroid $\rightsquigarrow X(M, s)$ scheme w/ pts.
 $X(M, s)(K) = \# \text{ rep's of } M \text{ in } K^s$

Matroids := $\mathcal{R}$ -mod gen. by $N_q(X(M, s))$ for all M and s

- relation between $X(M, s)$ and $J_r(s)$:

$\forall M$ matroid $\exists$ ^{finite} family Γ_i of graphs s.t.

$$N_q(X(M, s)) = \sum_i a_i N_q(J_{\Gamma_i}(s)) \quad a_i \in \mathcal{R}$$

$\Rightarrow \text{Matroids} \subset \text{Graphs}$

- Mnëv's universality theorem for matroids

$\Rightarrow \text{Matroids} = \text{Mot}$

($\Rightarrow \text{Graphs} = \text{Mot}$ proving theorem)

More detailed view of proof:

Lemma: subgroup of Mot^+ gen. by $N_q(\hat{Y}_r)$ same as

subgr. gen. by $N_q(Z_r)$

(4)

$S \subset E(\Gamma)$ subset of edges

$$\mathbb{A}^{\#S} \hookrightarrow \mathbb{A}^{\#E(\Gamma)}$$

$$n = \#E(\Gamma)$$

$$\mathbb{G}_m^{\#S}$$

as S varies over subsets of $E(\Gamma)$

these schemes stratify $\mathbb{A}^{\#E(\Gamma)}$

Given $X \subset \mathbb{A}^n$ and $S \subset E(\Gamma)$ let $X_S = X \cap \mathbb{A}^{\#(E(\Gamma) \setminus S)}$

$$X_S^+ = X \cap \mathbb{G}_m^{\#(E(\Gamma) \setminus S)}$$

X_S = intes of X w/ hyperplanes defined by equations $t_e = 0$ for $e \in S$

$X_\emptyset = X$ and X_S^+ stratify X :

$$N_q(X_S) = \sum_{T \supset S} N_q(X_T^+) \quad \text{and using inclusion-exclusion}$$

$$N_q(X_S^+) = \sum_{T \supset S} (-1)^{\#(T \setminus S)} N_q(X_T)$$

Take polynomials Q_Γ : one has $Z_{\Gamma, S} = Z_{\Gamma \setminus S}$ and

$$Z_{\Gamma, S}^+ = Z_{\Gamma \setminus S}^+$$

and dually

$$(\text{with } S \text{ forest}): \quad \hat{Y}_{\Gamma, S} = \hat{Y}_{\Gamma/S} \quad \hat{Y}_{\Gamma, S}^+ = \hat{Y}_{\Gamma/S}^+$$

Γ/S = graph obtained by contracting all edges in S

(if S not forest $\hat{Y}_{\Gamma, S}$ empty)

$$Q_\Gamma(t) = \Psi_\Gamma\left(\frac{1}{t}\right) \prod_{e \in \Gamma} t_e$$

so Z_Γ is the "Cremona transf. surface" that would corresp. to Ψ_Γ in planar case

Using previous fact on stratification:

$$N_q(\hat{Y}_T) = \sum_{\substack{S \subseteq E(T) \\ b_1(S)=0}} \sum_{T \subseteq \mathcal{AT}/S} (-1)^{\#E(T)} N_q(\hat{Z}_{T/S, T})$$

$$N_q(Z_T) = \sum_{S \subseteq E} \sum_{\substack{T \subseteq E \setminus S \\ b_1(T)=0}} (-1)^{\#E(T)} N_q(\hat{Y}_{(T, S)^\vee})$$

these suffices to prove that ~~Graphs~~ $\subset \text{Mot}^+$ span of $N_q(\hat{Y}_T)$ same as span of $N_q(Z_T)$

$\hookrightarrow \text{Graphs} = R\text{-submod of Mot spanned by } N_q(Z_T)$

(Note: R itself sits as submod of Graphs as T_n polygon $N_q(T_n) = q^n - q^{n-1}$)

Lemma:

$R = S^{-1}\mathbb{Z}[q]$ if $f \in R$ satisfies $f(q) \in \mathbb{Z} \quad \forall q=p^r \quad \forall p \text{ primes}$
then $f \in \mathbb{Z}[q]$

Pf: $f = \frac{a}{s} \quad a \in \mathbb{Z}[q], s \in S$

s monic so $f = d + \frac{r}{s}$
(Euclidean algorithm)

if $\frac{r(q)}{s(q)} \in \mathbb{Z} \quad \forall q \iff d, r \in \mathbb{Z}[q] \quad \deg(r) < \deg(s)$
must be $r=0 \Rightarrow f=d$

