

$$U(\Gamma, p_1, \dots, p_N) = \frac{\Gamma(n - \frac{Dl}{2})}{(4\pi)^{Dl/2}} \int \frac{P_\Gamma(t, p)}{\sigma_n \psi_\Gamma(t)^{-n + D(l+1)/2}} \omega_n$$

$$\psi_\Gamma(t) = \det M_\Gamma(t) = \sum_{T \subset \Gamma} \prod_{e \notin T} t_e$$

$$P_\Gamma(t, p) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e$$

$$s_C = \left(\sum_{v \in V(\Gamma_1)} P_v \right)^2 = \left(\sum_{v \in V(\Gamma_2)} P_v \right)^2$$

$$P_v = \sum_{\substack{e \in E_{\text{ext}}(\Gamma) \\ t(e) = v}} p_e$$

stable range:

$n \leq \frac{Dl}{2}$ then ψ_Γ in denom. and P_Γ in numerator.

$n = \frac{Dl}{2}$ log divergent case

$$\int \frac{\omega_n(t)}{\sigma_n \psi_\Gamma(t)^{D/2}} \quad \text{no dependence on } P_\Gamma(t, p)$$

unstable range:

$n > \frac{Dl}{2}$ have $P_\Gamma(t, p)$ in denominator

In all cases $\frac{P_\Gamma(t, p)^{-n + \frac{Dl}{2}}}{\psi_\Gamma(t)^{-n + D(l+1)/2}} \omega_n$ is an algebraic differential form (assume $P \in \mathcal{Q}^{ND}$)

②

form is defined on complement of a hypersurface

• stable range: $\hat{X}_P = \{t \in \mathbb{A}^{\#E(P)} \mid \psi_P(t) = 0\}$
graph hypersurfaces

• unstable range: $\hat{Y}_{P,P} = \{t \in \mathbb{A}^{\#E(P)} \mid P(t, P) = 0\}$
Landau varieties

Note: in both cases polynomials are homogeneous

$\Rightarrow$ can also consider projective hypersurfaces

$X_P = \{t \in \mathbb{P}^{n-1} \mid \psi_P(t) = 0\}$ ψ_P homog. deg $b_1, \dots, b_n$

$Y_{P,P} = \{t \in \mathbb{P}^{n-1} \mid P_P(t, P) = 0\}$

• form defined on $\mathbb{A}^n \setminus \hat{X}_P$ in stable range $(\mathbb{A}^n \setminus \hat{Y}_{P,P} \text{ or } \mathbb{A}^n \setminus \bigcup_{P \in \mathcal{P}} \hat{X}_P)$
otherwise in $\mathbb{P}^n \setminus X_P$

Integral in affine and in projective space

differential forms on $\mathbb{A}^n$ and on $\mathbb{P}^{n-1}$

$\mathbb{P}^{n-1} = \mathbb{A}^n \setminus \{0\} / \mathbb{G}_m$

Volume form $\omega_n = dt_1 \wedge \dots \wedge dt_n$ on $\mathbb{A}^n$

projective analog: $\Omega_n = \sum_{i=1}^n (-1)^{i+1} t_i dt_1 \wedge \dots \wedge \hat{dt}_i \wedge \dots \wedge dt_n$

relation: $\Omega_n = \Delta(\omega_n)$

$\Delta: \Omega^k \rightarrow \Omega^{k-1}$ is contraction with the Euler vector field $E = \sum_i t_i \frac{\partial}{\partial t_i}$

$$E = \sum_i t_i \frac{\partial}{\partial t_i}$$

$$\Delta(\omega)(v_1, \dots, v_{k-1}) = \omega(E, v_1, \dots, v_{k-1})$$

$f: \mathbb{A}^n \rightarrow \mathbb{A}$ homogeneous polynomial of $\deg(f)$

$$\pi: \mathbb{A}^n \setminus \{0\} \rightarrow \mathbb{P}^{n-1} \text{ proj. } (t_1: \dots: t_n) = \pi(t_1, \dots, t_n)$$

$$X_f = \{t \in \mathbb{A}^n \mid f(t) = 0\} \text{ affine hypersurface}$$

$$X_f = \{t \in \mathbb{P}^{n-1} \mid f(t) = 0\} \text{ projective hypersurface}$$

