

Grothendieck classes of graph hypersurfaces

Apr 9

①

QPT4 $K_0(\mathcal{V}_k)$ Grothendieck ring of varieties
field $(\mathbb{Q})$ (or $\mathbb{Z}$)

gen. by isom. classes $[X]$ of varieties (smooth q -proj. ; or arbitrary)
with relations $Y \hookrightarrow X$ closed embedding

$$[X] = [Y] + [X \setminus Y]$$

$$\text{and product } [X] \cdot [Y] = [X \times Y]$$

(can be gen. by smooth proj. w/ Bittner blowup relations)

~~Q~~ Lefschetz motive $L = [A^1]$

Tate subring $\mathbb{Z}[L] \subset K_0(\mathcal{V}_k)$

$[X] \in \mathbb{Z}[L] \rightsquigarrow$ polynomially countable

$$X \in \mathcal{V}_{\mathbb{Z}} \quad X_p \text{ red. mod } p \quad X_p(\mathbb{F}_q) \quad q = p^r$$

points over $\mathbb{F}_q$

$$N_q(X) = \# X_p(\mathbb{F}_q) \text{ counting function}$$

$X = \text{polyn. countable}$ iff $N_q(X) = \frac{P(q)}{q^k}$ $P_X = \text{polynomial}$

$$[X] = \sum_k a_k L^k \rightsquigarrow N_q(X) = \sum_k a_k q^k$$

Kontsevich conjecture: X_p graph hypersurfaces
are polynomially countable

Conjecture is false (Belkale - Brozos)

but true for all graphs up to 12 edges (Stenroos)

(2) Grothendieck classes of X_Γ and $\hat{X}_\Gamma$

$$[A^m, \hat{X}_\Gamma] = (\mathbb{L} - 1) [P^{m-1}, X_\Gamma]$$

because $\hat{X}_\Gamma$ is affine cone over X_Γ

$$\text{also } [A^m, \hat{X}_\Gamma] = \mathbb{L}^m - [\hat{X}_\Gamma]$$

So questions of whether $[X_\Gamma] \in \mathbb{Z}[\mathbb{L}]$ equiv. to whether $[\hat{X}_\Gamma] \in \mathbb{Z}[\mathbb{L}]$ or whether complements are

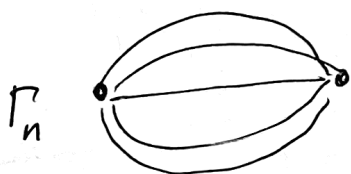
By last time discussion can reduce question to connected graphs and to 1PI graphs

$$[A^m, \hat{X}_\Gamma] = [A^m, \hat{X}_{\Gamma_1}] [A^{m_2}, \hat{X}_{\Gamma_2}] \text{ for connected components}$$

$$\text{and } [A^m, \hat{X}_\Gamma] = \mathbb{L}^{\#E(T)} \cdot \prod_{v \in V(T)} [A^{m_v}, \hat{X}_{\Gamma_v}] \text{ for 1PI decomposition}$$

Note: formally $[A^m, \hat{X}_\Gamma]$ has same algebraic properties as a Feynman rule and $\mathbb{L}$ as an "inverse propagator" ($\mathbb{L}^{-1}$ = Tate motive "propagator")

Example of explicit computation of a class $[X_\Gamma]$



2 vertices
 n -parallel edges

Graph polynomial:

3

$$\psi_{\Gamma_n}(t_1, \dots, t_n) = (t_1 \dots t_n) \left(\frac{1}{t_1} + \dots + \frac{1}{t_n} \right)$$

because spanning trees are just one edge

this is Cremona transformation formula for

$$\psi_{\Gamma_n^v} \quad \Gamma_n^v = \text{polygon with } n\text{-vertices}$$

$$\psi_{\Gamma_n^v}(s_1, \dots, s_n) = s_1 + \dots + s_n$$

So know that

$$X_{\Gamma_n} \setminus (X_{\Gamma_n} \cap \Sigma_n) \stackrel{\text{Cremona}}{\cong} X_{\Gamma_n^v} \setminus (X_{\Gamma_n^v} \cap \Sigma_n)$$