Determinantal schemes

$S = \text{Spec } A$ affine scheme

$T = \text{Spec } B$

set of maps $\phi: B^e \rightarrow B^f$ s.t. $\text{coker } \phi$ projective B -module
rank $f-r$

- ⑥ coker of condition same as requiring every $(r+1) \times (r+1)$ minor of ϕ vanishes but some $r \times r$ minor does not vanish (is invertible)

represent as an exf matrix $(y_{ij})_{\substack{i=1, \dots, e \\ j=1, \dots, f}}$

$A[y]$ polyn. ring in these variables

for a given k : $\{m_\ell^k\}_\ell = \text{list of all the } k \times k \text{ minors}$

then $I_k = \text{ideal generated by the } \{m_\ell^k\}$

~~$\text{Hom}_r(\mathcal{O}_S^e, \mathcal{O}_S^f)$~~

~~is defined by~~ $\Rightarrow$ scheme $\text{Hom}_r(\mathcal{O}_S^e, \mathcal{O}_S^f)$

~~whose points are~~

whose points are

$$\bigcap_i Z(m_i^{r+1}) \setminus \bigcap_i Z(m_i^r)$$

E, F $\mathcal{O}_S$ -modules:

$\text{Hom}_{\mathcal{O}_S, \leq r}(E, F)$ set of maps $\phi \in \text{Hom}_{\mathcal{O}_S}(E, F)$ s.t.
 $\text{rk coker}(\phi) \leq r$ at every pt $x \in S$

and

$$\text{Hom}_{\mathcal{O}_S, r}(E, F) = \text{Hom}_{\mathcal{O}_S, \leq r}(E, F) \setminus \text{Hom}_{\mathcal{O}_S, \leq r-1}(E, F)$$

map to Grassmannians:

$p: \text{Hom}_r(E, F) \rightarrow \text{Gr}(r, F)$ mapping ϕ to its image (coker?)

$q: \text{Hom}_r(E, F) \rightarrow \text{Gr}(e-r, E)$ mapping ϕ to kernel

in particular: $S = \text{Spec } \mathbb{Z}$ $E = \mathcal{O}_S^e$ $F = \mathcal{O}_S^f$ (7)

$$\text{Hom}(E, F) = \text{Spec } \mathbb{Z}[y] \quad y = (y_{ij})$$

$\text{Hom}_{\leq r}(E, F) \subset \text{Hom}(E, F)$ closed subscheme defined by $(r+1) \times (r+1)$ minors

$\text{Hom}_r(E, F)$ Zariski open subset of $\text{Hom}_{\leq r}(E, F)$ defined by requiring at least one $r \times r$ minor invertible

$\text{Sym } E$ closed subscheme of $\mathbb{A}^{\binom{e+f}{2}}(\mathbb{Z}[y])$ def by equations $y_{ij} = y_{ji}$

$\text{Sym}_{\leq r} E$ " " $(r+1) \times (r+1)$ minors

$\text{Sym}_r(E)$ Zariski open: $\exists r \times r$ minor invertible

K -pts of $\text{Sym}_r(E)$ are bilinear forms on K^e of $\text{rk} = r$

Some counting functions:

$$N_q(\text{GL}_n) = (q^n - 1)(q^n - q) \cdots (q^n - q^{n-1})$$

$$N_q(\text{GL}(a, b)) = N_q(\text{GL}_b) / (N_q(\text{GL}_a) \cdot N_q(\text{GL}_{b-a}) \cdot q^{a(b-a)})$$

$$N_q(\text{Hom}_r(E, F)) = N_q(\text{Gr}(r, e)) \cdot N_q(\text{Gr}(r, f)) \cdot N_q(\text{GL}_r)$$

all polynomially countable

The matrix-tree theorem

Already know for graph polynomial that

$$\chi_r(t) = \det M_r(t)$$

$$M_r(t) = \sum_{\substack{t \\ r}} 1(t) \downarrow_r$$

graph circuit matrix

⑧ but also for polynomial Q_Γ have a determinant interpretation

$$Q_\Gamma = \det(L_\Gamma^0)$$

where L_Γ^0 = reduced generic Laplacian of Γ

• rewrite the edge variables t_e of Γ as x_{ij} where
(assuming simple graph) $e \in E(\Gamma)$ with $\partial(e) = \{v_i, v_j\}$

$$i, j = 1, \dots, m = \#V(\Gamma)$$

setting $x_{ij} = 0$ when
no edge between v_i & v_j

• Laplacian (generic Laplacian)

$$L = L_{ij} \quad m \times m \text{ matrix}$$

$$L_{ij} = \begin{cases} \sum_{k=1}^m x_{ik} & i=j \\ -x_{ij} & i \neq j \end{cases}$$

