Equivalence relations on algebraic cycles

$Z = \sum_\alpha n_\alpha Z_\alpha \qquad Z_\alpha \subset X$ irreducible subvarieties

$\operatorname{codim}(Z_\alpha) = i \quad \forall \alpha \quad \Rightarrow \quad Z \in \mathcal{Z}^i(X)$

$\mathcal{Z}(X) = \bigoplus_i \mathcal{Z}^i(X)$

Good equivalence relation on cycles
(or "adequate")

1. compatible with grading and addition
2. compatible with products: if $Z \sim 0$ on X then $\forall Y \quad Z \times Y \sim 0$ in $\mathcal{Z}(X \times Y)$
3. compatible with intersection
   if $Z_1 \sim 0$ and $Z_1 \cdot Z_2$ is defined
   then $Z_1 \cdot Z_2 \sim 0$
4. compatible with projections:
   if $Z \sim 0$ in $X \times Y$ then
   $(pr_X)_* Z \sim 0$ in X
5. Moving lemma: given $Z, W_1, \dots, W_\ell \in \mathcal{Z}(X)$
   $\exists Z' \sim Z$ s.t. $Z' \cdot W_j$ is defined $\forall j = 1, \dots, \ell$

Then set $\mathcal{Z}^i_\sim(X) = \{ Z \in \mathcal{Z}^i(X) \mid Z \sim 0 \}$ subgroup of $\mathcal{Z}^i(X)$

and $\underset{\sim}{\mathcal{Z}}^{i}(X)_F = \underset{\sim}{\mathcal{Z}}^{i}(X) \otimes_{\mathbb{Z}} F$

then define quotient groups of cycles

$$\underset{\sim}{C}^{i}(X) = \mathcal{Z}^{i}(X) \Big/ \mathcal{Z}^{i}_{\sim}(X)$$

$$\underset{\sim}{C}(X) = \bigoplus_{i} \underset{\sim}{C}^{i}(X) \qquad \underset{\sim}{C}(X)_F = \underset{\sim}{C}(X) \otimes_{\mathbb{Z}} F$$

The properties required of a <u>good</u> equivalence relation ensure that

1) $\underset{\sim}{C}(X)$ is a <u>ring</u> product defined by intersection prod of cycles

2) $f : X \to Y$ morphism in $\mathrm{SmProj}(K)$

then have well defined push forward and pullback homomorphisms

$$f_* : \underset{\sim}{C}(X) \to \underset{\sim}{C}(Y) \quad \text{hom. abelian groups}$$

$$f^* : \underset{\sim}{C}(Y) \to \underset{\sim}{C}(X) \quad \text{(homom. of graded rings)}$$

3) Z correspondence of degree r from X to Y

induces $Z_* : \underset{\sim}{C}^{i}(X) \to \underset{\sim}{C}^{i+r}(Y)$ hom. abelian groups

equivalent correspondences $Z \sim Z'$ determine same homom. Z_*

Rational equivalence relation

- generalizes linear equivalence of divisors

rational function $f \in k(X)$ $\rightsquigarrow$ divisor $\mathrm{div}(f)$

$$\mathrm{div}(f) = \sum_{Y \subset X} \mathrm{ord}_Y(f) \cdot Y \qquad \mathrm{codim}(Y) = 1$$

where $\mathrm{ord}_Y : k(X)^* \longrightarrow \mathbb{Z}$ defined by

$$\mathrm{ord}_Y(f) = \underbrace{\ell_A}_{\text{length as } A\text{-mod}}(A/(f)) \qquad A = \mathcal{O}_{X,Y} \text{ local ring (one dimensional)}$$

$\Rightarrow$ for $f \in k(Y)^*$ $Y \subset X$ irred. subvariety

$\mathrm{div}(f)$ is a codim 1 cycle on Y

$\Rightarrow$ codim i cycle in X if $\mathrm{codim}(Y) = i-1$

Let $Z^i_{rat}(X)$ be the subgroup of $Z^i(X)$ generated by these cycles $\mathrm{div}(f)$ for (Y, f) $\mathrm{codim}(Y) = i-1$, $f \in k(Y)^*$

i.e. $Z \sim_{rat} 0$ iff $\exists$ finite collection (Y_α, f_α) $Y_\alpha \subset X$ irred. subvar, $\mathrm{codim}(Y_\alpha) = i-1$, $f_\alpha \in k(Y_\alpha)^*$

such that

$$Z = \sum_\alpha \mathrm{div}(f_\alpha)$$

Chow groups

$$CH^i(X) = Z^i(X) \Big/ Z^i_{rat}(X) = C^i_{rat}(X)$$

$$CH(X) = \bigoplus_i CH^i(X)$$

<u>Note</u>: $\sim_{rat}$ is a <u>good</u> equivalence relation (so $CH(X)$ is a ring not just a group)