$$X_{\Gamma_n^v} = \mathcal{L} \text{ hyperplane } \sum_v t_i = 0$$

$$[\mathcal{L} \setminus (\mathcal{L} \cap \Sigma_n)] = [\mathcal{L}] - [\mathcal{L} \cap \Sigma_n]$$

$$= \frac{\pi^{n-1} - (-1)^{n-1}}{\pi + 1}$$

$$\pi = [G_m] = \mathbb{L} - 1$$

$$= [A' \setminus A^0]$$

Also $X_{\Gamma_n} \cap \Sigma_n = S_n$ singularity locus of Σ_n

$$[S_n] = [\Sigma_n] - n\pi^{n-2}$$

$$[\Sigma_{n+1}] = \sum_{i=1}^r \binom{r+1}{i} \pi^{r-i} \quad \textcircled{*}$$

$$= \frac{(1+\pi)^{r+1} - 1 - \pi^{r+1}}{\pi} \quad \&$$

$$\Rightarrow [X_{\Gamma_n}] = [S_n] + [\mathcal{L} \setminus (\mathcal{L} \cap \Sigma_n)]$$

intersecting w/transv.
hyperplane

$$[\mathcal{L} \cap \Sigma_{n+1}] = \frac{\mathbb{L}^r - 1}{\mathbb{L} - 1} - \pi^{r-1} - \pi^{r-2} - \dots - \pi^0$$

use:
[P, \Sigma_{n+1}] = \pi^r then \textcircled{*} from [P, \Sigma_n]

$$[X_{\Gamma_n}] = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \frac{(\mathbb{L} - 1)^n - (-1)^n}{\mathbb{L}} - n(\mathbb{L} - 1)^{n-2}$$

④ Deletion contraction relation for the graph polynomial and algebro-geom. consequences

Polynomial invariants of graphs and deletion-contraction

• Tutte polynomial $T_\Gamma(x, y) \in \mathbb{C}[x, y]$

determined by properties

1) $T_\Gamma(x, y) = T_{\Gamma \setminus e}(x, y) + T_{\Gamma / e}(x, y)$ ~~deletion contraction~~

if $e \in E(\Gamma)$ neither bridge nor looping edge

2) if e looping edge

$$T_\Gamma(x, y) = y T_{\Gamma / e}(x, y)$$

3) if e bridge

$$T_\Gamma(x, y) = x T_{\Gamma \setminus e}(x, y)$$

4) (~~single vertex~~ ^{vertices} ~~vertex~~) no edges $T_\Gamma(x, y) = 1$

• Invariants satisfying deletion contraction relation

$$F(\Gamma) = \gamma^{\frac{b_0(\Gamma)}{\alpha} \#V(\Gamma) - b_0(\Gamma)} \beta^{b_1(\Gamma)} T_\Gamma\left(\frac{\gamma x}{\alpha}, \frac{y}{\beta}\right)$$

are all specializations of the Tutte polynomial

Tutte-Grothendieck invariants of graphs $F(\Gamma) \in \mathbb{C}[\alpha, \beta, \gamma, x, y]$

1) - $F(\Gamma) = \gamma^{\#V(\Gamma)}$ if $E(\Gamma) = \emptyset$

2) - $F(\Gamma) = x F(\Gamma \setminus e)$ if e bridge

3) - $F(\Gamma) = y F(\Gamma / e)$ if e looping edge

4) - $F(\Gamma) = \alpha F(\Gamma / e) + \beta F(\Gamma \setminus e)$ if e not bridge not looping edge

$$F(\Gamma) = \alpha F(\Gamma / e) + \beta F(\Gamma \setminus e)$$

the graph polynomial

$$\psi_{\Gamma}(t_1, \dots, t_n) = t_e \frac{\partial}{\partial t_e} \psi_{\Gamma} + \psi_{\Gamma}|_{t_e=0}$$