$$\hat{D}(f) = \mathbb{A}^n \setminus X_f \text{ hypersurf. complement } D(f) = \mathbb{P}^{n-1} \setminus X_f$$

forms $\Omega^k(D(f)) \ni \omega$

$$R = \mathbb{C}[t_1, \dots, t_n] \text{ or } \mathbb{K}[t_1, \dots, t_n]$$

ring of functions of $\mathbb{A}^n$

$$R_m = \text{homog. polyn. deg} = m$$

$$\Omega^k \text{ forms on } \mathbb{A}^n; \Omega_m^k \text{ } k\text{-forms homog. deg } m$$

$$\eta \in \Omega_{m \deg(f)}^k$$

Dimca-Dolgachev

characterization of forms on $\hat{D}(f)$ that are pullbacks of forms on $D(f)$

$$\omega \in \Omega^k(D(f)) \rightsquigarrow \pi^*(\omega) \in \Omega^k(\hat{D}(f)) \text{ characterized by}$$

1) invariant under G_m -action (hence)

2) $\Delta(\pi^*(\omega)) = 0$ under contraction with Euler vector field

$$\text{Sequence } 0 \rightarrow \Omega^n \xrightarrow{\Delta} \Omega^{n-1} \xrightarrow{\Delta} \dots \xrightarrow{\Delta} \Omega^1 \xrightarrow{\Delta} \Omega^0 \rightarrow 0$$

is exact in all but the last term

(4) So the condition $\Delta(\pi^*(w)) = 0$ equivalent to $\pi^*(w) = \frac{\Delta(\eta)}{f^m}$ and condition on G_m -action implies $\eta \in \Omega_{m \deg(f)}^K$

• in particular any $(n-1)$ -form on $D(f) \subset \mathbb{P}^{n-1}$ can be written as

$$\frac{h \cdot \Omega_n}{f^m} \quad h \in R_{m \deg(f) - n}$$

$\Omega_n = \Delta(w_n)$ projective volume form (homog. deg n)

• In log divergent case $n = \frac{Df}{2}$

$$\frac{w_n}{\psi_r^{\frac{Df}{2}}} \quad \deg n \quad \deg \frac{Df}{2} \quad (\psi_r \text{ homog. deg } l) \quad \Rightarrow G_m\text{-invariant}$$

~~pass from $\mathbb{P}^n$ to $\mathbb{P}^{n-1}$ along $\mathbb{A}^1$~~

$$\frac{\Delta(w_n)}{\psi_r^{D/2}}$$

On $\mathbb{P}^{n-1}$ consider this diff. form
(image under contraction with Euler vector field)

Note: domain of integration $\sigma_n = \{t \in \mathbb{R}_+^n \mid \sum_{i=1}^n t_i \geq 1\}$
is in a fundam. domain of
 G_m action on $\mathbb{A}^n \setminus \{0\} \rightsquigarrow$ can view in $\mathbb{A}^n$ or
in $\mathbb{P}^{n-1}$

$$\sum_i (-1)^{i+1} t_i dt_1 \wedge \dots \wedge \widehat{dt_i} \wedge \dots \wedge dt_n \quad \left[\begin{array}{l} dt_1 \wedge \dots \wedge dt_n \text{ w/ } dt_n = \sum_{i=1}^{n-1} dt_i \text{ would give } 0 \\ 1 \\ dt_n = -\sum_{i=1}^{n-1} dt_i \Rightarrow \left(\sum_{i=1}^{n-1} t_i \right) dt_1 \wedge \dots \wedge dt_{n-1} \end{array} \right] \left\{ \begin{array}{l} \text{correct} \\ \text{restriction of} \\ \text{volume form on } \sigma_n \end{array} \right.$$

focus on stable case: graph hypersurfaces

(5)

• X_Γ 's are singular hypersurfaces

if $\deg \psi_\Gamma > 2$ then singular locus of X_Γ
= intersection of cones over hypersurfaces X_{Γ_e}
 $e \in E(\Gamma)$ $\Gamma_e = \Gamma \setminus \{e\}$ removing edge

the cones $C(\Gamma_e)$ do not intersect transversely

Pf: $\psi_\Gamma(t)$ homog. deg $b(\Gamma)$ $\Rightarrow$ Euler formula:

$$L. \psi_\Gamma(t) = \sum_e t_e \frac{\partial}{\partial t_e} \psi_\Gamma(t)$$

$$\Rightarrow \bigcap_e \underbrace{Z(\partial_e \psi_\Gamma)}_{\{t \mid \partial_e \psi_\Gamma = 0\}} \subset X_\Gamma$$