- properties ①, ②, ③ of good equivn easy to verify
- property ④ follows from a more general fact:

$f : X \to Y$ proper morphism

$Z \sim_{rat} 0$ on X <u>then</u> $f_* Z \sim_{rat} 0$

<u>idea of why</u>: reduce to case of $Z = \mathrm{div}(\varphi)$

$\varphi \in k(X')^*$ $X' \subset X$ subvar.

replace $f : X \to Y$ with restriction to X'

$f : X' \to f(X')$

then $f_* Z = \mathrm{div}(N(\varphi))$ where

$N : k(X') \to k(f(X'))$ is the "norm" = determinant of linear map given by multipl. by φ

- property ⑤ <u>moving lemma</u>

$X \in \mathrm{SmProj}(k)$ $X \hookrightarrow \mathbb{P}^N$ some fixed $\mathbb{P}^N$

Z codim i subvariety of X $\quad \dim X = d$

and given another $W \subset X$ $\quad \mathrm{codim}(W) = j$

- if $\mathrm{codim}(W \cap Z) = i+j$ proper intersection: nothing to move
- if $\mathrm{codim}(W \cap Z) > i+j$ excess intersection

$$\mathrm{excess}(W \cap Z) = \mathrm{codim}(W \cap Z) - (i+j)$$

then
- if $X = \mathbb{P}^N$ use linear transformations:
  a generic linear transformation will move Z to a Z' that intersects W properly
- more generally $X \hookrightarrow \mathbb{P}^N$
  take a suff. general linear space $L \subset \mathbb{P}^N$ of $\mathrm{codim}(L) = d+1$

$C(L,Z)$ = cone over Z w/ vertex L

for L general, well defined intersection product with X

$$C(L,Z) \cdot X$$

this intersection prod will have Z & something

$$C(L,Z) \cdot X = Z + Z'$$

show then that $\mathrm{excess}(W \cap Z') < \mathrm{excess}(W \cap Z)$:

and then proceed by <u>induction on excess</u> using

as cycles in ambient $\mathbb{P}^N$ for some lin. transf. τ

$\tau(C(L,Z) \cdot X) \cdot W$ defined $\quad$ and $Z \sim_{rat} \tau(C(L,Z) \cdot X) - Z'$

Resulting properties of Chow groups: $X \in SmProj(k)$

1) $CH(X) = \bigoplus_i CH^i(X)$ graded commutative ring w/ inters. prod.

2) $f: X \to Y$ morphism in $SmProj(k)$

$f^*: CH(Y) \to CH(X)$ graded ring homom.

$f_*: CH(X) \to CH(Y)$ abelian grp. homomorphism degree $\dim Y - \dim X$

3) $X, Y \in SmProj(k)$ $Z \in Corr^r_{rat}(X, Y)$

$\Rightarrow$ abelian grp. homom. $Z_*: CH(X) \to CH(Y)$ degree r

Two additional properties for more general k-varieties:

$i: Y \hookrightarrow X$ closed embedding

$j: U = X \setminus Y \hookrightarrow X$

$\Rightarrow$ exact sequence

$$CH_q(Y) \xrightarrow{i_*} CH_q(X) \xrightarrow{j^*} CH_q(U) \to 0$$

& homotopy property

$pr_X: X \times \mathbb{A}^n \to X$ induces isomorphism

$$pr_X^*: CH^i(X) \xrightarrow{\sim} CH^i(X \times \mathbb{A}^n)$$

Note: $\exists$ different way of defining inters. prod on $CH(X)$ without using moving lemma but using deform. to normal cone (Fulton § 6) helpful when want to maintain control of support of intersection class

Algebraic equivalence $X \in SmProj(k)$

Another definition of rational equivalence:

$Z \in \mathcal{Z}^i(X)$ $\quad Z \sim_{rat} 0$ iff $\exists$ cycle $W \in \mathcal{Z}^i(X \times \mathbb{P}^1)$ and $a, b \in \mathbb{P}^1$ such that the cycles

$$(pr_X)_*(W \bullet (X \times \{t\})) =: W(t)$$

satisfy $W(a) = 0$ and $W(b) = Z$

Also stated as

$Z \in \mathcal{Z}(X)$ $\quad Z \sim_{rat} 0$ iff

$\exists\, W \in \mathcal{Z}^i(X \times \mathbb{P}^1)$ s.t. $Z = W(0) - W(\infty)$

see Fulton § 11.4 for equiv. to previous formulation

From rational to algebraic equivalence by generalize this definition of equiv. by replacing $\mathbb{P}^1$ with an arbitrary curve C