$\swarrow$ monomials with t_e $\swarrow$ monomials without t_e

(5)

if e neither bridge nor looping edge (otherwise one of the two terms missing)
 already seen $\frac{\partial}{\partial t_e} \psi_{\Gamma} = \psi_{\Gamma \setminus e}$ while

$\psi_{\Gamma}|_{t_e=0} = \psi_{\Gamma/e}$: spanning trees of Γ/e are spanning trees of Γ that contain e

then get deletion-contraction relation for ψ_{Γ}

$$\psi_{\Gamma}(t_1, \dots, t_n) = t_e \psi_{\Gamma \setminus e}(t_1, \dots, \hat{t}_e, \dots, t_n) + \psi_{\Gamma/e}(t_1, \dots, \hat{t}_e, \dots, t_n)$$

What does this relation imply for X_{Γ} 's?

- don't expect "deletion-contraction" relation for $[X_{\Gamma}]$'s otherwise would always get MT-classes if get for small graphs

Claim: e neither bridge nor looping edge

$$[A^n, \hat{X}_{\Gamma}] = \mathbb{L} \cdot [A^{n-1}, (\hat{X}_{\Gamma \setminus e} \cap \hat{X}_{\Gamma/e})] - [A^{n-1}, \hat{X}_{\Gamma \setminus e}]$$

if e bridge:

$$[A^n, \hat{X}_{\Gamma}] = \mathbb{L} \cdot [A^{n-1}, \hat{X}_{\Gamma/e}] = \mathbb{L} \cdot [A^{n-1}, \hat{X}_{\Gamma \setminus e}]$$

if e looping edge: $[A^n, \hat{X}_{\Gamma}] = (\mathbb{L}-1)[A^{n-1}, \hat{X}_{\Gamma \setminus e}] = (\mathbb{L}-1)[A^{n-1}, \hat{X}_{\Gamma/e}]$

⑥ If P contains at least two lines then have also projective version

$$[\mathbb{P}^{n-1} \setminus X_P] = L[\mathbb{P}^{n-2} \setminus (X_{P,e} \cap X_{P,e})] - [\mathbb{P}^{n-2} \setminus X_{P,e}]$$

and Euler characteristic satisfies

$$\chi(X_P) = n + \chi(X_{P,e} \cap X_{P,e}) - \chi(X_{P,e})$$

Consider projection $\mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-2}$

from pt. $p = (0:0:\dots:1)$ mapping

$$(t_1: \dots: t_n) \mapsto (t_1: \dots: t_{n-1})$$

have $\psi_P(t_1, \dots, t_n) = t_n \cdot F(t_1, \dots, t_{n-1}) + G(t_1, \dots, t_{n-1})$

- if F constant $\Rightarrow X_{P,e} = \emptyset$ then $\deg \psi_P = 1$

so X_P hyperplane and under projection $X_P \cong \mathbb{P}^{n-2}$

- if $\deg(F) > 0$ $F(p) = \psi_P(p) = 0 \Rightarrow p \in X_P \cap C(X_{P,e})$

so $p \notin X_P \setminus (X_P \cap C(X_{P,e}))$ so can look at image of this under projection

$\Rightarrow$ projection induces regular morphism

$$X_P \setminus (X_P \cap C(X_{P,e})) \longrightarrow \mathbb{P}^{n-2} \setminus X_{P,e}$$

Claim: this map is an isomorphism

(7)

take pt. $q = (q_1, \dots, q_{n-1})$ on the $\mathbb{P}^{n-2} \setminus X_{\Gamma, e}$
 and line $(q_1, \dots, q_{n-1}, t)$ from top then $F(q) \neq 0$

this line intersects $X_\Gamma \setminus (X_\Gamma \cap C(X_{\Gamma, e}))$ where
 $t F(q) + G(q) = 0$ but $F(q) \neq 0$ so polyn. deg 1 in t
 $\Rightarrow$ solution is a ~~pt~~ single pt

