Quantum field theory and motives

QFT 2

Momentum space picture:

Feynman rules: how to construct the asymptotic expansion

- $$\mathcal{G}_N(p_1,\dots,p_N) = \sum_\Gamma \frac{V(\Gamma, p_1,\dots,p_N)}{\# \mathrm{Aut}(\Gamma)}$$ momentum space Green functions

$\underline{p}$ = external momenta $\quad N = \# E_{ext}(\Gamma)$

Fourier transform of position space Green functions:

- $$\mathcal{G}_N(p_1,\dots,p_N) = \int G_N(x_1,\dots,x_N)\, e^{i(p_1x_1+\dots+p_Nx_N)} \frac{d^Dp_1}{(2\pi)^D}\cdots\frac{d^Dp_N}{(2\pi)^D}$$

- $$V(\Gamma, p_1,\dots,p_N) = \int I_\Gamma(p_1,\dots,p_N, k_1,\dots,k_M) \frac{d^Dk_1}{(2\pi)^D}\cdots\frac{d^Dk_n}{(2\pi)^D}$$

$M = \# E_{int}(\Gamma)$ $\quad k_e$ = momentum variable in $\mathbb{R}^D$ associated to internal edge $e \in E_{int}(\Gamma)$

- Feynman rules give a procedure to construct the densities $I_\Gamma(p_1,\dots,p_N, k_1,\dots,k_n)$

(1) For each vertex $v \in V(\Gamma) \rightsquigarrow$ factor $\lambda_v (2\pi)^D$

where λ_v coupling constant coeff (up to $k!$) of monomial ϕ^k in $P(\phi)$ s.t. $\mathrm{val}(v) = k$.

② and a conservation law at vertex :
linear constraint among variables of adjacent edges

$$\delta_v(k) := \delta\Big(\sum_{s(e)=v} k_e - \sum_{t(e)=v} k_e \Big)$$

if all adjacent edges internal

here fixing an orientation of the graph Γ

$$\delta_v(k,p) := \delta\Big(\sum_{i=1}^{M} \varepsilon_{v,i}\, k_i + \sum_{j=1}^{N} \varepsilon_{v,j}\, p_j \Big)$$

if both internal and external edges

$$\varepsilon_{v,i} = \begin{cases} 0 & e_i \text{ not adjacent to } v \\ +1 & v = s(e_i) \\ -1 & v = t(e_i) \end{cases}$$

$\varepsilon_{v,j}$ same for external edges e_j

(2) Each internal edge contributes an "inverse propagator" q_e^{-1}

$$q_e(k_e) = k_e^2 + m^2 \quad \text{(Euclidean)}$$

$$k_e^2 = \sum_{i=1}^{D} k_{e,i}^2$$

quadratic form in the momenta

Note: write as $k_e^2 = \sum_{i=1}^{D} k_{e,i}^2$ instead of $\sum_{i=1}^{D} |k_{e,i}|^2$ because same for $k_{e,i} \in \mathbb{R}$ and will be extending to $\mathbb{C}$ as algebraic expression, unlike $\|k_e\|^2$

Note: any expression defined for $k_e \in \mathbb{C}^D$ that agrees with q_e for $k_e \in \mathbb{R}^D$ also OK

(3) External edges p_e with $\Sigma p_e = 0$ summed over oriented ext. edges
(outgoing p_e − incoming p_e)

propagator of ext. edges

$$q_e(p_e) = p_e^2 + m^2 \rightsquigarrow q_e^{-1}(p_e) \text{ factor}$$

(4) Putting all together: $I_\Gamma(p_1, \dots, p_N, k_1, \dots, k_n)$ is

$$\prod_{v \in V(\Gamma)} \lambda_v (2\pi)^D \delta_v(k_{e_i}, p_{e_j}) \prod_{e_i \in E_{int}(\Gamma)} q_{e_i}(k_{e_i})^{-1} \prod_{e_j \in E_{ext}(\Gamma)} q_{e_j}(p_{e_j})^{-1}$$

linear relations among the momentum variables

$$V(\Gamma, p_1, \dots, p_N) = \varepsilon(p_1, \dots, p_N)\, U(\Gamma, p_1, \dots, p_N)$$

isolate contribution of propagators of external edges (not contributing to k_e - integration)

$$\varepsilon(p_1, \dots, p_N) = \prod_{e \in E_{ext}(\Gamma)} q_e(p_e)^{-1}$$

$$U(\Gamma, p_1, \dots, p_N) = C \int \frac{\delta\left(\sum_{i=1}^{n} \varepsilon_{v,i} k_i + \sum_{j=1}^{N} \varepsilon_{v,j} p_j\right)}{q_1(k_1) \cdots q_n(k_n)} \frac{d^D k_1}{(2\pi)^D} \cdots \frac{d^D k_n}{(2\pi)^D}$$

$$C = \prod_{v \in V(\Gamma)} \left(\lambda_v (2\pi)^D\right)$$

④ Simplifying the combinatorics of graphs:

- $V(\Gamma, p_1, \dots, p_N)$ multiplicative over disjoint unions of graphs $\Gamma = \Gamma_1 \sqcup \Gamma_2$

$$V(\Gamma, p_1, \dots, p_N) = V(\Gamma_1, p_1, \dots, p_{N_1}) \cdot V(\Gamma_2, p_{N_1+1}, \dots, p_N)$$

clear because no linear relation between momenta belonging to different components so just products of integrals

- The symmetry factors of graphs are also multiplicative over disjoint unions (up to keeping into account permutations)

$$\# \mathrm{Aut}(\Gamma) = \prod_j (m_j)! \prod_j \# \mathrm{Aut}(\Gamma_j)^{m_j}$$

symmetries of Γ that permute topologically isom. components

$\Gamma = \bigsqcup_j m_j \Gamma_j$ $\quad$ m_j connected components of Γ all isomorphic to Γ_j

- Vacuum bubbles: graphs with no external edges

from fin dim example: see $J=0$; vacuum bubbles are counted by $Z(0)$

So in $\frac{Z(J)}{Z(0)}$ because of multiplicative behavior of V and symmetries $\Rightarrow \frac{Z(J)}{Z(0)}$ no vacuum bubbles

- moreover need only use connected graphs because V of a multi-connected completely determined by V of components

technically pass to $W(J) = \log\left(\frac{Z(J)}{Z(0)}\right)$

also seen as generating function for "connected Green functions"

$$W(J) = \sum_{N=0}^{\infty} \frac{1}{N!} \int J(x_1) \cdots J(x_N) \; G_{N,c}(x_1, \ldots, x_N) \; d^D x_1 \cdots d^D x_N$$

where the Fourier transformed

$$\mathcal{G}_{N,c}(p_1, \ldots, p_N) = \sum_{\Gamma \text{ connected}} \frac{V(\Gamma, p_1, \ldots, p_N)}{\# \mathrm{Aut}(\Gamma)}$$

(⊗ in fact: for $\sum_j n_j N_j = N$

$$\frac{V(\Gamma, p_1, \ldots, p_N)}{\# \mathrm{Aut}(\Gamma)} = \prod_j \frac{V(\Gamma_j, p_{j1}, \ldots, p_{jN_j})^{n_j}}{(n_j)! \; \# \mathrm{Aut}(\Gamma_j)^{n_j}}$$

$$\frac{Z(J)}{Z(0)} = \sum_N \frac{1}{N!} \int J(x_1) \cdots J(x_N) \, G_N(x_1, \ldots, x_N) \prod_i d^D x_i$$

$$= \sum_N \sum_{\sum_j n_j N_j = N} \prod_j \frac{1}{n_j!} \left(\int J(x_1) \cdots J(x_{N_j}) \, G_{N_j,c}(x_1, \ldots, x_{N_j}) \right)^{n_j}$$

$$= \exp\left(\sum_N \frac{1}{N!} \int J(x_1) \cdots J(x_N) \, G_{N,c}(x_1, \ldots, x_N) \right)$$
)

6 Further simplification of combinatorics of graphs

k-connectivity for graphs:

- Γ k-edge-connected if cannot be disconnected by removal of any set of k or fewer edges

- Γ 2-vertex-connected if has no looping edges  and has at least 3 vertices and cannot be disconnected by removal of a single vertex

Note: vertex removal = vertex together with star of edges incident to that vertex

- Γ is k-vertex-connected (some $k \geq 3$) if it has no looping edges and has at least $k+1$ vertices
and no multiple edges 
and
cannot be disconnected by removal of any set of $k-1$ vertices