L^0 is ~~the~~ obtained from matrix L by
removing the first row & first column
↑ reduced Laplacian

Stanley proved that matrix-tree theorem implies

$$\mathbb{Z}_p^* \cong \text{set of } m \times m \text{ non-degen. symm. matrices} \\ \text{w/ } M_{ij} = 0 \text{ if } i \neq j \text{ and no } e_{ij} (x_{ij} = 0)$$

then $\hat{Z}_\Gamma =$ subscheme of Sym_m^m defined by
 equations $y_{ij} = 0 \quad \forall i, j \text{ s.t. } \exists e_{ij}$
 (and gives also $\hat{Z}_\Gamma = Z_{(\Gamma^0)^*}$ mentioned before)

Incidence schemes : $S = \text{Spec}(\mathbb{Z})$

W loc. free $\mathbb{Q}$ -mod Γ graph

$$A_\Gamma(W) \subset \text{Sym}(W) \times \text{Fun}(V(\Gamma), W) \text{ closed subscheme}$$

$$\{ (Q, f) \mid Q(f(v), f(w)) = 0 \text{ if } e_{vw} \in E(\Gamma) \}$$

($E(\Gamma)$ viewed as a subset of $\text{Sym}^2(V(\Gamma))$)
 $v, w \in V(\Gamma) \quad e_{vw} \in E(\Gamma) \text{ if } \exists \text{ edge between them}$

for any $r, k \in \mathbb{N}$

$$A_\Gamma(W, r, k) = A_\Gamma(W) \cap (\text{Sym}_r(W) \times \text{Fun}_k(V(\Gamma), W))$$

if $W = \mathbb{Q}_S^s$ write $A_\Gamma(s, r, k)$

functions whose
 span in W is
 of $\dim = k$

$A_\Gamma(W, r, k)$ stratification of $A_\Gamma(W)$ by loc. closed subschemes

$$(A_\Gamma(s, r, k) = \emptyset \text{ unless } 0 \leq k \leq n \text{ and } 0 \leq r, k \leq s)$$

$$A_\Gamma(s, r, 0) = \text{Sym}_r^s \quad A_\Gamma(W, 0, k) = \text{Fun}_k(V(\Gamma), W)$$

↑ polym. countable:

$$N_q(A_\Gamma(s, r, 0)) \text{ and } N_q(A_\Gamma(s, 0, k)) \in \mathbb{Z}[q]$$

(10)

Prop:

$$A_P(m, m, m) = \hat{Z}_P \times GL_m$$



Graphs* = R-mod gen. by $N_q(A_P(m, m, m))$
 (since $N_q(GL_m) \in S$)

$$W = \mathbb{O}_S^m \quad S = \text{Spec } \mathbb{Z} \Rightarrow \text{Fun}_r(\text{Vect}, W) \cong GL_m$$

Then $(Q, f) \mapsto (f^t Q f, f)$ identifies

$$A_P(m, m, m) \text{ with } \hat{Z}_P \times GL_m$$

~~Also proven argument shows $A_P(m, k, m) \cong \hat{Z}$~~

To find relations between the $A_P(s, r, k)$
 need to use a general fact about extending
 a bilinear form Q on $\mathbb{F}_q^{d_1}$ of rank r_1
 to a form of $\mathbb{F}_q^{d_2}$ rk r_2

$$C_Q(d_2, r_2, d_1, r_1) = \# \text{ ways of doing that}$$

$$\text{Prop: } C_q(d_1+1, r_2, d_1, r_1) = \begin{cases} q^{r_1} & r_2 = r_1 \\ q^{d_1+1} - q^{r_1} & r_2 = r_1 + 1 \\ q^{d_1+1} - q^{r_1+1} & r_2 = r_1 + 2 \\ 0 & \text{otherwise} \end{cases}$$

(MacWilliams
 Oring. matrices
 over finite fields
 '69)

$$C(d_2, r_2, d_1, r_1) = \sum_{j=0}^2 C(d_1, r_2, d_1+1, r_1+j) C(d_1+1, r_1+j, d_1, r_1)$$

$\Rightarrow$ hence $C(d_2, r_2, d_1, r_1)$ polynomial in q
 $\neq 0$ iff $d_2 \geq r_2$ $d_1 \geq r_1$ $0 \leq r_1 \leq r_2 \leq r_1 + 2(d_2 - d_1)$