$\Rightarrow$ classes in Grothendieck ring

$$[\mathbb{P}^{n-1} \setminus X_\Gamma] = [\mathbb{P}^{n-1} \setminus (X_\Gamma \cap C(X_{\Gamma, e}))] - \overset{X_{\Gamma, e}}{\cancel{[X_\Gamma \cap C(X_{\Gamma, e})]}}$$

$$= [\mathbb{P}^{n-1} \setminus (X_\Gamma \cap C(X_{\Gamma, e}))] - [\mathbb{P}^{n-2} \setminus X_{\Gamma, e}]$$

then check that if Γ has at least two loops then

$$[\mathbb{P}^{n-1} \setminus (X_\Gamma \cap C(X_{\Gamma, e}))] = 11 \cdot [\mathbb{P}^{n-2} \setminus (X_{\Gamma, e} \cap X_{\Gamma, e})]$$

(at least two loops $\Rightarrow \deg \psi_\Gamma > 1$ hence $\deg F \geq 1$)

in fact ideal of $X_\Gamma \cap C(X_{\Gamma, e})$ is gen by

$$(\psi_\Gamma, F) = (t_n F + G, F) = (F, G)$$

$$\Rightarrow X_\Gamma \cap C(X_{\Gamma, e}) = C(X_{\Gamma, e}) \cap C(X_{\Gamma, e})$$

all fibers are $\mathbb{A}^1$'s

note: $C(X_{\Gamma, e}) \cap C(X_{\Gamma, e})$ contains $p = (0, \dots, 0, 1)$

hence fibers of projection from $\mathbb{P}^{n-1} \setminus (C(X_{\Gamma, e}) \cap C(X_{\Gamma, e})) \rightarrow \mathbb{P}^{n-2} \setminus (X_{\Gamma, e} \cap X_{\Gamma, e})$

$$\textcircled{8} \Rightarrow [P^{n-1}((C(X_{re}) \cap (X_{re})))] = [A'] \cdot [P^{n-2}(X_{re} \cap X_{re})]$$

So get $[P^{n-1} X_P] = L \cdot [P^{n-2}(X_{re} \cap X_{re})] - [P^{n-2} X_{re}]$

Get affine case from projective case using rel. between classes of complements and treat remaining cases (1-loop & bridges and looping edges separately)

Euler characteristic and classes in $Ko(V)$:

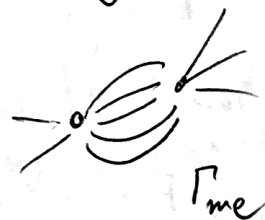
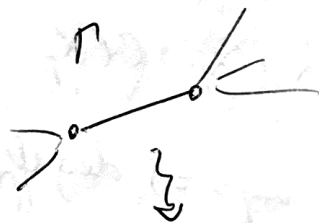
$\chi: Ko(V) \rightarrow \mathbb{Z}$ ring homom. $\chi(L) = 1$

$$\chi(P^n) = n+1 \Rightarrow n+1 - \chi(X_P) = n+1 - \chi(X_{re} \cap X_{re}) - n+1 + \chi(X_{re})$$

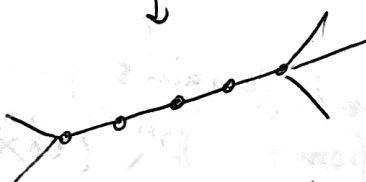
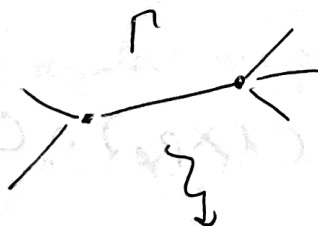
$$-\chi(X_{re}) + \chi(X_{re} \cap X_{re}) + n = \chi(X_P)$$

— Operations on graphs for which one can keep track of effect on $[X_P]$

• Multiplying edges:



• Splitting edges:



Toy model: what happens to the Tutte polynomial under this kind of operation

(9)