$\Rightarrow$ singular locus of X_Γ = equations $\partial_e \psi_\Gamma = 0$
(without having to also impose $\psi_\Gamma = 0$)

also: each variable t_e appears in ψ_Γ deg 1 (or deg 0 if not appearing)
each monomial of

$\Rightarrow \partial_e \psi_\Gamma$ = those monomials where t_e appears but without t_e (set $t_e = 1$)

this is same as $\psi_{\Gamma \setminus \{e\}}$ because spanning trees for $\Gamma \setminus \{e\}$ are those spanning trees for Γ that do not contain e

~~X_Γ~~ $X_{\Gamma \setminus \{e\}}$ hypersurface in $\mathbb{P}^{n-2}$ while $Z(\partial_e \psi_\Gamma) \subset \mathbb{P}^{n-1}$

(6) $\Rightarrow Z(2\psi_r) = C(X_{p_e})$ vertex at $v_e = (0, 0, 0, 1, 0, \dots)$

these cones do not intersect transversely;
if $\deg \psi_r > 2$ given two $C(X_{p_e})$ and $C(X_{p_{e'}})$
the vertex of one cone is in the hypersurface
spanning the other cone

↑
e-th
place
test

- in non log-divergent case when both P_r & ψ_r are present want a "generic" condition on external momenta
so that P_r & ψ_r have no common factors

Claim $\exists$ a dense open set of p 's st.
 $P_r(t, p)$ & ψ_r no common factor if r is 1PI

Rf: parameterizing space of external momenta:
affine subspace of $\mathbb{A}^{D \cdot \#E_{\text{ext}}(\Gamma)} = \mathbb{A}^{D \cdot N}$

given by equation $\sum_{e \in E_{\text{ext}}(\Gamma)} p_e = 0$ conservation

take simplest case where set all $p_e = 0$ except for
two edges $\{e_1, e_2\}$ with $p_{e_1} = p = -p_{e_2}$ with $v_1 \neq v_2$
 $v_i = \text{vertices} = \partial(e_i)$ $P_{v_1} = p = -P_{v_2}$

$$P_r(t, p) = p^2 \sum_T \left(\sum_{e' \in T_{v_1, v_2}} t_{e'} \right) \prod_{e \notin T} t_e$$

$C = T \cup \{e'\}$ cut sets

T_{v_1, v_2} = unique path in T connecting v_1 and v_2

cut set = complement of some T plus one edge in T_{v_1, v_2} (other terms = 0 because $p_e = 0$)

Polynomial ψ_Γ is irreducible under assumption (7)
 that Γ ~~is not~~ $\Gamma_1 \neq \Gamma_2$ joined at vertex
 (In particular if $\Gamma \text{ non } |\Gamma|$) for this fact see

Rewrite term in $P_\Gamma(t, p)$ as linear function

$$L_\Gamma(t) = p^2 \sum_{e \in T_{v_1, v_2}} t_e \quad \text{so that} \quad P_\Gamma(t, p) = \sum_T L_\Gamma(t) \prod_{e \in T} t_e$$

if ψ_Γ divides $P_\Gamma(t, p)$ would have

$$P_\Gamma(t, p) = \psi_\Gamma L(t) \quad \text{for some linear } L(t) \\ \text{because degs } l+1 \text{ and } l$$

$$\text{so that} \quad \left(\sum_T L_\Gamma(t) - L(t) \right) \prod_{e \in T} t_e \equiv 0 \quad \text{for all } t$$

~~but~~ for $\Gamma \text{ non } |\Gamma|$ can find a general (open dense)

$$\text{set of } p\text{'s s.t. } \sum_T L_\Gamma(t) \neq L(t)$$

So OK: ψ_Γ does not divide $P_\Gamma(t, p)$

(but for non $|\Gamma|$ may not be true
 and not true possibly when ψ_Γ not irreducible: $\Gamma = \Gamma_1 \cup \Gamma_2$)

