

• $\text{Mot}_{\sim}(k)$ additive pseudo-abelian

• $\text{Mot}_{\sim}(k)$ is a tensor category

$$(X, p, m) \otimes (Y, q, n) = (X \times Y, p \times q, m+n)$$

• Tate twists: $\mathbb{L} \otimes \mathbb{T} = \mathbb{1}$ inverses w/ resp. to $\otimes$

$$h_{\sim}^{2d}(X_d) \cong \mathbb{L}^{\otimes d}$$

$$(X, p, m) = (X, p, 0) \otimes \mathbb{T}^{\otimes m}$$

• Duality: $(X, p, m)^{\vee} = (X, {}^t p, d-m)$
 $\dim X = d$ involution

• $h_{\text{rat}}: \text{Sm Proj}(k) \rightarrow \text{Mot}_{\text{rat}}^{\text{eff}}(k)$

is not injective on objects

e.g. all projective bundles

have the same (isomorphic) motive

(see later why)

$$\begin{array}{ccc} \mathbb{P}^m \text{ bundles} & & \mathbb{P}(E) \\ E \text{ vector bundle} & & \\ \downarrow & & \\ X & & \end{array}$$

• Motivic interpretation of Chow groups

$$\text{CH}^i(M) \cong \text{Hom}_{\text{Mot}_{\text{rat}}(k)}(\mathbb{L}^{\otimes i}, M)$$

where $\text{CH}^i(M) := \text{Im } p_* \text{CH}^{i+m}(X)_{\mathbb{Q}}$ $M = (X, p, m)$

in fact have

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathrm{rat}}(k)}(\mathbb{L}^{\otimes i}, M) = \{ f = p \circ \Gamma \mid \Gamma \in \mathrm{CH}^{\mathrm{rat}}(X) \}$$

and (Lieberman's Lemma)

$$p \circ \Gamma = (\mathrm{id}_{\mathrm{Spec}(k)} \times p)_* (\Gamma) = \mathrm{Im} p_* \text{ in } \mathrm{CH}^{\mathrm{rat}}(X)$$

this uses $\mathbb{L}^{\otimes i} = (\mathrm{Spec} k, \mathrm{id}, -i)$

- Similarly for any good equiv. rel.

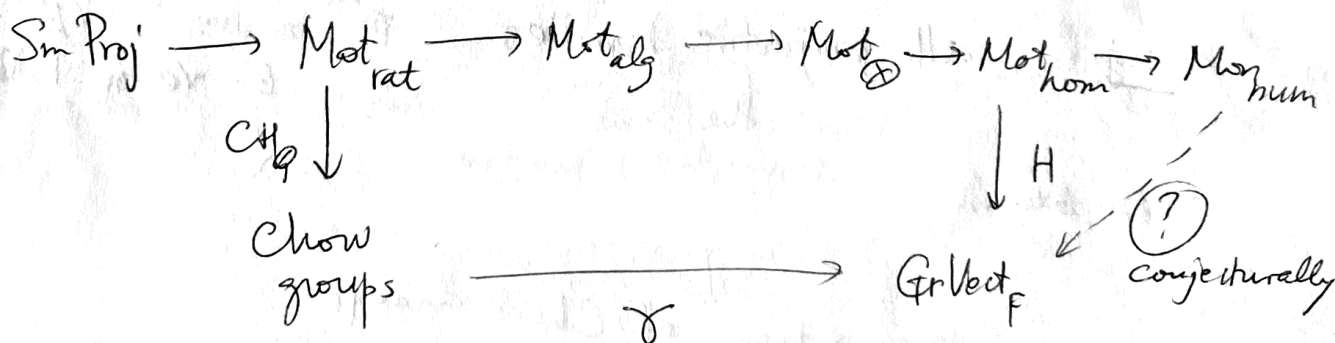
$$C_{\sim}^i(M) = \mathrm{Hom}_{\mathrm{Mot}_{\sim}(k)}(\mathbb{L}^{\otimes i}, M)$$

- Homology groups of motives (as opposed to of varieties)

$$H^i(M) = \mathrm{Im} p_* \subset H^{2+i}(X)$$

$M = (X, p, m)$

(have cohom. grps for $\mathrm{Mot}_{\mathrm{rat}}$, $\mathrm{Mot}_{\mathrm{alg}}$, $\mathrm{Mot}_{\otimes}$, $\mathrm{Mot}_{\mathrm{hom}}$
but not for $\mathrm{Mot}_{\mathrm{num}}$ unless standard conj. D holds)



Motives of curves

(3)

$\text{Chow}(k) \cong$ with integer instead of $\mathbb{Q}$ -coeffs
& Motrat equiv. rel.

