

Voisin-Voevodsky theorem

X smooth proj. / k then $Z_{\text{alg}}^1(X)_{\mathbb{Q}} \subset Z_{\text{hom}}^1(X)_{\mathbb{Q}}$

i.e. if $Z_{\text{alg}}^0 \Rightarrow \exists n > 0$ s.t. $Z^n = 0$ in $CH^*(X^n)_{\mathbb{Q}}$

Proof: outline of main steps

a) (because regularity for smooth by passing to $\mathbb{Q}$ coefficients)
can look at case of alg. closed $k = \bar{k}$

i) Reduce to case of smooth proj. curves;

$Z_{\text{alg}}^0 : \exists C$ sm. proj. curve,

$\Gamma \in \text{Corr}(C, X)$, $\nexists a, b \in C(k)$

s.t. $Z = \Gamma_*(a-b)$

Set $Z_0 = a-b \in A_0(C)$ $Z = \Gamma_*(Z_0)$

so $Z^n = (\Gamma^n)_*(Z_0^n)$

would suffice to show $Z_0^n = 0$ in $\underbrace{CH^*(X \times \dots \times X)}_{n\text{-times}}$

(in fact since Z_0^n inv. under S_n -action can see as cycle in symm. prod

$$CH_0(S^n(C))_{\mathbb{Q}} = CH_0(C^n)_{\mathbb{Q}}^{S_n}$$

Can show that
(see later)

$$(a-b)^n = 0 \quad \forall n \geq g+1$$

$$g = \text{genus}(C)$$

② C sm proj. curve $S^n(C)$ n -fold sym. product

$$[x_1, \dots, x_n] = \pi_n(x_1, \dots, x_n)$$

$$\pi_n: C^n \rightarrow S^n(C) \text{ proj.}$$

notation: write coords so as to collect together repeated entries

$$[\underbrace{a, \dots, a}_k, \dots, \underbrace{b, \dots, b}_l] = [ka, \dots, lb]$$

Then consider map $\varphi_n: S^n(C) \rightarrow J(C)$

$$\varphi_n[x_1, \dots, x_n] = \sum_{i=1}^n [x_i]$$

where pt $x_i \in J(C) \iff \text{divisor } (x_i) - (e)$
of deg 0 on C
for $e \in C(k)$ a base pt.

Claim: $(\varphi_n)_*: CH_0(S^n(C)) \xrightarrow{\sim} CH_0(J(C))$
for all $n \geq g$

— to see this first consider $n=g$ then known (Ful §16)

φ_g birational morphism & for X smooth & proper/ k
 $CH_0(X)$ birational invariant

(for $f: X \dashrightarrow Y$ birat. between smooth proper var/ k

$${}^t\Gamma_f \circ \Gamma_f = \text{id}_X + \text{consp that acts as zero on zero-cycles}$$

$$\Rightarrow f^* \circ f_* = \text{id on } CH_0(X)$$

— for $n \geq g$ $n-g=r$ embedding $S^g(C) \hookrightarrow S^n(C)$

$$[x_1, \dots, x_n] \mapsto [x_1, \dots, x_g, re]$$

$$(\varphi_g)_* = (\varphi_n)_* \circ \iota_* : CH_0(S^g(C)) \rightarrow CH_0(S^n(C)) \rightarrow CH_0(J(C))$$

Know $(\varphi_g)_*$ isom. so i_* injective
 if also know i_* surj. then $(\varphi_n)_*$ also isom.

$$y \in S^n(C) \quad z = \varphi_n(y) \in J(C)$$

$$\exists x \in S^g(C) \quad \varphi_g(x) = z$$

$i(x)$ and y in fiber $\varphi_n^{-1}(z)$: fiber is a proj. space
 so these pts are rationally equivalent

$$\Rightarrow i_*(x) = y$$

Consider the correspondences

$$\alpha_n : S^n(C) \rightarrow S^{n+1}(C)$$

$$\beta_n : S^{n+1}(C) \rightarrow S^n(C)$$

$$\alpha_n([x_1, \dots, x_n]) = [x_1, \dots, x_n, e] \quad (\text{morphism})$$

$$\beta_n([x_1, \dots, x_{n+1}]) = \sum_{k=0}^n \sum_{1 \leq i_1 < \dots < i_{n-k} \leq n+1} (-1)^k [x_{i_1}, \dots, x_{i_{n-k}}, ke] \quad (\text{correspondence but not morphism})$$

