

Origins I: enumerative geometry

- Bézout's theorem (projective geometry)

of points common to two plane curves (without common components)

$$C_1 = \{ P(x, y) = 0 \} \quad C_2 = \{ Q(x, y) = 0 \}$$

counted w/ multiplicity (and counting pts at ∞)

$$\# C_1 \cap C_2 = \deg P \cdot \deg Q$$

Poncellet: deform Q to a product of linear factors $\prod_i (a_i x + b_i y)$ in general position

then each line $a_i x + b_i y = 0$ meets $P = 0$ locus in $\deg P$ points

key: Conservation of # of intersections under deformation

- Modern viewpoint on this story:

algebraic cycles in a variety equivalence

algebraically equivalent $Z \sim_{\text{alg}} Z'$

if can be transformed one into other by algebraic deformation

numerical equivalence $Z \sim_{\text{num}} Z'$ if same intersection numbers with any other cycle

algebraic equiv. $\Rightarrow$ numerical equivalence

many enumerative geometry problems about counting geometric (algebra-geom.) objects in a certain family with assigned incidence relations to a given config. of (other) geom. objects

→ modern developments of enumerative geometry:

- intersection theory (Fulton, MacPherson)
- quantum cohomology, (mirror symmetry relations to string theory)
- one of the origins of the theory of motives

• Other enumerative problem at the origin of motives

• Counting points over finite fields

Solutions of polynomial equations over finite fields $\mathbb{F}_q$ $q = p^r$ char p

X algebraic variety defined over $\mathbb{F}_q$

$v_n = \#$ points of X over $\mathbb{F}_{q^n}$ ($= \# X(\mathbb{F}_{q^n}) = N_{q^n}(X)$)
notation

Generating function for these numbers

Z_X defined by

$$\log Z_X(t) = \sum v_n \frac{t^n}{n}$$

intersection # computed in $X \times X$

v_n 's numbers are related to action of Frobenius on X
acting as $\text{Fr}: x \mapsto x^q$ on algebra of functions on X

$v_n = \langle \Delta \cdot \Gamma_{\text{Fr}_X^n} \rangle$ $\Delta \subset X \times X$ diag. (fixed pts) $\Gamma_{\text{Fr}_X^n} = \text{graph of } \text{Fr}^n$ n^{th} power of Frobenius

- Numerical motives provide natural categorical setting for this kind of enumerative problems

(see later the precise construction of the category)

• Origine II : cohomology theories

X smooth projective algebraic variety

$H^*(X)$ cohomology (different types of cohomology existed at the time of Grothendieck's idea of motives (early '60s))

- de Rham, Betti (char 0)
- étale (for each $l \neq \text{char } p$)
- crystalline (char p)

coeffs in $\mathbb{Q}$

Z alg. cycle codim $r \rightsquigarrow$ fund. class of $Z \in H^{2r}(X)$

cup product in $H^*(X)$ compatible w/ inters. prod of cycles

$Z \sim_{\text{hom}} Z'$ homological equivalence

if fundam. class same in $H^*(X)$

(intermediate between algebraic and numerical equiv.)
[see later conjectures]

Bézout revisited : $H^2(\mathbb{P}^2, \mathbb{Q}) \cong H^4(\mathbb{P}^2, \mathbb{Q}) = \mathbb{Q}$

cup product $H^2(\mathbb{P}^2, \mathbb{Q}) \otimes H^2(\mathbb{P}^2, \mathbb{Q}) \rightarrow H^4(\mathbb{P}^2, \mathbb{Q})$
multiplication (polynomial ring)

$C \subset \mathbb{P}^2$ curve $\rightsquigarrow$ fundam. class $\deg C \cdot u$

$\{P=0\}$ then $C_1 \cap C_2 \rightsquigarrow \deg C_1 \cdot \deg C_2 \cdot u^2$

Counting pts over finite fields revisited
in cohomological setting

$\alpha: X \rightarrow X$ endomorphism w/ fin. # isolated fixed pts
Lefschetz formula:

$$\# \text{Fix}(\alpha) = \sum_{i=0}^{2 \dim X} (-1)^i \text{tr}(\alpha^* / H^i(X))$$