Note: 1-vertex-connected = 1-edge-connected = connected

$k \geq 2$ k-vertex-connected stronger than k-edge-connected

Physics terminology: for $k \geq 2$

$(k+1)$-edge-connected $=$ k-particle-irreducible (kPI)

for $k=1$:

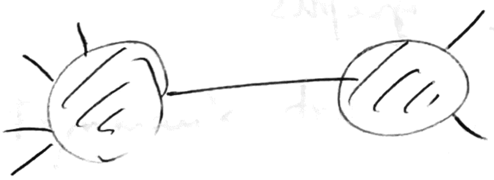
a 2-edge-connected graph (not a tree) is called a 1-particle-irreducible graph

Any connected graph can be obtained from a tree T and 1PI graphs Γ_v for $v \in V(T)$ w/ $\# E_{ext}(\Gamma_v) = \mathrm{val}(v)$

if Γ is 1PI $\quad T =$ single vertex with $E_{ext}(T) = E_{ext}(\Gamma)$

Γ $\qquad$ T

if Γ not 1PI find an edge whose removal disconnects



then repeat operation in each blob until left w/ blobs that are 1PI graphs

Contract each blob to vertex

$\rightsquigarrow$ get tree T

⑧ $\Rightarrow$ behavior of $V(\Gamma, p_1, \ldots, p_N)$
over a non 1-PI graph Γ:

$\{\Gamma_v\}$ collection of 1PI graphs
and $e \in E_{int}(T)$ disconnecting edges of Γ between different Γ_v's

inverse propagators along connecting edges

$$V(\Gamma, p) = \prod_{\substack{v \in V(T) \\ e \in E_{int}(T) \\ v \in \partial(e)}} V(\Gamma_v, p_v)\, q_e(p_v)^{-1}\, \delta((p_v)_e - (p_{v'})_e)$$

external momentum of Γ_v along edge e of T

matched to $p_{v'}$ external momentum along same edge of T

$\Rightarrow$ $V(\Gamma, p)$ completely determined by $V(\Gamma_v, p_v)$ (w/ matching conditions on p_v's when glued along edges of T)

$\Rightarrow$ enough to consider only 1PI graphs

(Effective action)

Schwinger and Feynman parameters

"parametric Feynman integral"

Gamma function: $\Gamma(t+1) = \int_0^\infty s^t e^{-s}\, ds$

$$\Gamma(t+1) = t\,\Gamma(t)$$

Exponential integral: $\frac{1}{q} = \int_0^\infty e^{-sq}\, ds$ $\quad s =$ Schwinger parameter

More general:

$$\frac{1}{q_1 \cdots q_n} = \int_{\mathbb{R}_+^n} e^{-(s_1 q_1 + \cdots + s_n q_n)}\, ds_1 \cdots ds_n$$

and also

$$\frac{1}{q_1^{k_1} \cdots q_n^{k_n}} = \frac{1}{\Gamma(k_1) \cdots \Gamma(k_n)} \int_{\mathbb{R}_+^n} e^{-(s_1 q_1 + \cdots + s_n q_n)}\, s_1^{k_1 - 1} \cdots s_n^{k_n - 1}\, ds_1 \cdots ds_n$$

Feynman's trick:

$$\frac{1}{ab} = \int_0^1 \frac{1}{(ta + (1-t)b)^2}\, dt$$

(10) A more general form of the Feynman trick

$$\frac{1}{q_1^{k_1}\cdots q_n^{k_n}} = \frac{\Gamma(k_1+\cdots+k_n)}{\Gamma(k_1)\cdots\Gamma(k_n)} \int_{[0,1]^n} \frac{t_1^{k_1-1}\cdots t_n^{k_n-1}\,\delta(1-\sum_i t_i)}{(t_1 q_1+\cdots+t_n q_n)^n}\, dt_1\cdots dt_n$$

(performing $s_i = S t_i$ with $S = s_1+\cdots+s_n$ change of variables in previous int for $q_1^{-k_1}\cdots q_n^{-k_n}$)

In particular for

$$\frac{1}{q_1\cdots q_n} = (n-1)! \int_{\sigma_n} \frac{dt_1\cdots dt_n}{(t_1 q_1+\cdots+t_n q_n)^n}$$

$$\sigma_n = \{ t = (t_1\cdots t_n) \in \mathbb{R}_+^n \mid \sum_i t_i = 1 \}$$

Use this to rewrite the Feynman integral

$$U(\Gamma, p_1,\dots,p_N) = \int \frac{\delta\left(\sum_{i=1}^{n} \varepsilon_{v,i} k_i + \sum_{j=1}^{N} \varepsilon_{v,j} p_j\right)}{q_1(k_1)\cdots q_n(k_n)}\, d^D k_1 \cdots d^D k_n$$

in terms of an integral involving the Feynman parameters $t = (t_i)$ with $t \in \sigma_n$