C smooth projective curve

$e \in C(k)$ base point

$$\pi_0(C) = e \times C$$

$$\pi_2(C) = C \times e$$

$$\pi'_1(C) = \Delta(C) - \pi_0(C) - \pi_2(C)$$

$\Delta \subset C \times C$ diagonal correspondence

for curve $\pi_1(C) := \pi'_1(C)$

$\Rightarrow$ integral Chow-Künneth decomposition

$$\text{ch}_Z(C) = \text{ch}_Z^0(C) \oplus \text{ch}_Z^1(C) \oplus \text{ch}_Z^2(C)$$

$$\text{where } \text{ch}_Z^2(C) = (C, \pi_0, 0)$$

under a Weil cohomology

$$\text{Know } H(\text{ch}_Z^0(C)) = H^0(C) \quad H(\text{ch}_Z^2(C)) = H^2(C)$$

hence $H(\text{ch}_Z^1(C)) = H^1(C)$ only remaining piece

but $\text{ch}_Z^1(C)$ more refined geometric information than the cohomology $H^1(C)$

related to the Jacobian $J(C)$

Theorem (Grothendieck)

① The only non-trivial Chow group of $\text{ch}_Z^1(C)$ is $\text{CH}^1(\text{ch}_Z^1(C)) \cong J(C)(k) \cong \text{CH}_{\text{alg}}^1(C)$

(2) C, C' smooth projective curves

$$\text{Hom}_{\text{Chow}(K)}(\text{ch}'_2(C), \text{ch}'_2(C'))$$

$$\cong \text{Hom}_{AV}(J(C), J(C'))$$

in category of
abelian
varieties ~~to isogeny~~

(3) in $\text{Mot}_{\text{rat}}(K) \otimes \mathbb{Q}$ the full subcategory
gen. by motives $\text{ch}'(C)$ of curves
equiv. to category of abelian varieties up to isogeny

$$\text{Hom}_{AV\text{-isog}}(A, A') = \text{Hom}_{AV}(A, A') \otimes \mathbb{Q}$$

Comments on theorem :

(2) depends on a theorem of A. Weil

$$\text{Hom}_{AV}(J(C), J(C')) \cong \text{CH}'(C \times C') / \text{CH}'_{\equiv}(C \times C')$$

where $\text{CH}'_{\equiv}(X \times Y) =$ subgroup of $\text{CH}'(X \times Y)$
of classes of degenerate divisors

D divisors s.t.

$$\text{pr}_X(D) \neq X \text{ or } \text{pr}_Y(D) \neq Y$$

i.e. degenerate iff

$$D = D_1 \times Y + X \times D_2$$

with D_1 div in X , D_2 div. in Y

then if full subcategory $\mathcal{M}'$ of $\text{Chow}(k)_{\mathbb{Z}}$
of motives isom to $\text{ch}'_{\mathbb{Z}}(C)$
and functn $F: \mathcal{M}' \rightarrow \{\text{cat. of Jacobians of curves}\}$
 $F(\text{ch}'_{\mathbb{Z}}(C)) = J(C)$

ess. surjective and fully faithful
 $\Rightarrow$ equivalence of categories

$\Rightarrow$ also equiv. after passing to $\mathbb{Q}$ -coeff's

• then known facts about abelian varieties: A abelian var.
 $\exists C$ alg curves s.t. $J(C) \twoheadrightarrow A$

• Poincaré reducibility $J(C) = A_1 + A_2$ A_1, A_2 ab. subvarieties of $J(C)$
 A_1, A_2 finite group
& $A_1 \sim A$ isogeny

$\Rightarrow$ up to isogeny
 $J(C) = A_1 \oplus A_2$

$\Rightarrow$ each A a direct summand of some $J(C)$
 $\Rightarrow$ category of ab. varieties is pseudo-abelian
 $\Rightarrow$ envelope of cat. of Jacobians of curves

(adding images & kernels of projectors)

$\Rightarrow (\mathcal{M}')_{\mathbb{Q}} = \text{Ab}_{\mathbb{Q}}$ abelian varieties up to isogeny

Motives of curves

$$\text{ch}_{\mathbb{Z}}(C) = 1 \oplus J(C)_{\mathbb{Q}} \oplus L$$

in general "motivic decompositions" for other kinds of motives?