• Know that the integrals satisfy

$$U(\Gamma, p) = U(\Gamma_1, p_1) \cdot U(\Gamma_2, p_2)$$

for disjoint unions $\Gamma = \Gamma_1 \sqcup \Gamma_2 \quad p = p_1 \sqcup p_2$

What about the hypersurface complements where
 the differential forms live: different for affine
 or projective case

⑧ Affine hypersurface complements $\Gamma = \Gamma_1 \sqcup \Gamma_2$

$$\mathbb{A}^n \setminus \hat{X}_\Gamma = (\mathbb{A}^{n_1} \setminus \hat{X}_{\Gamma_1}) \times (\mathbb{A}^{n_2} \setminus \hat{X}_{\Gamma_2}) \quad \left(\sum_{i=1}^k n_i = n \right)$$

more generally for $\Gamma = \sqcup_i \Gamma_i$: $\mathbb{A}^n \setminus \hat{X}_\Gamma = \prod_i \mathbb{A}^{n_i} \setminus \hat{X}_{\Gamma_i}$

Pf: $\psi_\Gamma(t) = \det M_\Gamma(t)$

if disjoint union of Γ_i matrix splits into blocks

$$\text{so } \psi_\Gamma(t) = \prod_{i=1}^k \psi_{\Gamma_i}(t_{i,1}, \dots, t_{i,n_i}) \quad (*)$$

each factor depends only on variables of that component

$$\text{so } \hat{X}_\Gamma = \bigcup_{i=1}^k \mathbb{A}^{n_1} \times \dots \times \mathbb{A}^{n_{i-1}} \times \hat{X}_{\Gamma_i} \times \mathbb{A}^{n_{i+1}} \times \dots \times \mathbb{A}^{n_k}$$

$\Rightarrow$ the hypersurface complement is a product of complements (simple set theoretic identity)

Different situation with projective hypersurfaces

$$\Gamma = \Gamma_1 \sqcup \Gamma_2$$

Complement $\mathbb{P}^{n_1+n_2-1} \setminus X_\Gamma$ is a $\mathbb{G}_m$ -bundle over $(\mathbb{P}^{n_1-1} \setminus X_{\Gamma_1}) \times (\mathbb{P}^{n_2-1} \setminus X_{\Gamma_2})$

In fact polynomials as in (*) but in proj. spaces

this means $\psi_\Gamma(t) = \psi_{\Gamma_1}(t_1) \cdot \psi_{\Gamma_2}(t_2) \Rightarrow$

$$X_\Gamma = C^{n_2}(X_{\Gamma_1}) \cup C^{n_1}(X_{\Gamma_2})$$

(9)

Union of cones in $\mathbb{P}^{n_1+n_2-1}$
over X_{Γ_i} w/ vertices $\mathbb{P}^{n_2-1}$ & $\mathbb{P}^{n_1-1}$, respectively

Assume neither graph is a forest (if forest one X_{Γ_i} empty; not def. everywhere) then have a well defined map

$$\mathbb{P}^{n_1+n_2-1} \setminus X_\Gamma \rightarrow (\mathbb{P}^{n_1-1} \setminus X_{\Gamma_1}) \times (\mathbb{P}^{n_2-1} \setminus X_{\Gamma_2})$$

$$(t_{1,1} : \dots : t_{1,n_1} : t_{2,1} : \dots : t_{2,n_2}) \mapsto ((t_{1,1} : \dots : t_{1,n_1}), (t_{2,1} : \dots : t_{2,n_2}))$$

surjective and fiber over a pt is set of all

$$(u t_{1,1} : \dots : u t_{1,n_1} : v t_{2,1} : \dots : v t_{2,n_2}) \text{ where}$$

$$(u:v) \in \mathbb{P}^1 \quad (uv \neq 0) \quad \text{i.e. a pt in } G_m$$

1PI graphs versus connected graphs and (affine) graph hypersurface

Γ non 1PI connected obtained by inserting 1PI graphs Γ_v at vertices of a tree T

$$\Rightarrow \mathbb{A}^n \setminus X_\Gamma = \mathbb{A}^{\#E(T)} \times \prod_{v \in V(T)} (\mathbb{A}^{\#E_{\text{int}}(\Gamma_v)} \setminus X_{\Gamma_v})$$

