

QFT & Motives

(1)

QFT I

- Scalar (massless) perturbative QFT Euclidean
- Momentum space
- Config space
- Amplitudes, twist integrals, $SU(N)$

Why relation to motives?
 — view Feynman integrals (Feynman graphs) as periods of alg. var. (occurrence of MZVs)
 — organize structure of divergences and renormalization categorically (diff. Galois theory)

1. Feynman integrals and Feynman diagrams

Finite dimensional model:

• Gaussian integral $\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx = \left(\frac{2\pi}{a}\right)^{1/2} \quad a > 0$

polar coords: $\int_{\mathbb{R}^2} e^{-\frac{1}{2}a(x^2+y^2)} dx dy = 2\pi \int_0^\infty e^{-\frac{1}{2}ar^2} r dr = \frac{2\pi}{a} \int_0^\infty e^{-u} du$

• Gaussian integral with a source term:

$$\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + Jx} dx = \left(\frac{2\pi}{a}\right)^{1/2} e^{\frac{J^2}{2a}}$$

as above completing the square $-\frac{ax^2}{2} + Jx = -\frac{a}{2}\left(x^2 - \frac{2Jx}{a}\right) = -\frac{a}{2}\left(x - \frac{J}{a}\right)^2 + \frac{J^2}{2a}$

② expectation values (example):

$$\langle x^{2n} \rangle := \frac{\int_{\mathbb{R}} x^{2n} e^{-\frac{1}{2}ax^2} dx}{\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx} = \frac{(2n-1)!!}{a^n}$$

$$(2n-1)!! = (2n-1)(2n-3) \dots 5 \cdot 3 \cdot 1$$

obtained inductively from previous integral ~~via the same~~

$$\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2} dx = \left(\frac{2\pi}{a}\right)^{1/2} \text{ by repeatedly applying}$$

$$-2 \frac{d}{da} \text{ to } \curvearrowright$$

Note: $(2n-1)!!$ counts all different ways of connecting in pairs the $2n$ linear terms in

$$x^{2n} = \underbrace{x \dots x}_{2n} \rightsquigarrow \text{Wick contractions}$$

• Interaction terms: (Gaussian case: free field)

$$(*) \quad Z(J) = \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + P(x) + Jx} dx$$

$P(x)$ = polynomial of $\deg \geq 3$

treated as perturbation of the Gaussian integral

$$Z(J) = \int_{\mathbb{R}} \left(\sum_{n=0}^{\infty} \frac{P(x)^n}{n!} \right) e^{-\frac{1}{2}ax^2 + Jx} dx$$

so "perturbative expansion" of $(*)$ defined to be $\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} P(x)^n e^{-\frac{1}{2}ax^2 + Jx} dx$
(note: parameter $\hbar$ of order of expansion included in P)

• monomial contributions:

$$\int_{\mathbb{R}} x^k e^{-\frac{1}{2}ax^2 + Jx} dx = \left(\frac{d}{dJ}\right)^k \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + Jx} dx$$

$$\Rightarrow \text{using } \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + Jx} dx = \left(\frac{2\pi}{a}\right)^{1/2} e^{\frac{J^2}{2a}}$$

obtain

$$\int_{\mathbb{R}} x^k e^{-\frac{1}{2}ax^2 + Jx} dx = \left(\frac{2\pi}{a}\right)^{1/2} \left(\frac{d}{dJ}\right)^k e^{\frac{J^2}{2a}}$$

$\Rightarrow$ if $P(x) = \frac{\lambda}{k!} x^k$ single monomial
rewrite perturbative expansion as

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}} \left(\frac{\lambda}{k!} x^k\right)^n e^{-\frac{1}{2}ax^2 + Jx} dx =$$

$$\sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{\lambda}{k!} \left(\frac{d}{dJ}\right)^k\right)^n \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + Jx} dx$$

$$\Rightarrow Z(J) = \left(\frac{2\pi}{a}\right)^{1/2} \exp\left(\frac{\lambda}{k!} \left(\frac{d}{dJ}\right)^k\right) \exp\left(\frac{J^2}{2a}\right)$$

Two frequently used monomial examples $P(x) = \frac{g}{6} x^3$

$$P(x) = \frac{\lambda}{4!} x^4$$

④ • Combinatorial labeling in terms of graphs

- identify term of order λ^α and J^β in $Z(J)$

= terms of order α in the Taylor expansion
of $\exp\left(\frac{\lambda}{k!} \left(\frac{d}{dJ}\right)^k\right)$

and terms of order $\beta + k\alpha$ in J
in the Taylor expansion of $\exp\left(\frac{J^2}{2a}\right)$

all terms contain

- a combinatorial factor (ratio of products of factorials)
- a power of J
- a power of λ
- a power of $2a$ in denominator