Multiplying edges: $T_{\Gamma_{me}}(x, y)$ as a function of $T_{\Gamma}(x, y)$

e not bridge not looping edge:

$$\sum_{m \geq 0} T_{\Gamma_{me}}(x, y) \frac{s^m}{m!} = e^s \left(T_{\Gamma \setminus e}(x, y) + \frac{e^{(y-1)s} - 1}{y-1} T_{\Gamma/e}(x, y) \right)$$

$$\sum_{m \geq 0} T_{\Gamma_{me}}(x, y) s^m = \frac{1}{1-s} \left(T_{\Gamma \setminus e}(x, y) + \frac{s}{1-y s} T_{\Gamma/e}(x, y) \right)$$

Pf: use deletion-contraction:

$$T_{\Gamma_{me}}(x, y) = T_{\Gamma_{(m-1)e}}(x, y) + y^{m-1} T_{\Gamma/e}$$

produce $m-1$ looping edges when contract e in Γ_{me}

$$\begin{aligned} \text{repeat: } T_{\Gamma_{me}}(x, y) &= T_{\Gamma_e} + (1 + y + \dots + y^{m-1}) T_{\Gamma/e} \\ &= T_{\Gamma_e} + \frac{y^m - 1}{y - 1} T_{\Gamma/e} \end{aligned}$$

then generating function obtained from this: coeff of T_{Γ_e} is $\exp(s)$ or $\frac{1}{1-s}$ and other

(other cases (bridge/looping edge) similar)

$$\text{from } \sum \frac{y^m s^m}{m!} = e^{ys} \text{ or}$$

$$\sum_{m \geq 0} (y^m - 1) s^m = \frac{1}{1-ys} - \frac{1}{1-s}$$

Now try to do something

similar for $[A^n, \hat{X}_{\Gamma}]$

not deletion-contraction but formula w/ $\hat{X}_{\Gamma_e} \cap \hat{X}_{\Gamma/e}$ term
key: recursion where bad term cancels

(10)

 Γ_{2e}

$$U(\Gamma) = [A^\vee, \hat{X}_\Gamma]$$

$$U(\Gamma_{2e}) = (T-1) U(\Gamma) + T U(\Gamma \setminus e) + (T+1) U(\Gamma/e)$$

$$T = [G_m] \quad \text{for } e \text{ not bridge not looping edge}$$

$$\text{for } e \text{ looping edge: } U(\Gamma_{2e}) = T^2 U(\Gamma \setminus e)$$

$$e \text{ bridge: } U(\Gamma_{2e}) = T(T+1) U(\Gamma \setminus e)$$

check first case: (others simple to see)

$$\psi_\Gamma = t_e F + G$$

$$F = \psi_{\Gamma \setminus e}$$

$$G = \psi_{\Gamma/e}$$

e replaced by two parallel edges

$$\psi_{\Gamma_{2e}} = t_e t_{e'} F + (t_e + t_{e'}) G$$

neither in
spanning tree
(t_e was not)

either one in
spanning tree
(t_e was)

$$= t_{e'} (t_e F + G) + t_e G = t_{e'} \psi_\Gamma + t_e G$$

then formula applied to Γ_{2e} (deleting e' gives back Γ)

$$\text{gives } U(\Gamma_{2e}) = [A^\vee, (\hat{X}_\Gamma \cap \hat{X}_{\Gamma_0})] \cdot L - U(\Gamma) \quad (*)$$

where $\Gamma_0 = \text{graph } \Gamma/e \text{ with a looping edge attached}$

$$\hat{X}_{\Gamma_0} = \{ t_e G = 0 \} \quad \text{ideal of intersection } (\psi_\Gamma, t_e G)$$

so inters is union of loc defined by
ideals (ψ_F, t_e) and (ψ_F, G)

(11)

$$(t_e F + G, t_e) = (G, t_e)$$

$$(t_e F + G, G) = (t_e F, G)$$

union of loc def by

$$\hat{X}_F \cap \hat{X}_G = (H \cap \hat{X}'_{F,e}) \cup (\hat{X}'_{F,e} \cap \hat{X}'_{G,e}) \quad (t_e, G) \text{ and } (F, G)$$