(for $k=n$ only one term $(-1)^n [ne]$)

$$\textcircled{3}: \quad \beta \circ \alpha = \text{id}_{S^n(C)}$$

$$\beta \alpha([x_1, \dots, x_n]) = \beta([x_1, \dots, x_n, e])$$

$$= \sum_{k=0}^n \sum_{1 \leq i_1 < \dots < i_{n-k} \leq n} (-1)^k [x_{i_1}, \dots, x_{i_{n-k}}, ke]$$

$$+ \sum_{k=0}^{n-1} \sum_{1 \leq i_1 < \dots < i_{n-k-1} \leq n} (-1)^k [x_{i_1}, \dots, x_{i_{n-k-1}}, (k+1)e]$$

all terms cancel except $k=0$ in first sum

(4) maps $\alpha_x: CH_0(S^n(C)) \rightarrow CH_0(S^{n+1}(C))$
 $\beta_#: CH_0(S^{n+1}(C)) \rightarrow CH_0(S^n(C))$
 are isomorphisms for all $n \geq g$

in fact have diagrams commuting:

$$\begin{array}{ccc} S^n(C) & \xrightarrow{\alpha} & S^{n+1}(C) \\ \varphi_n \downarrow & & \downarrow \varphi_{n+1} \\ J(C) & \xrightarrow[\text{id}]{\cong} & J(C) \end{array}$$

$\Rightarrow (\varphi_n)_* = (\varphi_{n+1})_* \circ \alpha_*$ since already know $(\varphi_n)_*$ isom's for $n \geq g$
 get α_* also isom
 then because $\beta_* \circ \alpha_* = \text{id}$ & α_* isom. $\Rightarrow \beta_*$ isom.

(5) next check that zero-cycle
 $Z_0 = a - b$ satisfies $\beta_{g*}(Z_0^{g+1}) = 0$
 (then done since β_* isom.)

to see this: assume $a = e$ base pt.

apply same trick $(z+z')^n = \sum_{r+s=n} \binom{n}{r} z^r \times z'^s$
 and get

$$Z_0^{g+1} = (b-e)^{g+1} = \sum_{k=0}^{g+1} (-1)^k \binom{g+1}{k} b^{g-k+1} \times e^k$$

cycle is symmetric so can view in
 Chow of $S^{g+1}(C)$

write them as $\sum_{k=0}^{g+1} (-1)^k \binom{g+1}{k} [(g-k+1)b, ke]$

$$= [(g+1)b] + \sum_{k=1}^{g+1} (-1)^k \binom{g+1}{k} \alpha_{g*} ([g-k+1]b, (k-1)e]$$

(5)

using $\beta_{g*} \circ \alpha_{g*} = \text{id}_{S^g(C)}$ get

$$\beta_{g*} (Z_0^{g+1}) = \beta_{g*} ([(g+1)b]) + \sum_{k=1}^{g+1} (-1)^k \binom{g+1}{k} [(g-k+1)b, (k-1)e]$$

// by def of β
 $\sum_{k=0}^g (-1)^k \binom{g+1}{k+1} [(g-k)b, ke]$ so cancels other term

Construction of category of pure motives (Grothendieck)

Correspondences (w/ rational equivalence)

$$\text{Corr}(X, Y) := CH(X \times Y) \otimes \mathbb{Q}$$

(also $\text{Corr}_{\sim \text{rat}}(X, Y)$)