• circuit matrix of the graph : (11)

$$\eta_{ik} = \begin{cases} +1 & e_i \in \gamma_k \text{ matching orientations} \\ -1 & e_i \in \gamma_k \text{ reverse orientation} \\ 0 & e_i \notin \gamma_k \end{cases}$$

choose a basis $\gamma_1, \dots, \gamma_\ell$ of $H_1(\Gamma, \mathbb{Z})$ and representatives given by loops γ_i in Γ

incidence matrix $\varepsilon_{v,i}$ and circuit matrix $\eta_{i,k}$

satisfy $\quad \varepsilon \eta = 0$

(indep of choice of orientation & of basis of H_1)

in fact: suppose e_i, e_j both incident to v and both in loop γ_k then $\varepsilon_{v,i}\eta_{i,k} = -\varepsilon_{v,j}\eta_{j,k}$

so entries of $\varepsilon\eta$ are sums with terms that cancel in pairs

Def: Kirchhoff - Symanzik matrix of Γ
$\ell \times \ell$ matrix

$$(M_\Gamma(t))_{kr} = \sum_{i=1}^{n} t_i \, \eta_{ik} \eta_{ir}$$

$$M_\Gamma(t) = \eta^t \Lambda(t) \eta \qquad \Lambda(t) = \mathrm{diag}(t_1, \dots, t_n)$$

$$M_\Gamma : \mathbb{A}^n \longrightarrow \mathbb{A}^{\ell^2} \qquad t \mapsto (M_\Gamma(t))_{k,r=1,\dots,\ell}$$

12 The Kirchhoff polynomial (or first Symanzik polynomial or just graph polyn.)

$$\psi_\Gamma(t) = \det M_\Gamma(t)$$

Note: $\psi_\Gamma(t)$ indep. of orientation choice and choice of basis of $H_1(\Gamma)$

in fact: change of orient of one edge e_i
$\rightsquigarrow$ sign change of one column of η_{ik}
and one row η_{ki} of η^t

$\rightsquigarrow$ det unchanged

change of basis of H_1 conjugation by $A \in GL_\ell(\mathbb{Z})$

Change of variables in Feynman integral:

$$k_i = u_i + \sum_{r=1}^{\ell} \eta_{ir} x_r$$

where u_i required to satisfy condition

$(**)$ $\displaystyle\sum_{i=1}^{m} t_i u_i \eta_{ir} = 0$

i.e. columns of $\Lambda(t) u$ orthog. to corresp. rows of η

the momentum conservation constraint

$$\sum_{i=1}^{n} \varepsilon_{v,i} k_i + \sum_{j=1}^{N} \varepsilon_{v,j} p_j = 0$$

becomes after change of var. (~~with u-constraint~~)

$$\sum_{i=1}^{n} \varepsilon_{v,i} u_i + \sum_{j=1}^{N} \varepsilon_{v,j} p_j = 0 \quad (*)$$

(*) together with (**) Kirchhoff law ~~relations~~ for circuits applied to momenta

Note: $u_i = u_i(p)$ uniquely determined as a function of external momenta by these relations (explicitly see below)

Fact: $\psi_\Gamma(t) = \det(M_\Gamma(t))$ also written as

$$\psi_\Gamma(t) = \sum_{\substack{T \subset \Gamma \\ \text{spanning trees}}} \prod_{e \notin E(T)} t_e$$

homogeneous polynomial of deg. $b_1(\Gamma)$

Sketch of idea: expand det showing written as

$$\sum_S \prod_{e \in S} t_e$$

where S ranges over maximal subsets $S \subset E_{int}(\Gamma)$ s.t. removal of all edges in S leaves a connected graph

