

5) Lefschetz type standard conjecture

①

Polarization: $D \in CH^1(X)$ class of an ^{ample} divisor

$\gamma = \gamma_X^1(D)$ image in $H^2(X)(1)$ = polarization

Lefschetz operator $L = L_\gamma$ = cup product with γ

strong Lefschetz theorem

$$L^{d-i}: H^i(X)(r) \rightarrow H^i(X)(d-i+r)$$

is an isomorphism $\forall i \leq d$ and all $r \in \mathbb{Z}$
for H^* a Weil cohom. (one of the classical cohomologies)

where $\gamma_X^r: CH^r(X) \rightarrow H^r(X)(r)$ ^{to keep weights into account}

sl_2 -triplets V = finite dim. K -vector space

(x, h, y) sl_2 -triplet = image in $\text{End}_K(V)$ of
triplet $\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right)$

through a representation of Lie algebra sl_2 on V

i.e. relations $[h, x] = 2x, [h, y] = -2y, [x, y] = -h$

$\Rightarrow x, y$ nilpotent and y determined completely by (x, h)

primitive decomposition of V : $V = \bigoplus_{i,j} x^i P_{r,j}^y$ where

$P_{r,j}^y$ = inters. of Ker of y and
eigenspace of h w/ eigenvalue j

$$x^i: Gr_i^X \xrightarrow{\sim} Gr_{-i}^X$$

canonical filtration

$$F_i^X := \{ v \in V \mid h(v) = jv \}$$

$$x(F_i^X) \subset F_{i-2}^X$$

$$Gr_i^X \simeq \text{Eigenspace of } h \text{ } \lambda = -i$$

Lefschetz involution $*_{L, (x, h)} \in \text{Aut}_k(V)$
 is x^v on Gr_i^x for $v \geq 0$
 or x^{-i} for $i < 0$

Strong Lefschetz theorem in terms of sl_2 -triplets:

L nilpotent endomorphism of $H(X)$

$\exists!$ representation of sl_2 lie alg on $H(X)$ s.t.

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto L \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \sum_{i=0}^{2d} (i-d) \pi_X^i$$

- $H^i(X)$ is eigenspace of h w/ eigenvalue $d-i$
- Pr_{d-i}^y gives primitive cohomology $P_i^y(X)$ s.t. that

$$H(X) = \sum_{0 \leq i, j \leq d} L^j P_i^j(X)$$

Standard conjecture of Lefschetz type $B(X)$:

The Lefschetz involution is given by an algebraic correspondence (with rational coeff's)

- the conjecture is stable under products $X \times X'$ using the product polarization $\eta \otimes [X'] + [X] \otimes \eta'$
- Known to hold in $\dim \leq 2$; for abelian varieties.

Relations between standard conj's

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$$D(X^2) \Rightarrow B(X) \Rightarrow C(X)$$

$$B(X) + I(X) \Rightarrow D(X)$$

$$B(X) \Rightarrow B(X^n) \quad \forall n \geq 1$$

Main application of standard conjectures so far:
(conditional) motivic Galois groups

Tannakian categories and the Tannakian formalism

Affine group schemes and commutative Hopf algebras

H unital commutative algebra over $\mathbb{K}$

coproduct $\Delta: H \rightarrow H \otimes H$ homom. of $\mathbb{K}$ -alg's

counit: $\varepsilon: H \rightarrow \mathbb{K}$

antipode: $S: H \rightarrow H$ anti-homom.

$$(\Delta \otimes \text{id}) \Delta = (\text{id} \otimes \Delta) \Delta$$

$$(\text{id} \otimes \varepsilon) \Delta = \text{id} = (\varepsilon \otimes \text{id}) \Delta$$

$$m(\text{id} \otimes S) \Delta = m(S \otimes \text{id}) \Delta = 1 \cdot \varepsilon$$

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$$\text{Hom}_{\text{Alg}_F}(\mathcal{H}, A) =: G(A)$$

commutative alg's over F

$$\phi: \mathcal{H} \rightarrow A \quad \phi(xy) = \phi(x)\phi(y) \quad \phi(1) = 1$$

Product in $G(A)$:

$$(\phi_1 * \phi_2)(x) = \langle \phi_1 \otimes \phi_2, \Delta(x) \rangle$$

$$\text{inverse in } G(A) \quad \phi^{-1}(x) = (\phi \circ S)(x)$$

$$\text{unit in } G(A) \quad \varepsilon: \mathcal{H} \rightarrow F \rightarrow A$$