if other equivalence relation

$$\text{Corr}_{\sim}(X, Y) = C_{\sim}(X \times Y) \otimes \mathbb{Q}$$

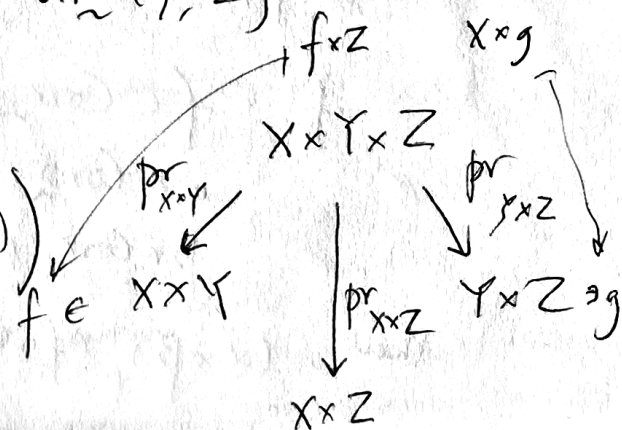
Composition of correspondences

$$f \in \text{Corr}_{\sim}(X, Y)$$

$$g \in \text{Corr}_{\sim}(Y, Z)$$

$g \circ f \in \text{Corr}_{\sim}(X, Z)$ defined by

$$g \circ f = \text{pr}_{X \times Z} ((f \times Z) \bullet (X \times g))$$



— Since using rat. equiv. already know
 at least inters. prod. in $X \times Y \times Z$ always
 well defined

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projectors $p \in \text{Corr}_{\sim}(X, X)$ satisfying
 $p \circ p = p$ (note depends on $\sim$)

degree of a correspondence

$$\dim X = d$$

$$\text{Corr}_{\sim}^r(X, Y) = \bigcup_{\sim}^{d+r} (X \times Y) \otimes \quad \text{degree } r \text{ correspondences}$$

if $\phi: X \rightarrow Y$ morphism $\dim X = d$ $\dim Y = e$
 then $\Gamma_{\phi} \in X \times Y$ gives corresp.

$$\phi_* = \Gamma_{\phi} \in \text{Corr}^{e-d}_{\sim}(X, Y)$$

$$\phi^* = \Gamma_{\phi} \in \text{Corr}_{\sim}^0(Y, X) \quad \leftarrow \text{"cohom. theory" contravariant}$$

A useful property (easy to verify check projections
 from various product spaces)

$$\begin{aligned} \text{if } f &\in \text{Corr}_{\sim}(X, Y) \\ \alpha &\in \text{Corr}_{\sim}(X, X') \\ \beta &\in \text{Corr}_{\sim}(Y, Y') \end{aligned}$$

(Lieberman's Lemma)

$$\text{then } (\alpha \times \beta)_* (f) = \beta \circ f \circ {}^t \alpha$$

Pure motives

⑦

Construction in steps:

$$\text{SmProj}(k)^{\text{opp}} \rightarrow C_{\sim} \text{SmProj}(k) \rightarrow \text{Mot}_{\sim}^{\text{eff}}(k) \hookrightarrow \text{Mot}_{\sim}(k)$$

- ① $C_{\sim} \text{SmProj}(k)$ same objects as $\text{SmProj}(k)$
but morphisms = degree zero correspondences
w/ composition of corresp.

for f morphism in $\text{SmProj}(k) \mapsto \Gamma_f \in \text{Corr}_{\sim}^0(Y, X)$

Note unlike $\text{SmProj}(k)$, $C_{\sim} \text{SmProj}(k)$ is an additive category
(first improvement on categorical properties)

- ② effective motives $\text{Mot}_{\sim}^{\text{eff}}(k)$

Objects: pairs (X, p) w/ $X \in \text{SmProj}(k)$
 p a projector

Morphisms: $(X, p) \rightarrow (Y, q)$

$$\text{Hom}_{\text{Mot}_{\sim}^{\text{eff}}}((X, p), (Y, q)) = q \circ \text{Corr}_{\sim}^0(X, Y) \circ p$$

$q \circ f \circ p$ f deg zero correspondence

further improvement of categorical properties

$$\text{Mot}_{\sim}^{\text{eff}}(k) = (\text{C}_{\sim} \text{SmProj}(k))^{\wr}$$

the pseudo-abelian (or Karoubian)
envelope of $\text{C}_{\sim} \text{SmProj}(k)$

$\mathcal{C}$ = additive category

$\mathcal{C}^{\wr}$ = smallest cat. containing $\mathcal{C}$
where projectors in $\mathcal{C}$ have images and kernels in $\mathcal{C}^{\wr}$
pseudo-abelian

functor $A \mapsto (A, \text{id})$ from $\mathcal{C}$ to $\mathcal{C}^{\wr}$
is universal for functors from $\mathcal{C}$ to pseudo-abelian categories