↑ induced map on cohomology

A. Weil:

$$N_n = \# X(\mathbb{F}_{q^n}) = \sum_{i=0}^{2 \dim X} (-1)^i \text{tr}((F_{r_X}^n)^* / H^i(X))$$

↑ suitable cohom.
 $H_{\text{ét}}^i(X)$

$$Z_X(t) = \frac{\det(1 - t F_{r_X}^* / H^-(X))}{\det(1 - t F_{r_X}^* / H^+(X))}$$

in fact étale cohomology (Grothendieck, Artin ~1963)
constructed to the purpose of making this idea precise

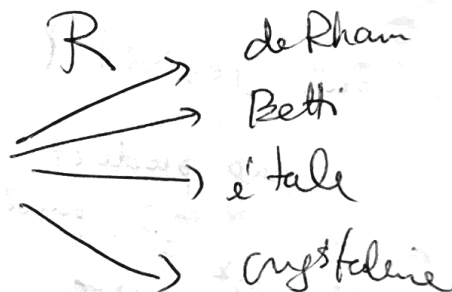
- have different étale cohom. for each $l \neq \text{char}(k) = p$
~~the~~ coeffs in $\mathbb{Q}_l$: relations between them?
relations to/between Betti, de Rham, crystalline
when each applies ---

Grothendieck's idea:

existence of a
universal cohomology

theory underlying
all of these

Motives



realization
functors

Origins III: Galois theory

$K = \text{field}$ $\bar{K} = \text{separable closure}$ $\text{Gal}(\bar{K}/K) = \text{absolute Galois grp.}$

(e.g. $\text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$
w/ $\bar{\mathbb{Q}}$ alg. closure)

$K' = \text{finite dimensional commutative } K\text{-algebra}$

$Z = \text{Spec}(K')$ is a ~~finite~~ K -scheme

it is étale if:
equivalent
properties
are
satisfied

- ① $K' \otimes \bar{K} = \bar{K}^{[K':K]}$
- ② $K' \cong \prod_{\alpha} K_{\alpha}$
- ③ $|Z(\bar{K})| = [K':K]$

K_{α}/K (separable finite extensions)

if K perfect also equiv. to K' reduced

Functor $\{ \text{étale } K\text{-schemes (finite)} \} \longleftrightarrow \{ \text{finite sets w. action of } \text{Gal}(\bar{K}/K) \}$

$Z \longmapsto Z(\bar{K})$

is an equivalence of categories (Galois-Grothendieck correspondence)

• Question: linearization of the GG correspondence

$\{ ? \} \longleftrightarrow \{ \text{fin. dim. } \mathbb{Q}\text{-linear vector spaces w/ linear action of } \text{Gal}(\bar{K}/K) \}$

"linearization" of category of finite étale K -schemes

$\rightsquigarrow$ Artin motives $\rightsquigarrow$ "motivic Galois groups" as generalization of Galois theory

• (Origins) II: analytic aspects

- Hodge theory pure and mixed Hodge structures
- Period integrals; classes of transcendental numbers related to periods & motives like multiple zeta values etc.

Two main branches of theory of motives

- Pure motives (smooth projective varieties)
- Mixed motives (arbitrary varieties: singular...)

- Theory of pure motives better developed $\rightarrow$ but basic questions about algebraic cycles still unsolved (standard conjectures)

- mixed motives more difficult to construct a good category (triangulated category: only in a few cases (Tate motives over number fields) have abelian category, Galois groups, etc...)

• More recent development: Noncommutative motives

dg-categories replace varieties

Algebraic cycles and equivalence relations

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$k = \text{field}$

$\text{SmProj}(k)$ category of smooth projective varieties over k

Variety: reduced scheme (not nec. irreducible: can have finite disj-union of irred components)

Algebraic cycle Z on X variety

$$Z = \sum_{\alpha} n_{\alpha} Z_{\alpha} \quad \text{fin. } \mathbb{Z}\text{-linear combination of irreducible subvarieties of } X$$

if $\dim(Z_{\alpha}) = k$ for all α :