= complements of spanning trees

changing variables from k_i to x_i

$$\int \frac{d^D x_1 \cdots d^D x_\ell}{\left(\sum_{i=0}^n t_i q_i\right)^n} = \int \frac{d^D x_1 \cdots d^D x_\ell}{\left(\sum_{i=0}^n t_i(u_i^2+m^2) + \sum_{k,r} (M_\Gamma(t))_{k,r} x_k x_r\right)^n}$$

changing variables again diagonalizing M_Γ

$$\int \frac{d^D y_1 \cdots d^D y_\ell}{(a + \sum_k \lambda_k y_k^2)^n} = C_{\ell,n} \; a^{-n+D\ell/2} \prod_{k=1}^{\ell} \lambda_k^{-D/2}$$

$$C_{\ell,n} = \int \frac{d^D x_1 \cdots d^D x_\ell}{(1 + \sum_k x_k^2)^n}$$

for this term need explicit expression of $\sum_i t_i u_i^2$ as function of external momenta

this in terms of $\det M_\Gamma$

so $\Psi_\Gamma(t)^{-D/2}$

this ~~[illegible]~~ uses fact that u_i determined by p_e's through the Kirchhoff law

Claim: the term $\sum_i t_i u_i^2 = p^t R_\Gamma(t) p$

$p = (p_e)_{e \in E_{ext}(\Gamma)}$ external momenta

and R_Γ $N \times N$ - matrix

$$p^t R_\Gamma(t) p = \sum_{v,v' \in V(\Gamma)} P_v \, (D_\Gamma(t)^{-1})_{v,v'} \, P_{v'}$$

where

$$(D_\Gamma(t))_{v,v'} = \sum_{i=1}^{n} \varepsilon_{v,i}\, \varepsilon_{v',i}\, t_i^{-1}$$

$$P_v = \sum_{\substack{e \in E_{ext}(\Gamma) \\ t(e) = v}} p_e$$

<u>Sketch of proof</u>: (cf. Björken-Drell; Nakanishi; Itzykson-Zuber)

by Kirchhoff law
$\exists$ matrix $A_{i,v}$ s.t. $- \sum_v A_{i,v} P_v = t_i u_i$

hence write

$$\sum_i t_i u_i^2 = \sum_v \sum_{v,v'} P_v P_{v'} A_{i,v} A_{i,v'} t_i^{-1}$$

constraints on the u_i in Kirchhoff law:
$\Rightarrow A_{i,v} = \varepsilon_{i,v}$ so matrix has to be of form $D_\Gamma(t)$

Set $V_\Gamma(t,p) := p^t R_\Gamma(t) p + m^2$

<u>massless case</u>: $V_\Gamma(t,p)\big|_{m=0} = \dfrac{P_\Gamma(t,p)}{\psi_\Gamma(t)}$

homogeneous polynomial of deg $b_1(\Gamma)+1$

graph polyn. as above

$$P_\Gamma(t,p) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e$$

compl. of a spanning tree plus one edge

sum over <u>cut-sets</u>
C = collections of $b_1(\Gamma)+1$ edges whose deletion dividing Γ into two components $\Gamma_1 \sqcup \Gamma_2$

where s_C function of external momenta

$$s_C = \Big(\sum_{v\in V(\Gamma_1)} P_v\Big)^2 = \Big(\sum_{v\in V(\Gamma_2)} P_v\Big)^2 \qquad \Gamma \setminus C = \Gamma_1 \sqcup \Gamma_2$$

Sketch of proof: $\det D_\Gamma(t) = \sum_T \prod_{e\in T} t_e^{-1}$ sum over spanning trees

relation to $\psi_\Gamma(t)$:

$$\psi_\Gamma(t) = \Big(\prod_e t_e\Big)\cdot \det D_\Gamma(t)$$

(later interpretation in terms of Cremona transformation)

for $m=0$ massless case:

$$p^t R_\Gamma(t)\, p = \sum_{v,v'\in V(\Gamma)} P_v\, (D_\Gamma(t))^{-1}_{v,v'}\, P_{v'}$$

$$= \frac{1}{\psi(t)} \sum_{C\subset\Gamma} s_C \prod_{e\in C} t_e$$

Outcome: after all changes of variables and initial reformulation w/ Feynman parameters

$$U(\Gamma, p_1,\dots,p_N) = \frac{\Gamma\left(n-\frac{D\ell}{2}\right)}{(4\pi)^{D\ell/2}} \int_{\sigma_n} \frac{\omega_n}{\psi_\Gamma(t)^{D/2}\, V_\Gamma(t,p)^{n-D\ell/2}}$$

residue

$$= \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\,\omega_n}{\psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

pole: ultraviolet divergence