$Z \sim_{alg} 0$ if $\exists$ smooth irreducible curve C and a cycle $W \in \mathcal{Z}(X \times C)$

and $a, b \in C$ s.t. $W(a) = 0$, $W(b) = Z$

$\mathcal{Z}^i_{alg}(X) = \{ Z \in \mathcal{Z}^i(X) \mid Z \sim_{alg} 0 \}$

$CH^i_{alg}(X) = \dfrac{Z^i_{alg}(X)}{Z^i_{rat}(X)} \subset CH^i(X)$ measures the difference between alg. & rational equivalence

Note: sufficient to use C curve instead of general V varieties for deformation: given $a, b \in V$ always $\exists\, C \subset V$ passing through a, b

difference between $Z^i_{rat}(X)$ and $Z^i_{alg}(X)$:

X elliptic curve $a, b \in X$ pts

$Z = a - b$ is <u>not</u> the divisor of a function

but is algebraically ~ 0

– Known that $\dfrac{CH^i(X)}{CH^i_{alg}(X)} = \dfrac{Z^i(X)}{Z^i_{alg}(X)}$ is a countable group (Chow van der Waerden '37)

Smash nilpotence equivalence $X \in SmProj(k)$

$X^n = \underbrace{X \times \dots \times X}_{n\text{-times}}$ $Z \subset X$ $Z^n = \underbrace{Z \times \dots \times Z}_{n\text{-times}}$

$$\boxed{Z \sim_{\otimes} 0 \quad \text{iff} \quad \text{for some } n \geq 1 \quad Z^n \sim_{rat} 0 \text{ on } X^n}$$

This is also a good equivalence relation

$-\ \mathcal{Z}^i_{\otimes}(X) = \{ Z \in \mathcal{Z}^i(X) \mid Z \sim_{\otimes} 0 \}$ is a subgroup of $\mathcal{Z}^i(X)$

in fact take $Z, Z' \in \mathcal{Z}^i_{\otimes}(X)$ want to check $Z + Z' \in \mathcal{Z}^i_{\otimes}(X)$

then for $n >> 1$

$$(Z+Z')^n = \sum_{r+s=n} \binom{n}{r} Z^r \times Z'^s$$

If n large enough then in each term either Z^r or Z'^s is $\sim_{rat} 0$ and so is the product $Z^r \times Z'^s \sim_{rat} 0$

$\Rightarrow (Z+Z')^n \sim_{rat} 0 \Rightarrow Z + Z' \sim_{\otimes} 0$

Similar argument shows $\sim_{\otimes}$ preserved by correspondences

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Theorem (Voevodsky, Voisin)

$$Z^i_{alg}(X)_{\mathbb{Q}} \subset Z^i_{\otimes}(X)_{\mathbb{Q}}$$

(idea of proof next time)

Homological equivalence

Weil cohomologies

F field $\mathrm{char}(F)=0$

$$H : \mathrm{SmProj}(k) \longrightarrow \mathrm{GrVect}_F \quad \text{functor}$$

① $\exists$ cup product

$$\cup : H(X) \times H(X) \longrightarrow H(X) \quad \text{graded super-commutative}$$

$$b \cup a = (-1)^{d(b)d(a)} a \cup b$$

② Poincaré duality:

$\exists$ tr isomorphism $\quad \mathrm{tr} : H^{2d}(X) \xrightarrow{\sim} F$

($\dim X = d$) such that

$$H^i(X) \times H^{2d-i}(X) \xrightarrow{\cup} H^{2d}(X) \xrightarrow{\mathrm{tr}} F$$

is a perfect pairing

(in particular $H^0(\mathrm{pt}) = F$)

③ Künneth formula

$$H(X) \otimes H(Y) \xrightarrow[\cong]{(pr_X)^* \otimes (pr_Y)^*} H(X \times Y)$$

graded isomorphism

④ cycle class maps

$$\gamma_X : CH^i(X) \longrightarrow H^{2i}(X) \quad \text{satisfying}$$

- functoriality: $f : X \to Y$

$$f^* \circ \gamma_Y = \gamma_X \circ f^*$$

$$f_* \circ \gamma_X = \gamma_Y \circ f_*$$

- compatibility with intersection product

$$\gamma_X(\alpha \cdot \beta) = \gamma_X(\alpha) \cup \gamma_X(\beta)$$

- compatible with points

$$\mathrm{Tr} \circ \gamma_P = \deg$$

$$\begin{tikzcd} CH^0(P) \arrow[r, "\gamma_P"] \arrow[d, "\deg"'] & H^0(P) \arrow[d, "\mathrm{Tr}"] \\ \mathbb{Z} \arrow[r, hook] & F \end{tikzcd}$$ commutes

<u>Notation</u> $A^i(X) = \gamma_X(CH^i(X)) \subset H^{2i}(X)$

$\qquad\qquad\quad \| $

$\qquad\quad H^i_{alg}(X)$ part of the cohomology that is realized by algebraic cycles