Then relations between the $A_P(s, r, k)$

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$$N_q(A_P(s, r, k)) = N_q(\text{Gr}(k, s)) \cdot \sum_j C(s, r, k, j) N_q(A_P(k, j, k))$$

reduces to
combination of
simpler ones

Pf: $W = \mathbb{F}_q^s$

$\text{few}^{V(r)} \langle f \rangle := \text{span of } f(v_i)$

$(Q, f) \mapsto \langle f \rangle$ fibers set $A_P(s, r, k)$ over $\text{Gr}(k, s)$

w/ fiber over a subspace $U \subset W$ given by set

$A_P(s, r, U)$ of $(Q, f) \in A_P(s, r, k)$ s.t. $\langle f \rangle = U$

• GL_s acts transitively on $\text{Gr}(k, s) \Rightarrow$ fibers all with same # of pts

$$\Rightarrow \#(A_P(s, r, k)) = \#(\text{Gr}(k, s)) \cdot \#(A_P(s, r, U))$$

now to compute this take

• $A_G(s, r, j) = \{(Q, f) \in A_P(s, r, U) \mid Q|_U \text{ rank } j\}$

disjoint subsets decomposition

• $P_U: A_P(s, r, U, j) \rightarrow A_P(U, j, k)$

$(Q, f) \mapsto (Q|_U, f)$

fiber above pt $(\bar{Q}, f)$ has cardinality $C(s, r, k, j)$

$$\Rightarrow N_q(A_P(s, r, U, j)) = C(s, r, k, j) N_q(A_P(k, j, k))$$

Another identity proved similarly

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$$N_q(A_P(s, r, s)) = N_q(G_P(m-s, m-r)) N_q(G_{L_{s-r}})$$

$$r \leq s \quad q^{r(s-r)} N_q(A_P(r, r, r))$$

Also relation between incidence schemes for Γ and for $D\Gamma = \Gamma$ w/ added disjoint vertex

$$N_q(A_{D\Gamma}(s, r, k)) = q^k N_q(A_P(s, r, k)) + (q^k - q^{k-1}) N_q(A_P(s, r, k-1))$$

Since span $f|_{V(\Gamma) \subset V(D\Gamma)}$ within k or $k-1$ dim subspace
in each case count # of possibilities

Set $M(\Gamma) = R$ -submodule of Mat gen. by $N_q(A_P(s, r, k))$
has finite filtration $M(\Gamma) = M(\Gamma)_m \supset M(\Gamma)_{m-1} \supset \dots \supset M(\Gamma)_0 = R$
 $M(\Gamma)_t$ submod gen by $N_q(A_P(s, r, k))$ with $s \leq t$

Want to show $M(\Gamma) = \text{Graphs}_*$

Notation: $K_P(s) = A_P(s, s, s)$

$$J_P(s) = \bigcup_k A_P(s, s, k)$$

$$H_P(s) = A_P(m, s, m) \quad (K_P(m) = H_P(m))$$

← scheme of all $(Q, f) \in A_P(s)$
w/ $Q \in \text{Sym}_s^s$ no restriction
on rank
of f

• first result: $N_q(A_P(s, r, k)) \in M(\Gamma)_d$ $d = \min(s, r, k)$

$M(\Gamma)_t$ spanned by $N_q(K_P(s))$ $s \leq t$

" " " $N_q(J_P(s))$ $s \leq t$

" " " $N_q(H_P(s))$ $s \leq t$

$$N_q(J_{DP}(s)) = q^s N_q(J_P(s)) \quad (*)$$

because map $J_{DP}(s) \rightarrow J_P(s)$ restricting f from $V(DP)$ to $V(P)$ has fibres $\mathbb{F}_q^s$

$$N_q(H_{DP}(r)) = a_r(r) N_q(H_P(r)) + b_r(r) N_q(H_P(r-1)) + c_r(r) N_q(H_P(r-2)) \quad (**)$$

where

$$\left. \begin{aligned} a_r(r) &= q^{m_r+r} (q^{m_r+1} - 1) \\ b_r(r) &= q^{m_r+r-1} (q^{m_r+1} - 1) (q-1) \\ c_r(r) &= q^{m_r} (q^{m_r+1} - 1) (q^{m_r+1} - q^{r-1}) \end{aligned} \right\} \text{all in } S$$

from relation between A_{DP} and A_P

$$\bullet \text{ then also } M(DP) = M(P) \text{ (e.g. from } (*))$$

$$\bullet M(P) \text{ R-mod gen. by } N_q(\hat{\sum}_{D(P)}) \text{ hence } M(DP) \subset \text{Graphs}_*$$

$$\text{Use } N_q(K_P(m_P)) = N_q(\hat{\sum}_P) N_q(GL_{m_P})$$

$$\Rightarrow M(P) \text{ hence span of } \downarrow \subset M(DP)$$

for converse direction inductive argument using $(**)$
starting w/ $N_q(H_P(m_P)) = N_q(K_P(m_P))$