G affine group scheme

$G(A)$
group

A -points of G

$$G = \text{Hom}_{\text{Alg}}(\mathcal{H}, \cdot)$$

covariant representable functor $\text{Alg}_F \rightarrow \text{Groups}$

Examples: $G = G_a$ additive grp $\mathcal{H} = F[t]$
 $\Delta(t) = t \otimes 1 + 1 \otimes t$

$G = G_m$ multpl. grp $\mathcal{H} = F[t, t^{-1}]$
 $\Delta(t) = t \otimes t$

$G = GL_n$ $\mathcal{H} = F[x_{ij}, t]_{i,j=1,\dots,n}$
 $\Delta(x_{ij}) = \sum_k x_{i,k} \otimes x_{k,j}$
 $(\det(x_{ij})t - 1)$

$\text{Rep}_F G$ fin. dim F -linear representations of G

$\cong$ comodules finite $\dim_F$ for coalg. structure on $\mathcal{H}$

algebra structure of $\mathcal{H}$ manifests itself
in monoidal str. of category $\otimes: \text{Rep}_F G \times \text{Rep}_F G \rightarrow \text{Rep}_F G$

Tensor category: F now just commut. ring

$\mathcal{C}$ F -linear category with

- a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$
- a unit object 1
- functorial isomorphisms

$$a_{LMN}: L \otimes (M \otimes N) \xrightarrow{\sim} (L \otimes M) \otimes N$$

$$c_{MN}: M \otimes N \xrightarrow{\sim} N \otimes M \text{ with } c_{MN}^{-1} = c_{NM}$$

$$u_M: M \otimes 1 \xrightarrow{\sim} M \quad u'_M: 1 \otimes M \xrightarrow{\sim} M$$

with compatibility conditions expressed by diagrams

Pentagon diagram:

$$\begin{array}{ccccc} (A \otimes B) \otimes C \otimes D & \xrightarrow{a_{ABC} \otimes 1_D} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{a_{A, B \otimes C, D}} & A \otimes ((B \otimes C) \otimes D) \\ \downarrow a_{A \otimes B, C, D} & & & & \swarrow 1_A \otimes a_{BCD} \\ (A \otimes B) \otimes (C \otimes D) & \xrightarrow{a_{A, B, C \otimes D}} & A \otimes (B \otimes (C \otimes D)) & & \end{array}$$

Triangle diagram

$$\begin{array}{ccc} (A \otimes 1) \otimes B & \xrightarrow{a_{A, 1, B}} & A \otimes (1 \otimes B) \\ u_{A \otimes B} \searrow & & \swarrow 1 \otimes u'_B \\ A \otimes B & & A \otimes B \end{array}$$

Note: monoidal category
if only with
 a & u, u' no c

Hexagonal diagrams :

$$\begin{array}{ccc}
 A \otimes (B \otimes C) & \xrightarrow{c} & (B \otimes C) \otimes A \\
 \nearrow a & & \searrow a \\
 (A \otimes B) \otimes C & & B \otimes (C \otimes A) \\
 \searrow c \otimes 1 & & \nearrow 1 \otimes c \\
 (A \otimes B) \otimes C & \xrightarrow{a} & B \otimes (A \otimes C)
 \end{array}$$

$$\begin{array}{ccc}
 (A \otimes B) \otimes C & \xrightarrow{c} & C \otimes (A \otimes B) \\
 \nearrow a^{-1} & & \searrow a^{-1} \\
 A \otimes (B \otimes C) & & (C \otimes A) \otimes B \\
 \searrow 1 \otimes c & & \nearrow c \otimes 1 \\
 A \otimes (C \otimes B) & \xrightarrow{a^{-1}} & (A \otimes C) \otimes B
 \end{array}$$

Coherence diagram :

$$\begin{array}{ccc}
 A \otimes \mathbb{1} & \xrightarrow{c} & \mathbb{1} \otimes A \\
 \searrow u & & \swarrow u' \\
 & A &
 \end{array}$$

Note : without requiring

with $c_{NM} = c_{MN}^{-1}$
symmetric

is braided monoidal category
 (tensor category)

Rigid Tensor category :