Visual way of keeping track of power counting
for these terms using graphs

Set of vertices
Set of half-edges

Pairings between half-edges
(unpaired = external edges
paired = internal edges)

$$\frac{\lambda^\alpha J^\beta}{a^\delta}$$

(5)

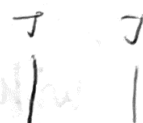
$\leadsto$ construct corresponding graphs

- To each factor $\lambda \leadsto$ a vertex of valence
= degree of monomial $P(x) = \frac{\lambda}{k!} x^k$
= k

a corolla given
by a vertex with
 k half-edges attached



- To each factor $J \leadsto$ an external edge



- The ^{resulting} ~~each~~ power of $a^{-1} \leadsto$ # of internal edges
obtained by pairing
all remaining
(non-external) half-edges
(with half-edges of corollas
or external)

for each monomial
that occurs
in the expansion

$$\frac{\lambda^\alpha J^\beta}{a^\delta}$$

$\leadsto$ a finite collection of
graphs given by all
possible ways of constructing
a graph with rules above

identify all isomorphic graphs:

multiplicity factor: all assignments that produce
isomorphic graphs

"symmetry factors of graphs"

⑥ To see how to compute explicitly the symmetry factors:

- Green functions: $Z(J)$ as a generating function

$$Z(J) = \sum_{N=0}^{\infty} \frac{J^N}{N!} \int_{\mathbb{R}} x^N e^{-\frac{1}{2}ax^2 + P(x)} dx$$

$$= Z \cdot \sum_{N=0}^{\infty} J^N G_N$$

with $Z = \int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + P(x)} dx$ (partition function)

and Green functions

$$G_N = \frac{\int_{\mathbb{R}} \frac{x^N}{N!} e^{-\frac{1}{2}ax^2 + P(x)} dx}{\int_{\mathbb{R}} e^{-\frac{1}{2}ax^2 + P(x)} dx}$$

expectation value for $\frac{x^N}{N!}$

- Then further expand the exponential $\exp(P(x))$ of the interaction term $P(x) = \frac{\lambda}{k!} x^k$

+ formally exchange summation & integration

$$G_N = \frac{\sum_{n=0}^{\infty} \int \frac{x^N}{N!} \frac{(\lambda x^k)^n}{(k!)^n n!} e^{-\frac{1}{2}ax^2} dx}{\sum_{n=0}^{\infty} \int \frac{(\lambda x^k)^n}{(k!)^n n!} e^{-\frac{1}{2}ax^2} dx}$$

(7)

Thus coefficient of $\frac{\lambda^\alpha J^\beta}{a^\gamma}$ in expansion of $Z(J)$ obtained by counting all pairings (Wick contractions) in

$$\int_{\mathbb{R}} x^N x^{kn} e^{-\frac{1}{2}ax^2} dx$$

seen that $(N+kn-1)!!$ when $N+kn$ even and vanishing for odd

then combine w/ other coeffs in expansion in Green functions and get

$$\frac{(N+kn-1)!!}{N! n! (k!)^n} \quad (\otimes) (\otimes)$$

Meaning of these terms for graphs:

$(N+kn-1)!! = \#$ possible pairings of half-edges

$N! =$ factoring out $\#$ permutations of external edges

$(k!)^n =$ " " $\#$ ~~$k!$~~ permutations of half-edges attached to each of the n corollas

$n! =$ " " $\#$ permutations of the n corollas

8

- Asymptotic expansion in terms of graphs

$$\sum_{\Gamma \text{ graphs}} \frac{\lambda^{\#V(\Gamma)} J^{\#E_{\text{ext}}(\Gamma)}}{\#Aut(\Gamma) a^{\#E_{\text{int}}(\Gamma)}}$$

- Note: in matching combinatorial terms obtained in terms of expansion in Green functions with those obtained by expanding

$$\mathcal{Z}(J) = \left(\frac{2\pi}{a}\right)^{1/2} \exp\left(\frac{\lambda}{k!} \left(\frac{d}{dJ}\right)^k\right) \exp\left(\frac{J^2}{2a}\right)$$

use identity $(2n-1)!! = \frac{(2n)!}{2^n n!}$

expression $\textcircled{*}$ has a more transparent meaning in terms of the graphs

- Higher dimensions (still finite dimensional)

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2} x^t A x + J x} dx_1 \dots dx_N = \frac{(2\pi)^{N/2}}{(\det A)^{1/2}} e^{\frac{1}{2} J A^{-1} J^t}$$