$Z \in \mathbb{Z}^k(X)$ abelian group of codim k cycles

(Note: want cohom. theory grading on cycles by codim)

or $\mathbb{Z}_{d-k}(X)$ if graded by dim & $\dim X = d$

$$\mathbb{Z}^k(X)_F = \mathbb{Z}^k(X) \otimes_{\mathbb{Z}} F$$

coeff's in a field
(often will use $F = \mathbb{Q}$)

e.g. • $\mathbb{Z}^1(X) = \text{Div}(X)$ divisors
codim 1 cycles

• $\mathbb{Z}^d(X) = \mathbb{Z}_0(X)$ zero-cycles formal sums of pts

$$Z = \sum_{\alpha} n_{\alpha} P_{\alpha}$$

$\deg(P_{\alpha}) = [k(P_{\alpha}) : k]$ deg of extension

$$\deg Z = \sum n_{\alpha} \deg(P_{\alpha})$$

Operations on Algebraic cycles

(1) Product (Cartesian product of varieties extended linearly to cycles)

(2) Push forward
 $f: X \rightarrow Y$ morphism of varieties
 $Z \subset X$ irreducible subvariety

$$\deg(Z/f(Z)) = \begin{cases} [K(Z): K(f(Z))] & \dim f(Z) = \dim Z \\ 0 & \dim f(Z) < \dim Z \end{cases}$$

degree of extension

$[K(Z): K(f(Z))]$ counts # sheets of ~~branched~~ covering

$$f_* (Z) = \deg(Z/f(Z)) \cdot f(Z) \quad \text{extended linearly to combinations}$$

(3) Intersection number

$V, W \subset X$ subvarieties $\text{codim } V = i$ $\text{codim } W = j$

$V \cap W = \bigcup_{\alpha} Z_{\alpha}$ intersect in a union of subvarieties

with $\text{codim } Z_{\alpha} \geq i+j$ ~~to get~~ (assuming X smooth)
(Hartshorne p. 48)

intersection is proper if $\forall \alpha$ $\text{codim } Z_{\alpha} = i+j$

$\Rightarrow$ intersection number

length of A -mod

$$i(V \cdot W; Z_{\alpha}) := \sum_r (-1)^r \ell_A(\text{Tor}_r^A(A/I(V), A/I(W)))$$

where $A = \mathcal{O}_{X, Z_{\alpha}}$
and intersection product

$I(V)$ = ideal defining V in ring A
 $V \cdot W = \sum_{\alpha} i(V \cdot W; Z_{\alpha}) Z_{\alpha}$

Lefschetz trace formula and intersection numbers

$$V = V^+ \oplus V^- \quad W = W^+ \oplus W^-$$

$$\check{V}, \check{W} \quad K\text{-duals}$$

$\mathbb{Z}/2\mathbb{Z}$ -graded vector spaces / K
(fin. dim.)

$\mathbb{Z}/2\mathbb{Z}$ -parity

$$\delta(v) \delta(w)$$

$$V \otimes W \cong W \otimes V$$

$c_{V,W}$ Koszul sign rule

$$c_{V,W}(v \otimes w) = (-1)^{w \otimes v}$$

$$V \otimes \check{W} \text{ dual to } \check{V} \otimes W : \langle \check{V} \otimes W, v \otimes \check{w} \rangle = (-1)^{\delta(v) \delta(w)} \langle \check{v}, v \rangle \langle w, \check{w} \rangle$$

$$(\delta(v) = \delta(\check{v}), \delta(w) = \delta(\check{w}))$$

$$\check{v} \otimes w \mapsto (v' \mapsto \langle \check{v}, v' \rangle w)$$

$$\check{V} \otimes W \cong \text{Hom}(V, W)$$

$$\check{W} \otimes V \cong \text{Hom}(W, V)$$

$$\text{Endomorphism of } V \text{ given by } (\check{W} \otimes V) \circ (\check{V} \otimes W) = \langle w, \check{w} \rangle \check{v} \otimes v$$

$$\text{str}(\check{v} \otimes v) = \langle \check{v}, v \rangle$$

$$\text{Then get } \langle \check{v} \otimes w, c_{\check{W},V}(\check{w} \otimes v) \rangle = \text{str}((\check{W} \otimes V) \circ (\check{V} \otimes W))$$