$H = \{t_e = 0\}$ hyperplane

$\hat{X}' =$ cylinder over $\hat{X} \subset \mathbb{A}^m$
in $\mathbb{A}^n$

$$H \cap \hat{X}' = \hat{X}$$

~~so get~~ so get

$$\begin{aligned} [\hat{X}_F \cap \hat{X}_G] &= [H \cap \hat{X}'_{F,e}] + [\hat{X}'_{F,e} \cap \hat{X}'_{G,e}] - [H \cap (\hat{X}'_{F,e} \cap \hat{X}'_{G,e})] \\ &= [\hat{X}_{F,e}] + (\mathbb{L} - 1) [\hat{X}'_{F,e} \cap \hat{X}'_{G,e}] \end{aligned}$$

so

$$\begin{aligned} [\mathbb{A}^n, (\hat{X}_F \cap \hat{X}_G)] &= \mathbb{L}^n - ([\mathbb{A}^{n-1}, \hat{X}_{F,e}] - \mathbb{L}^{n-1}) + (\mathbb{L} - 1) ([\mathbb{A}^{n-1}, (\hat{X}'_{F,e} \cap \hat{X}'_{G,e})] - \mathbb{L}^{n-1}) \\ &= \mathbb{L}^n - [\mathbb{A}^{n-1}, \hat{X}_{F,e}] + (\mathbb{L} - 1) [\mathbb{A}^{n-1}, (\hat{X}'_{F,e} \cap \hat{X}'_{G,e})] \end{aligned}$$

but also have $\mathbb{L}(\Gamma) = \mathbb{L} \cdot [\mathbb{A}^{n-1}, (\hat{X}'_{F,e} \cap \hat{X}'_{G,e})] - \mathbb{L}(\Gamma_{F,e})$

so putting all in (*) get cancellation of "bad terms"

$$\begin{aligned} \mathbb{L}(\Gamma_{2e}) &= \mathbb{L} \mathbb{L}(\Gamma_{F,e}) + (\mathbb{L} - 1) (\mathbb{L}(\Gamma) + \mathbb{L}(\Gamma_{F,e})) - \mathbb{L}(\Gamma) \\ &= (\mathbb{L} - 2) \mathbb{L}(\Gamma) + (\mathbb{L} - 1) \mathbb{L}(\Gamma_{F,e}) + \mathbb{L} \mathbb{L}(\Gamma_{F,e}) \end{aligned}$$

(12) Can then obtain a generating function:

$$\sum_{m \geq 0} U(\Gamma_{me}) \frac{s^m}{m!} = \frac{e^{\pi s} - e^{-s}}{\pi + 1} U(\Gamma) + \frac{e^{\pi s} + \pi e^{-s}}{\pi + 1} U(\Gamma_e) + \left(s e^{\pi s} - \frac{e^{\pi s} - e^{-s}}{\pi + 1} \right) U(\Gamma_e)$$

e neither bridge nor looping edge

looping edge $\sum_{m \geq 0} U(\Gamma_{me}) \frac{s^m}{m!} = e^{\pi s} U(\Gamma_e)$

bridge $\sum_{m \geq 0} U(\Gamma_{me}) \frac{s^m}{m!} = \left(\pi \frac{e^{\pi s} - e^{-s}}{\pi + 1} + s e^{\pi s} + 1 \right) U(\Gamma_e) \quad (**)$

looping edge ~~easy~~: $U(\Gamma_{me}) = \pi^m U(\Gamma_e)$ prove directly

bridge: by multiplicative property for graphs glued at a vertex

$$U(\Gamma_{me}) = \underbrace{E_m(\pi)}_{\text{class of } m\text{-th banana graph}} \cdot U(\Gamma_e)$$