In fact if $M = (X, p) \in \text{Mot}_{\sim}^{\text{eff}}(X)$

a projector $q \in \text{End}(M)$

is $q = p \circ q' \circ p$ $q' \in \text{Corr}_{\sim}^0(X, X)$

with $q^2 = q \circ q = q$

$N = (X, q)$ then is the image of q

and $N' = (X, p - q)$ is kernel of q

Since $q = p \circ q = q \circ p$

So indeed all projectors have images and kernels

then $M = N \oplus N'$ hence in $\mathcal{C}^{\wr}$

can split off Ker and images of projectors
as direct summands

final step: $\text{Mot}_{\sim}(k)$

Objects: (X, p, m) $X \in \text{SmProj}(k)$ p projector
 $m \in \mathbb{Z}$

morphisms

$$\text{Hom}_{\text{Mot}_{\sim}(k)}((X, p, m), (Y, q, n)) \\ = q \circ \text{Corr}_{\sim}^{n-m}(X, Y) \circ p$$

this allows for all possible degrees of correspondences as morphisms

— Composition as composition of correspondences

— faithful embedding $\text{Mot}_{\sim}^{\text{eff}} \hookrightarrow \text{Mot}_{\sim}$

— The additional integer m : Tate twist

Notation: $X \in \text{SmProj}(k)$ $h_{\sim}(X) \in \text{Mot}_{\sim}(k)$

image under contravariant functor

$$h_{\sim}: \text{SmProj}(k) \longrightarrow \text{Mot}_{\sim}(k)$$

$$f: X \rightarrow Y \longmapsto h_{\sim}(f) = {}^t \Gamma_f \in \text{Hom}_{\text{Mot}_{\sim}(k)}(h_{\sim}(Y), h_{\sim}(X))$$

Sometimes use notation $ch(X) = h_{\text{rat}}(X)$

In general $(X, p, 0)$ is a direct summand of $h_{\sim}(X)$

so it represents a "piece" of the motive (corresp to piece of cohomology)

Examples

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1) $\mathbb{1} = (\text{Spec}(k), \text{id}, 0) = h_{\sim}(pt)$

2) if $X(k) \neq \emptyset$ (always up to passing to an extension)

$e \in X(k) \quad \pi_0(X) = e \times X$

$\pi_{2d}(X) = X \times e$

$\pi_0 \circ \pi_{2d} = \pi_{2d} \circ \pi_0 = 0$

$\leadsto$ corresp. motives $h_{\sim}^0(X) = (X, \pi_0, 0)$

$h_{\sim}^{2d}(X) = (X, \pi_{2d}, 0)$

Note: have $h_{\sim}^0(X) \cong \mathbb{1}$

and $h_{\sim}^{2d}(X)$ isomorphic for all X w/ $\dim X = d$

using $X \rightarrow \text{Spec}(k)$ structural map
and $\text{Spec}(k) \rightarrow X$ point e

3) Direct sum of motives: $M = (X, p, m) \quad N = (Y, q, m)$

$M \oplus N = (X \sqcup Y, p \sqcup q, m)$

same

general case see scholl

4) $M = (X, p, 0)$ $1-p$ also projector and $h_{\sim}(X) = (X, p, 0) \oplus (X, 1-p, 0)$

5) Lefschetz motive and Tate motive

$\mathbb{L}_{\sim} := (\mathbb{P}^1, \pi^2, 0) = h_{\sim}^2(\mathbb{P}^1)$

$\pi := (\text{Spec}(k), \text{id}, 1)$

by viewing (X, p, m) as direct summand of $h_{\sim}(X \times \mathbb{P}^1 \rightarrow \mathbb{P}^1)$

(Scholl)

Note $\mathbb{L}_{\sim} = (\text{Spec}(k), \text{id}, -1)$

(in general $h_{\sim}^{2d}(X) = (\text{Spec}(k), \text{id}, -d)$)