Pf: if Γ forest then $\mathbb{A}^n \setminus X_\Gamma = \mathbb{A}^n$; if Γ_1 & Γ_2 are

(10) graphs joined at a single vertex $\Gamma_1 * \Gamma_2$

then $A^{\wedge}, \hat{X}_{\Gamma} = (A^{\wedge}, \hat{X}_{\Gamma_1}) \times (A^{\wedge}, \hat{X}_{\Gamma_2})$ (if not irreducible)
 Same as for a disjoint union

- so applying to construction of non IPI T
 from IPI Γ_v and edges of T get formula

Graphs and dual graphs (Cremona transformation)

If Γ planar graph

dual graph Γ^v (depends on planar embedding)

- choose $i: \Gamma \hookrightarrow S^2$ embedding

- take as vertices $v \in V(\Gamma^v)$ a pt in each
 connected component of $S^2 \setminus i(\Gamma)$

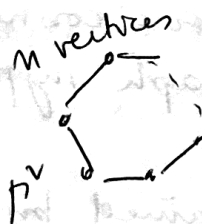
- edges $e \in E(\Gamma^v)$ between v & v' for each
~~there is an~~ edge of Γ in the boundary
 between the regions containing v & v'

So $\#E(\Gamma^v) = \#E(\Gamma)$

Example:

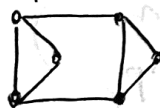


n parallel
edges



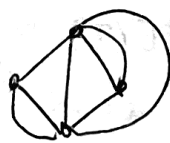
polygon

Example: dependence on embedding



same Γ two different
planar embeddings

Γ^v



combinatorially
non-isomorphic
graphs

Claim! Planar graph Γ $n = \#E(\Gamma)$
 $\Gamma^\vee$ dual graph (for chosen embedding)

(11)

$$\psi_\Gamma(t_1, \dots, t_n) = \left(\prod_{e \in E(\Gamma)} t_e \right) \psi_{\Gamma^\vee}(t_1^{-1}, \dots, t_n^{-1})$$

i.e. graph hypersurfaces related by Cremona transformation

$$\mathcal{C}(X_\Gamma \cap (\mathbb{P}^{n-1} \setminus \Sigma_n)) = X_{\Gamma^\vee} \cap (\mathbb{P}^{n-1} \setminus \Sigma_n)$$

$\Sigma_n = \{t \mid \prod_e t_e = 0\}$ divisor of coordinate hyperplanes

$$e: (t_1, \dots, t_n) \mapsto \left(\frac{1}{t_1}, \dots, \frac{1}{t_n} \right)$$

$e: \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ a pair defined on complement of Σ_n

Notation:

$$G(e) = \text{graph of } e \subset \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$$

$$\begin{array}{ccc} & G(e) & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathbb{P}^{n-1} & \xrightarrow{e} & \mathbb{P}^{n-1} \end{array}$$

with projections

$S_n \subset \mathbb{P}^{n-1}$ locus defined by ideal

$$I_{S_n} = (t_1 \cdots t_{n-1}, t_1 \cdots t_{n-2} t_n, \dots, t_2 \cdots t_n)$$