Manin's identity principle

How to check if a correspondence f from X to Y is trivial

• where is the subtlety?

example: $X=Y=E$ elliptic curve
 $a, b, c, d \in E(k)$ different pts in general position

$$((a)-(b)) \times ((c)-(d)) \in CH^2(E \times E)$$

is not zero but it acts as zero on all the Chow groups of E

So don't necessarily have $f=0$ if $f_*: CH(X) \rightarrow CH(Y)$ is the zero map

for $T \in \text{SmProj}(k)$ set $X(T) := \text{Corr}(T, X)$

and if $f \in \text{Corr}_{\text{rat}}(X, Y)$ set

$$f_T \text{ to be } f_T: X(T) \rightarrow Y(T)$$

$$\alpha \mapsto f_T(\alpha) = f \circ \alpha$$

Theorem (Manin) · Manin's identity principle

for $f, g \in \text{Corr}_{\text{rat}}(X, Y)$ the following are equivalent

- $f = g$
- $f_T = g_T$ for all $T \in \text{SmProj}(k)$
- $f_X = g_X$

in particular $f \text{ isom} \Leftrightarrow f_T \text{ isom} \forall T \Leftrightarrow f_X \text{ isom}$ (7)

(1) $\Rightarrow$ (2) $\Rightarrow$ (3) clear

and for (3) $\Rightarrow$ (1) take $\alpha = \Delta(X)$

Note (using Lieberman's lemma)

$$f_T(\alpha) = f \circ \alpha = (\text{id}_T \times f)_*(\alpha)$$

So can rephrase as

$$\begin{aligned} \text{ch}(X) \longrightarrow \text{ch}(Y) &\iff \text{CH}(T \times X) \longrightarrow \text{CH}(T \times Y) \\ \text{ch}(f) = \text{ch}(g) &\quad (\text{id}_T \times f)_* = (\text{id}_T \times g)_* \end{aligned}$$

as correspondences on two fibres

$f = g$ as correspondences if f & g act in same way not only on $\text{CH}(X) \rightarrow \text{CH}(Y)$ but after arbitrary "base change to T -points"

$$\text{CH}(T \times X) \longrightarrow \text{CH}(T \times Y)$$

Applications of Manin's identity principle

E vector bundle
 $\downarrow$ or $m+1$
 S

$\pi: \mathbb{P}(E) \rightarrow S$ projective bundle

then motive

$$\text{ch}(\mathbb{P}(E)) \simeq \bigoplus_{i=0}^m \text{ch}(S)(-i)$$

tate twist

$$\checkmark \quad \chi_{(P, D)}^{(m)} = \chi_{(P, D)}^{(m)} \otimes \pi^m$$

to see this use isom. on Chow groups

$$\text{CH}(\mathbb{P}(E)) \simeq \bigoplus_{i=0}^m \text{CH}(S)[\xi^i]$$

ξ = tautological line bundle

and this still holds after base change $T \rightarrow \text{Spec}(k)$

A very useful application of Mami's identity principle
Blowup formula for motives

$Z \subset X$ smooth proj. subvar. codim $m+1$

$$\mathrm{Bl}_Z(X) = \tilde{X} \quad \text{blowup}$$

E exceptional divisor of blowup is proj. bundle $\mathbb{P}(E)$
 with $E \rightarrow Z$ rk $m+1$

$$\mathrm{ch}(\tilde{X}) \oplus \mathrm{ch}(Z) = \mathrm{ch}(X) \oplus \mathrm{ch}(E)$$

$$\mathrm{ch}(\tilde{X}) = \mathrm{ch}(X) \oplus \bigoplus_{i=1}^m \mathrm{ch}(Z)(-i)$$

simplest form
 for projective spaces

$$\mathrm{ch}(\mathbb{P}^n) = \bigoplus_{i=0}^n \mathbb{1}(-i) = \mathbb{1} \oplus L \oplus L^2 \oplus \dots \oplus L^n$$

using fact that

$$\mathbb{Z}_{\sim}^*(Y \times \mathbb{P}^n)_F \cong \mathbb{Z}_{\sim}^*(Y)_F[t] / (t^{n+1})$$



t class of a hyperplane in $\mathbb{P}^n$

$[\Delta] \in \mathbb{Z}_{\sim}^n(\mathbb{P}^n \times \mathbb{P}^n)$
 diagonal

$$[\Delta] = \sum t^i \times t^{n-i} \quad \text{K\"umeth. decomp.}$$

Grothendieck standard conjectures

(9)

- type C : Künneth conjecture $C(X)$
 Künneth components of diagonal are algebraic
- type B : Lefschetz conjecture $B(X)$ (also type $A(X, L)$)
 Lefschetz involution $*_{L, X}$ is alg. ($\mathbb{Q}$ -coeffs)
- ~~type A~~
 type I : Hodge type conjecture
 quadratic form defined by Hodge involution $*_H$ is positive definite
- type D : numerical & homological coincide

- fix a choice of a Weil cohomology H^*
 $A^i(X) = \text{Im}(\gamma_X) \subset H^{2i}(X) \quad \gamma_X : CH^i(X)_{\mathbb{Q}} \rightarrow H^{2i}(X)$