$\left\{ \begin{array}{l} \text{this known explicitly} \\ \text{from previous} \end{array} \right.$

so check using $\nearrow$ that satisfies recursion

$$U(\Gamma_{(m+1)e}) = ((\pi - 1) E_m(\pi) + \pi E_{m-1}(\pi) + (\pi + 1) \pi^{m-1}) U(\Gamma_e)$$

$$E(s) = \sum_{m \geq 0} E_m(\pi) \frac{s^m}{m!}$$

$$\begin{aligned} E_{m+1}(\pi) &= (\pi - 1) E_m(\pi) \\ &\quad + \pi E_{m-1}(\pi) + (\pi + 1) \pi^{m-1} \end{aligned}$$

so generating series is

$E(s) \cdot U(\Gamma_e)$ express recursion as diff. equation

$$E''(s) - (\pi - 1) E'(s) - \pi E(s) = (\pi + 1) e^{\pi s} - \pi \quad (*)$$

recursion for banana graphs

Finish deletion-contraction argument

(13)

$se^{\pi s} + 1$ is a solution of $(*)$ (direct check)

then general solution $Ae^{\pi s} + Be^{-s} + se^{\pi s} + 1$

initial conditions $\varepsilon_0(\pi) = 1$ $\varepsilon_1(\pi) = \pi + 1$

gives $A = \frac{\pi}{\pi+1}$ $B = -\frac{\pi}{\pi+1}$ which gives $(**)$

Case of neither bridge nor looping edge:

- $V(\Gamma_{me}) = f_m(\pi) V(\Gamma) + g_m(\pi) V(\Gamma \setminus e) + h_m(\pi) V(\Gamma/e)$
- Know the coefficients satisfy (from $V(\Gamma_{2e})$ case)
 and know also f_2, g_2, h_2

$$\begin{cases} f_0(\pi) = 0 \\ g_0(\pi) = 1 \\ h_0(\pi) = 0 \end{cases} \begin{cases} f_1(\pi) = 1 \\ g_1(\pi) = 0 \\ h_1(\pi) = 0 \end{cases} \quad [*]$$
- for $m \geq 1$ have (from the Γ_{2e} formula:)

$$V(\Gamma_{(m+1)e}) = (\pi-1) V(\Gamma_{me}) + \pi V(\Gamma_{(m-1)e}) + (\pi+1)\pi V(\Gamma_k)$$
- gives a recursion for the coefficients f_m, g_m, h_m :

$$f_{m+1} = (\pi-1) f_m + \pi f_{m-1}$$

$$g_{m+1} = (\pi-1) g_m + \pi g_{m-1}$$

$$h_{m+1} = (\pi-1) h_m + \pi h_{m-1} + (\pi+1)\pi^{m-1}$$

- Define series $F(s) = \sum_{m \geq 0} f_m(\pi) \frac{s^m}{m!}$

$$G(s) = \sum_{m \geq 0} g_m(\pi) \frac{s^m}{m!}$$

$$H(s) = \sum_{m \geq 0} h_m(\pi) \frac{s^m}{m!}$$

So generating series satisfies

$$\sum_{m \geq 0} U(\Gamma_m) \frac{s^m}{m!} = F(s) U(\Gamma) + G(s) U(\Gamma_e) + H(s) U(\Gamma_e)$$

- translate recursions into differential equations

$$\begin{cases} F'(s) - (\pi-1)F'(s) - \pi F(s) = 0 \\ G'(s) - (\pi-1)G'(s) - \pi G(s) = 0 \\ H'(s) - (\pi-1)H'(s) - \pi H(s) = (\pi+1)e^{\pi s} \end{cases}$$

$\nearrow se^{\pi s}$ is a solution of this

homogeneous part same as previous case for $E(s)$ so get solutions

$$F(s) = A_1 e^{\pi s} + B_1 e^{-s}$$

$$G(s) = A_2 e^{\pi s} + B_2 e^{-s}$$

$$H(s) = A_3 e^{\pi s} + B_3 e^{-s} + s e^{\pi s}$$

A_i, B_i are functions of π determined by initial conditions
[*]

get stated solution

Similar procedure for edge splitting

Can combine to compute classes for arbitrary
chains of polygons