extending by linearity

$$\text{for all } \phi \in \text{Hom}(V, W) \quad \text{for all } \psi \in \text{Hom}(W, V)$$

$$\langle \phi, c_{\check{W},V}(\psi) \rangle = \text{str}(\psi \circ \phi)$$

$$\text{Apply to } V^+ = H^+(X) = H^{\text{even}}(X) \quad V^- = H^-(X) = H^{\text{odd}}(X)$$

$$W^+ = H^+(Y) = H^{\text{even}}(Y) \quad W^- = H^-(Y) = H^{\text{odd}}(Y)$$

$$\phi \in H^{2d_Y + i}(X \times Y)$$

$$\psi \in H^{2d_X - i}(Y \times X)$$

$$c_{\check{W},V}(\psi) = (c_{Y,X})^*(\psi) = (c_{X,Y})^*(\psi) = \tau \psi$$

$$\Rightarrow \langle \phi, \tau \psi \rangle = \sum_{j=0}^{2d_X} (-1)^j \text{tr}(\psi \circ \phi | H^j(X))$$

$$\text{for } X=Y \quad \psi = \pi_X^j \quad \tau \psi = \pi_X^{2d_X - j} \Rightarrow \text{tr}(\phi | H^j(X)) = (-1)^j \langle \phi, \pi_X^{2d_X - j} \rangle$$

intersection numbers as traces

• Further operations on algebraic cycles

(10)

— Pullback: $f: X \rightarrow Y$ morphism of smooth proj. var.

$Z \subset Y$ subvariety

$\Gamma_f = \text{graph of } f = \text{subvar. of } X \times Y$

if Γ_f meets $X \times Z$ properly then

$$f^*(Z) = (\text{pr}_X)_* (\Gamma_f \cdot (X \times Z))$$

$\text{pr}_X: X \times Y \rightarrow X$ projection
 $\uparrow$ push forward
 $\uparrow$ intersection product

Note: not always defined
will need equivalence relations on cycles

— Using pullback can describe intersection product (Weil) as

$$V \cdot W = \Delta_X^* (V \times W)$$

$V, W \subset X$ subvar. (cycles)

$\Delta_X: X \hookrightarrow X \times X$ diagonal

— Correspondences = cycles in $X \times Y$
from X to Y

$Z \in \mathcal{Z}^k(X \times Y)$ acts on cycles

image of cycle under the correspondence

$T \in \mathcal{Z}^i(X)$

$\dim X = d$

$$Z(T) = (\text{pr}_Y)_* (Z \cdot (T \times Y))$$

$\uparrow$
 $\mathcal{Z}^{i+k-d}(Y)$

if $k = d$ correspondence preserves codim of cycles
 $k - d = \text{degree of the correspondence}$

- I homogeneous ideal of a polynomial ring
 $\mathbb{C}[x_0, x_1, \dots, x_n]$ (or replace $\mathbb{C}$ with K)

- $I = \sqrt{I}$ reduced ideal

$$\sqrt{I} = \{ f \in \mathbb{C}[x_0, x_1, \dots, x_n] \mid f^k \in I \text{ for some } k \in \mathbb{N} \}$$

- Projective algebraic sets $\longleftrightarrow$ reduced homogeneous ideals

- Projective algebraic varieties $\longleftrightarrow$ " with ideal prime
 (i.e. homog. coord. ring is integral domain)

- reducible $X = Y \cup Z$ union of proper closed subvarieties
 $Y, Z \subsetneq X$
 otherwise irreducible

length of module $\asymp$ max length of chain of submodules

Note on the Tor def. of intersection number

$$i(V, W; Z) = \sum_{k=0}^{\infty} (-1)^k \text{length}_{\mathcal{O}_{X,z}} \text{Tor}_{\mathcal{O}_{X,z}}^k (\mathcal{O}_{X,z}/I, \mathcal{O}_{X,z}/J)$$

the first term

$$\mathcal{O}_{X,z}/I \otimes_{\mathcal{O}_{X,z}} \mathcal{O}_{X,z}/J = \mathcal{O}_{Z,z}$$

is naive notion of intersection multiplicity

$\mathcal{O}_{X,z}$ - local ring at z