$N \times N$ real matrix w/ $A^t = A$
and positive eigenvalues

$J = N$ -vector

obtained by diagonalizing A
and using 1-dim case

$\leadsto$ asymptotic expansion of

$$\int_{\mathbb{R}^N} e^{-\frac{1}{2} x^t A x + P(x) + Jx} dx$$

where interaction term = polyn. in coordinates x_i

e.g. $P(x) = \frac{\lambda}{4!} \sum_{i=1}^4 x_i^4$

→ again obtain asymptotic series where can label terms by graphs

- instead of factors a^{-1} to internal edges have factors $(A^{-1})_{ij}$ for edge connecting vertex v_i to vertex v_j (Wick contraction of a linear term x_i and x_j in integral)

Feynman integrals : perturbative expansion

formally adapting argument above to an ∞ -dim integral

classical scalar field

$$\phi \in C^\infty(\mathbb{R}^D, \mathbb{R})$$

- Lagrangian density (Lorentzian signature)

(10)

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{P}(\phi).$$

$$(\partial\phi)^2 = g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \quad \begin{array}{l} g_{\mu\nu} \text{ metric tensor of} \\ \text{signature} \\ (1, -1, -1, \dots, -1) \text{ in } \mathbb{R}^D \end{array}$$

interaction term $\mathcal{P}(\phi)$ = polynomial in the field ϕ $\deg(\mathcal{P}) \geq 3$

e.g. monomial $\mathcal{P}(\phi) = \frac{\lambda}{k!} \phi^k$

• action functional

$$S(\phi) = \int_{\mathbb{R}^D} \mathcal{L}(\phi) \, d\mathbf{x}$$

can
volume form $\sqrt{|g|} \, d\text{vol}(g)$
in metric tensor

$$S(\phi) = S_{\text{free}}(\phi) + S_{\text{int}}(\phi)$$

$$S_{\text{free}}(\phi) = \int_{\mathbb{R}^D} \left(\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 \right) d\mathbf{x}$$

$$S_{\text{int}}(\phi) = - \int_{\mathbb{R}^D} \mathcal{P}(\phi) \, d\mathbf{x}$$

• Probability amplitude $e^{\frac{i}{\hbar} S(\phi)}$ (set $\hbar=1$ units)

- expectation values

$\mathcal{O}(\phi)$ observable = functional on $C_c^\infty(\mathbb{R}^D, \mathbb{R})$
(nonlinear in general)

formally:

$$\langle \mathcal{O}(\phi) \rangle = \frac{\int \mathcal{O}(\phi) e^{iS(\phi)} \mathcal{D}(\phi)}{\int e^{iS(\phi)} \mathcal{D}(\phi)}$$

"integrated over a space of classical fields"
 ∞ -dimensional

- N -point Green functions

$$G(x_1 \rightarrow x_N) = \frac{\int \phi(x_1) \dots \phi(x_N) e^{iS(\phi)} \mathcal{D}(\phi)}{\int e^{iS(\phi)} \mathcal{D}(\phi)}$$

for which generating function is
functional integral with source term

$$\int e^{iS(\phi) + \langle J, \phi \rangle} \mathcal{D}(\phi)$$

J = linear functional (distribution)
over $C_c^\infty(\mathbb{R}^D, \mathbb{R})$

$$\langle J, \phi \rangle = \int_{\mathbb{R}^D} J(x) \phi(x) dx$$

if distribution given by function

⑫ Euclidean version

$$e^{iS(\phi)} \longrightarrow e^{-S(\phi)}$$

$$S(\phi) = \int_{\mathbb{R}^D} \left(\frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + P(\phi) \right) d^D x$$

euclidean w/ $g_{\mu\nu}$ Riemannian

$$G_N(x_1, \dots, x_N) = \frac{\int \phi(x_1) \dots \phi(x_N) \cdot e^{-S(\phi)} \mathcal{D}(\phi)}{\int e^{-S(\phi)} \mathcal{D}(\phi)}$$

$$Z(J) = \int e^{-\int_{\mathbb{R}^D} \left(\frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + P(\phi) + J(\phi) \right) d^D x} \mathcal{D}(\phi)$$

$$\frac{Z(J)}{Z(0)} = \sum_{N=0}^{\infty} \frac{1}{N!} \int J(x_1) \dots J(x_N) G_N(x_1, \dots, x_N) d^D x_1 \dots d^D x_N$$

$$Z(0) = Z = \int e^{-\int_{\mathbb{R}^D} \left(\frac{1}{2} (\partial\phi)^2 + \frac{m^2}{2} \phi^2 + P(\phi) \right) d^D x} \mathcal{D}(\phi)$$

partition
function

Make sense of ∞ -dim integral not directly
but by defining the asymptotic expansion