MATROID THEORY

Matroid: Set $E = E(M)$ "edges" and rank function

$$p: 2^E \rightarrow \mathbb{N} \text{ satisfying}$$

$\{ \text{Subsets of } E \}$

$$1) \rho(E) \leq \#E$$

$$(\rho(E) = \text{rk of matroid } M = (E, \rho))$$

$$2) \text{ for } X \subset Y \subset E \Rightarrow \rho(X) \leq \rho(Y)$$

$$3) \forall X, Y \subset E \quad \rho(X \cup Y) + \rho(X \cap Y) \leq \rho(X) + \rho(Y)$$

Simultaneous generalization of graphs & matrices

• Representability: a matroid M is representable over a field K if $\exists$ function $f: E \rightarrow K^r$ s.t.

$$\dim \text{span}(f(X)) = \rho(X) \quad \text{for all } X \subset E$$

• Matroids from matrices:

given $E \subset K^S \quad \exists M$ matroid (repres. over K)

given by setting $\rho(X) = \dim \text{span}(X) \quad \forall X \subset E$

$$K^{n+1} \setminus \{0\} \rightarrow \mathbb{P}^n(K) \quad E \subset \mathbb{P}^n(K)$$

any lift $\sigma(E) \subset K^{n+1}$ defines matroid (indep. of choice)

• Representation schemes of matroids

M matroid W loc. free sheaf over some S (will take $S = \text{Spec } \mathbb{Z}$)

$X(M, W)$ subscheme of $\text{Fun}(E, W)$

all f s.t. restriction to X is $\text{Fun}_{\rho(X)}(X, W)$

for all $X \subset E$

$\uparrow$ $\dim$ of span

• for $S = \text{Spec } \mathbb{Z}$ $W = \mathbb{Q}^S$ $X(M, S) = X(M, W)$

(15)

• points over K $X(M, S)(K) = \{ f: E \rightarrow K^S \mid \dim \text{span } f(X) = \rho(X) \}$
 $\forall X \subseteq E$

• Matroids = R -module generated by $N_q(X(M))$
 $\parallel$
 $X(M, \text{rk}(M))$

• Note: $N_q(X(M, S)) = N_q(\text{Gr}(r, S)) N_q(X(M))$

• Next show: Matroids $\subset$ Graphs*

set of subsets

$2^{V(r)}$

$\pi: \text{Dom}(\pi) \rightarrow \mathbb{N}$

• Γ graph $V(\Gamma)$ set of vertices

$J_r(s, \pi) = \{ (Q, f) \in J_r(s) \mid f|_H \text{ has rank } \pi(H) \text{ for all } H \in \text{Dom}(\pi) \}$

• then $N_q(J_r(s, \pi)) \in \text{Graphs}_*$

(induction on cardinality of $\text{Dom}(\pi)$)
 express $N_q(J_r(s, \pi)) = \sum_i q^{t_i} N_q(J_r(s, \pi_i))$
 adding disj. vertices w/ edges only to vertices in subset H

• Next step: Γ w/ m vertices and no edges

then $N_q(J(s, \pi)) = N_q(\text{Sym}_S^r)$

$\uparrow$
Graphs*

$\uparrow$
 S

$N_q(L(s, \pi))$

$\uparrow$
all $f \in \text{Fun}(V(\Gamma), \mathbb{Q}_{\text{Spec } \mathbb{Z}}^S)$
 $f \text{ rk } \pi(H) \text{ on all } H \in \text{Dom}(\pi)$

$\Rightarrow$ also in Graphs_*

- Matroids $\subset$ Graphs_{*} from last
 because if M matroid w/ $\rho: 2^E \rightarrow \mathbb{N}$
 $X(M, \rho) = L(\rho, \rho)$ ~~etc~~

(16)

Last step: Matroids = Mot (hence Graphs_{*} = Mot)

theorem of (Mn̄ev) & (Bokowski & Strumfels) (indep.)

$\forall X$ quasi-proj. scheme finite type / $\mathbb{Z}$

$\exists$ set V ; set^{w/} of subsets ~~of~~ of V ;
 function $\pi: W \rightarrow \mathbb{Z}$; element $\sigma \in S$

s.t. $\sigma \cdot N_q(X) = N_q(L(\mathbb{Z}, \pi))$

Universality of matroids