(5) <u>Weak Lefschetz</u> $i : Y_{d-1} \hookrightarrow X_d$

Smooth hyperplane section

then $H^i(X) \xrightarrow{i^*} H^i(Y)$ isomorphism $i < d-1$

injective $i = d-1$

6 **Hard Lefschetz** : X, Y as previous

Lefschetz operator $L(\alpha) = \alpha \cup \gamma_X(Y)$

induces isomorphisms

$$L^{d-i} : H^{d-i}(X) \xrightarrow{\sim} H^{d+i}(X) \qquad 0 \le i \le d$$

List of classical Weil cohomology theories

- $\mathrm{char}(k) = 0$ & $k \subset \mathbb{C}$
  - Betti $H^i_B(X)$ (singular cohomology of X_{an} complex mfld)
  - de Rham $\begin{cases} \text{analytic} & H_{dR}(X_{an}, \mathbb{C}) \\ \text{algebraic} & H^i(X_{Zar}, \Omega^\bullet_{X/k}) \end{cases}$

- étale cohomology $H^i_{ét}(X_{\bar{k}}, \mathbb{Q}_\ell)$ ℓ prime, $\ell \neq \mathrm{char}(k)$

$$H^i_{ét}(X_{\bar{k}}, \mathbb{Z}_\ell) = \varprojlim_n H^i_{ét}(X_{\bar{k}}, \mathbb{Z}/\ell^n\mathbb{Z})$$

$$H^i_{ét}(X_{\bar{k}}, \mathbb{Q}_\ell) = H^i_{ét}(X_{\bar{k}}, \mathbb{Z}_\ell) \otimes_{\mathbb{Z}_\ell} \mathbb{Q}_\ell$$

(highly nontrivial to show it is a Weil cohomology e.g. that it satisfies hard Lefschetz etc.)
Deligne

- k perfect crystalline cohomology $H^i_{crys}(X/W(k)) \otimes K$ — Witt ring — its field of fractions

(again lots of work Berthelot, Katz, Messing in showing Weil cohom.

Comparisons

1) $k \subset \mathbb{C}$ (period isom & de Rham theorem)

$$H_{dR}(X) \otimes_k \mathbb{C} \xrightarrow{\sim} H_{dR}(X_{an}; \mathbb{C}) \xrightarrow{\sim} H_B(X) \otimes_{\mathbb{Q}} \mathbb{C}$$

alg. de Rham — analytic de Rham

2) $k = \mathbb{C}$ Artin's theorem

$$H_{ét}(X, \mathbb{Q}_\ell) \xrightarrow{\sim} H_B(X) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$$

Homological equivalence on cycles

$$Z \sim_{hom} 0 \iff \gamma_X(Z) = 0$$

depends on choice of Weil cohomology (in principle)

Numerical equivalence $X \in SmProj(k)$ $\dim X = d$

for $Z \in \mathcal{Z}^i(X)$ $Z \sim_{num} 0$

if $\forall W \in \mathcal{Z}^{d-i}(X)$ for which $\underbrace{Z \cdot W}_{\text{0-cycle}}$ is defined

have $\deg(Z \cdot W) = 0$

Relation of numerical to other equivalences

$$Z^i_{alg}(X) \subset Z^i_{hom}(X) \subset Z^i_{num}(X)$$

Conjecture ($k=\bar{k}$) (Grothendieck standard conjecture of type D)

$D(X)$ holds : $\mathcal{Z}^i_{hom}(X) = \mathcal{Z}^i_{num}(X)$

homological and numerical equivalence agree

- Known for divisors (arbitrary characteristic) Matsusaka's theorem
- char $=0$: also known for codim $=2$; dim $=1$; & for abelian varieties

Conjecture ($k=\bar{k}$) (Voevodsky nilpotence conjecture)

$$\mathcal{Z}^i_{\otimes}(X) = \mathcal{Z}^i_{num}(X)$$

implies standard conj. $D(X)$ for all X (and for all Weil cohom.) because

$$\mathcal{Z}^i_{\otimes}(X) \subset \mathcal{Z}^i_{hom}(X) \subset \mathcal{Z}^i_{num}(X)$$

in fact: if $Z^n \sim 0$ then $\gamma_{X^n}(Z^n) = \gamma_X(Z)^{\otimes n} = 0 \in H^{2ni}(X^n)$
$\Leftarrow \gamma_X(Z) = 0$

for 0-cycles $\deg(Z) = \mathrm{Tr}\,\gamma_X(Z)$ numerical = homol and in general if $Z \in \mathcal{Z}^i_{hom}(X)$, $i<d$, and $W \in \mathcal{Z}^{d-i}(X) \Rightarrow \deg(Z \bullet W) = \mathrm{Tr}(\gamma_X(Z) \cup \gamma_X(W)) = 0$