prods of $n-1$ coords

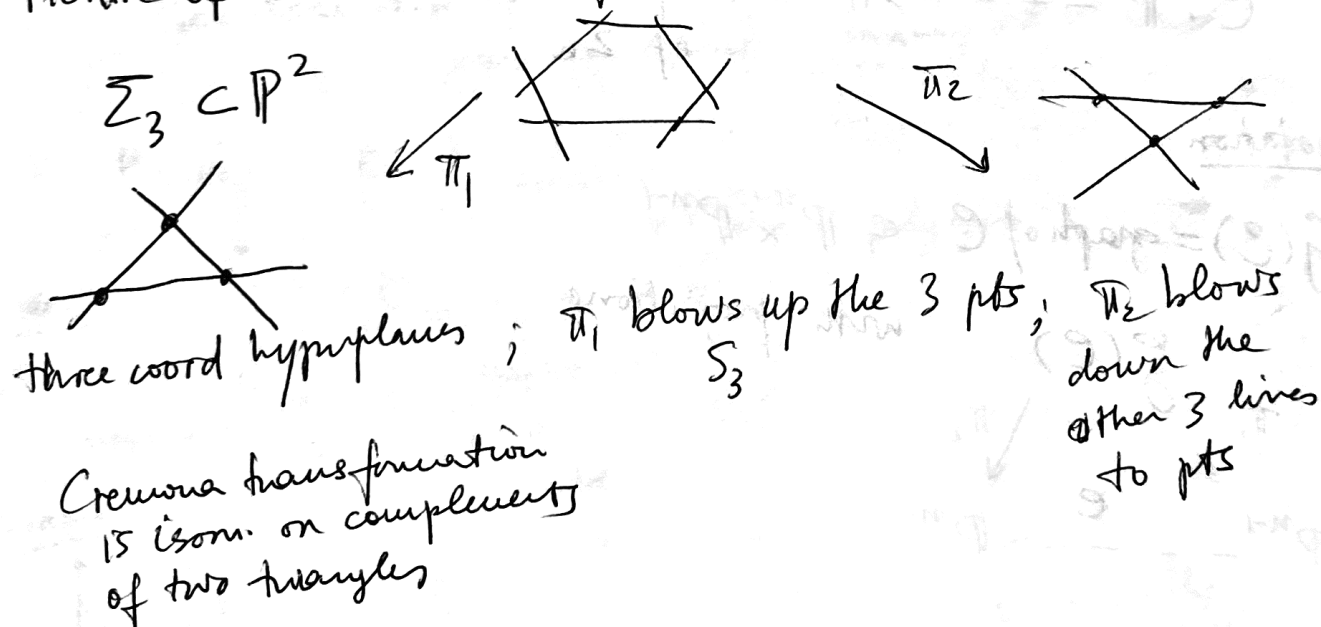
- (12) Then
- indeterminacy locus of $\mathcal{C}$ is S_n ← is singular locus of Σ_n (not layer Σ_n)
 - $\pi_1: \mathcal{G}(\mathcal{C}) \rightarrow \mathbb{P}^{n-1}$ is blowup along S_n
 - $L = \{t \in \mathbb{P}^{n-1} \mid \sum t_i = 0\}$ hyperplane intersects every component of S_n transversely
 - Σ_n ^{cuts out a} _{divisor} w/ simple normal crossings on L

Pf: rewrite components of $\mathcal{C}$ as

$$\mathcal{C}_i: (t_1: \dots: t_n) \mapsto (t_2 - t_n, \dots, t_1 - t_{n-1})$$

(384 follow from $S_n = \text{singular locus of } \Sigma_n$)

Picture of Cremona transformation in $\mathbb{P}^2$



Relation between the graph hypersurfaces

$$\psi_\Gamma(t_1, \dots, t_n) = \sum_{T \subseteq E(\Gamma)} \prod_{e \in T} t_e = \left(\prod_{e \in E(\Gamma)} t_e \right) \sum_{T \subseteq E(\Gamma)} \prod_{e \in T} t_e^{-1}$$

$$= \left(\prod_{e \in E(\Gamma)} t_e \right) \sum_{T' \subseteq \Gamma^\vee} \prod_{e \in T'} t_e^{-1}$$

$$= \left(\prod_{e \in E(\Gamma)} t_e \right) \psi_{\Gamma^\vee}(t_1^{-1}, \dots, t_n^{-1})$$

using $\#E(\Gamma) = \#E(\Gamma^\vee)$ $\#V(\Gamma^\vee) = b_0(S^2 \setminus \Gamma)$ so that
 $\deg \psi_\Gamma + \deg \psi_{\Gamma^\vee} = \#E(\Gamma)$

and $\exists$ bijection between complements of spanning
 trees in Γ and spanning trees in $\Gamma^\vee$

by shrink edges of T in Γ and take
 dual graph of resulting graph

If Γ not necessarily planar can still define
 a "dual hypersurface" $X_\Gamma^\vee$ s.t. under Cremona

$$X_\Gamma \setminus (X_\Gamma \cap \Sigma_n) \cong_c X_\Gamma^\vee \setminus (X_\Gamma^\vee \cap \Sigma_n)$$

though $X_\Gamma^\vee$ not
 a graph hypersurface
 in general