• Künneth conjecture: $C(X)$ holds if

$\Delta \subset X \times X$ diagonal

$$\gamma_{X \times X}(\Delta) \in H^{2d}(X \times X) = \bigoplus_{i=0}^{2d} H^{2d-i}(X) \otimes H^i(X)$$

$$\Delta_i^H \in H^{2d-i}(X) \otimes H^i(X)$$

Künneth components of the diagonal in the Weil cohomology H

$\rightarrow \exists$ alg. cycle classes

$$\Delta_i \in CH^d(X \times X)_{\mathbb{Q}} \quad \text{s.t.} \quad \gamma_{X \times X}(\Delta_i) = \Delta_i^H$$

"the Künneth components of the diagonal are algebraic"

- True for varieties with an algebraic cell decomposition (projective spaces, Grassmannians, flag var.)
- True for curves
- also known for surfaces and for abelian varieties
- over finite fields (follows from Deligne's pf. of Weil conjectures)
- for $k = \mathbb{C}$ would follow from Hodge conj.

equivalent formulation:

Künneth projectors

$$\pi_X^i: H^*(X) \longrightarrow H^i(X)$$

$C(X)$: π_X^i are given by algebraic cycles

$$\exists \pi_X^i \in \mathbb{Z}_{\text{alg}}(X \times X)_{\mathbb{Q}}$$

$$\text{s.t. } \gamma_{X \times X}(\pi_X^i) = \pi_X^i \in H^*(X \times X)$$

- Künneth conj. stable under products
if $C(X)$ & $C(Y)$ hold then since

$$\pi_{X \times Y}^n = \sum_{i+j=n} \pi_X^i \otimes \pi_Y^j \quad \text{also } C(X \times Y) \text{ holds}$$

Weight filtration ^{graduation (grading)} of homological & numerical motives

(11)

Assume $C(X)$ then have

$$\pi_X^i \in \text{Mor}_{\text{hom}}^{(k)}(h(X), h_c(X))$$

$$h^v(X) := \pi_X^i h(X)$$

$$M = p h(X)(r) \quad (= (X, p, r))$$

has a grading $M = \bigoplus_i \text{Gr}_i(M)$ (where

$$\text{Gr}_i(M) := p h^{2r+i}(X)(r)$$

— induces grading on numerical motives

$$\text{Gr}_n(M \otimes N) = \sum_{i+j=n} \text{Gr}_i(M) \otimes \text{Gr}_j(N)$$

$$\text{Gr}_i(M^\vee) = (\text{Gr}_{-i}(M))^\vee$$

— The projector π_X^i are orthogonal so

$$\text{Hom}_{M_{\text{hom}}(k)_F}(\text{Gr}_i(M), \text{Gr}_j(N)) = 0 \quad \text{when } i \neq j$$

— $\mathbb{1}(1)$ has weight -2

Utt

Weaker form of the Lefschetz conjecture
sign conjecture $(C^+(X))$

$$\pi_X^+ = \sum_{i=0}^d \pi_X^{2i} \quad \text{is algebraic} \quad \left(\text{hence also } \pi_X^- = \sum_{i=1}^d \pi_X^{2i-1} \right)$$

(consider only odd/even cohomology)

$\Rightarrow \mathbb{Z}/2\mathbb{Z}$ - weight grading

Note: assuming $(C^+(X))$ holds

then $\text{Ker } C_{\text{hom}}(X \times X)_F \rightarrow C_{\text{mm}}(X \times X)_F$
 homom of algebras (prod. by composition)
 is a nilpotent ideal

Pf: $f \in \text{Ker} \quad \forall n \in \mathbb{N}$

$$\langle f^n, \pi_X^+ \rangle = \langle f^n, \pi_X^- \rangle = 0$$

using Lefschetz formula write these inters. #'s as
 traces

$$\text{tr}_{H^\pm(X)}(f^n) = 0$$

all powers of f have zero trace in cohomology

$\Rightarrow f$ nilpotent

(as a consequence $\text{Mot}_{\text{hom}}(V) \xrightarrow{f} \text{Mot}_{\text{mm}}(V)$ is a map of nilpotents)
 subject on which $(C^+(X))$ holds $\rightarrow$ conservative & abs. surj.
 isom. come from isom. $\rightarrow$ can lift idempotents

$$M = (X, p, m) = (X, p, n) \otimes \mathbb{L}^{n-m} \quad \text{if } m < n$$

$$= (X, p, n) \otimes h^2(\mathbb{P}^1)^{n-m}$$

$$= (X \times (\mathbb{P}^1)^{n-m}, p', n) \quad \text{where } p' \text{ new projector combining } p \text{ and } \pi_{\mathbb{P}^1}^2$$

if $N = (Y, q, n)$ then direct sum

$$(X, p, m) \oplus (Y, q, n) =$$

$$(X \times (\mathbb{P}^1)^{n-m} \sqcup Y, p' \oplus q, n)$$

direct sum of motives