Additional structure : duality

$$v: \mathbb{1} \rightarrow \mathbb{1}^{\text{op}}$$

s.t. $- \otimes M^{\vee}$ is left adjoint of $- \otimes M$
 and $M^{\vee} \otimes -$ is right adjoint of $M \otimes -$

i.e. there are adjunction morphisms

$$\varepsilon: M \otimes M^{\vee} \rightarrow \mathbb{1} \quad \text{and} \quad \eta: \mathbb{1} \rightarrow M \otimes M^{\vee}$$

evaluation and coevaluation

$$\varepsilon \otimes 1_M \circ 1_M \otimes \eta = 1_M \quad \text{and} \quad 1_{M^{\vee}} \otimes \varepsilon \circ \eta \otimes 1_{M^{\vee}} = 1_{M^{\vee}}$$

In a rigid tensor category
all $f \in \text{End}(M)$ have a categorical trace

$$\mathbb{1} \xrightarrow{f} M^\vee \otimes M \xrightarrow{c} M \otimes M^\vee \xrightarrow{\varepsilon} \mathbb{1}$$

" $\dim(M) = \text{Tr}(1_M) = \varepsilon \circ c_{M^\vee M} \circ \gamma \in \text{End}(\mathbb{1}) (\cong F)$
or $\chi(M)$
Euler char.

$$\underline{\text{Hom}}(X, X) \cong X^\vee \otimes X$$

(in a rigid tensor category
 $\underline{\text{Hom}}(X, Y)$ exists $\forall X, Y$
given by $X^\vee \otimes Y$ with

$f \in \text{Hom}(M, M)$ seen as

$$f \in \text{Hom}(\mathbb{1}, \underline{\text{Hom}}(M, M)) = \text{Hom}(\mathbb{1}, M^\vee \otimes M)$$

$$\text{Hom}(\mathbb{1}, \underline{\text{Hom}}(X, Y)) = \text{Hom}(X, Y) \quad (= \text{Hom}(\mathbb{1} \otimes X, Y))$$

A functor $\otimes$ -functor between $\otimes$ -categories
is an F -linear functor

$$\omega: \mathcal{C} \rightarrow \mathcal{C}'$$

together w/ a collection of functorial isom's

$$\phi_{MN}: \omega(M \otimes N) \xrightarrow{\sim} \omega(M) \otimes \omega(N)$$

$$\phi_{\mathbb{1}}: \omega(\mathbb{1}) \xrightarrow{\sim} \mathbb{1}'$$

compatible w/ a, c, u, u' morphisms

for rigid $\phi_{M, N}$ compatible w/ duality

Tannakian categories

F field

$\otimes$ -category $\mathcal{C}$

assume $\mathcal{C}$ abelian

also assume $\text{End}(1) = F$

K an extension of F

A fiber functor

$$\omega: \mathcal{C} \rightarrow \text{Vect}_K$$

is an exact

faithful

tensor functor

- if such a functor exists then $\mathcal{C}$ is Tannakian
- if it exists with $K = F$ it is neutral Tannakian

$\Downarrow$

Affine group scheme

$$G = \text{Aut}^{\otimes}(\omega)$$

group of invertible natural transformations of the functor ω

$\exists$ equivalence of categories (neutral case)

$$\mathcal{C} \xrightarrow{\omega} \text{Rep}_F(G)$$

determined by the fiber functor ω

Note:

G depends on $\mathcal{C}$ and ω (not only on $\mathcal{C}$)

Deligne's characterization of Tannakian categories (10)

Theorem (Deligne 1990)

$\mathcal{C}$ rigid $\otimes$ -category over a field F
 $\text{char}(F) = 0$; $\mathcal{C}$ abelian with $\text{End}(1) = F$

then $\mathcal{C}$ is Tannakian iff the $\chi(M)$ of all objects of $\mathcal{C}$ is in $\mathbb{N}$ $\text{Tr}(1_M)$
(positive integer)

Making motives into a Tannakian category

Steps

- 1) Abelian category (unconditional)
Jannsen's result on numerical equivalence
- 2) Deligne's criterion fails $\chi(M) = \text{Euler characteristic}$
can be negative
- 3) Correct commutativity constraint
(conditional) so Deligne's criterion holds
- 4) Explicit fiber functor (conditional)
and neutral Tannakian

$\Rightarrow$ Motivic Galois group $\text{Gal}(\text{Num}^+(\mathbb{K})_{\mathbb{Q}})$