

(Algebraic varieties)

- polynomial equations over a field K (or a ring R)
 e.g. $\mathbb{Q}$ e.g. $\mathbb{Z}$
 (or $\mathbb{F}_q, \dots$)

affine space A^n over $K \leftrightarrow$ polynomial ring $K[x_1, \dots, x_n]$

- ~~algebraic~~ algebraic set

$$Z = \{x \in A^n \mid f(x) = 0 \ \forall f \in S\}$$

some (finite) set of polynomials
 $f \in K[x_1, \dots, x_n]$

- irreducible Z
 if $Z \neq Z_1 \cup Z_2$

$$V \subset A^n : I(V) \subset K[x_1, \dots, x_n] \text{ ideal of polyn. vanishing on } V$$

$$f(x) = 0 \ \forall x \in V$$

$$K[x_1, \dots, x_n] / I(V) = \text{structure ring of } V = R(V)$$

$I \subset K[x_1, \dots, x_n]$ ideal, radical $\sqrt{I} = \{f \in K[x_1, \dots, x_n] \mid f^k \in I \text{ for some } k \in \mathbb{N}\}$

- reduced ideal: $I = \sqrt{I}$ • $V(I) = \{x \in A^n \mid f(x) = 0 \ \forall f \in I\}$

- if K alg. closed field $I(V(I)) = \sqrt{I}$ (Hilbert nullstellensatz)

- affine algebraic variety: V affine alg. set $\leftrightarrow$ radical ideal $I(V)$

$$V \text{ affine alg. variety} \leftrightarrow \frac{\text{prime ideal}}{I(V)} \leftrightarrow \frac{\text{integral domain}}{R(V)}$$

reduced ring $R(V)$
 (prod of any two nonzero elements is nonzero: no zero divisors)

$$V(I) = V(J) \cup V(K)$$

$$I = JK \text{ (not prime)}$$

- projective alg. variety: $\mathbb{P}^n$ projective space $\mathbb{P}^n = (A^n - \{0\}) / \sim$
 $Z = \{x(x_0, \dots, x_n) \in \mathbb{P}^n \mid f(x) = 0 \ \forall f \in S\}$
 $S \subset K[x_0, \dots, x_n]$ homogeneous polynomials

Algebraic cycles: in an algebraic variety V

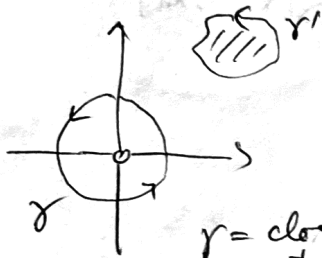
formal linear combinations (w/ coeff's in a vector space or an abelian group) of subvarieties $V_i \subset V$ closed immersion

$$C = \sum_i a_i V_i$$

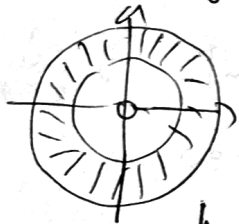
correspond to prime ideals of $R[V]$ ring

homology - cohomology:

basic simple example: $\mathbb{R}^2 \setminus \{0\}$



γ = closed curve
not "homologous" to trivial
i.e. = boundary of surface
 γ_1 = boundary of surface



homologous curves
 $\gamma_1 - \gamma_2 = \partial S$

homology

C_k = formal combination of k -dim elements (simplices, cubes, ...)

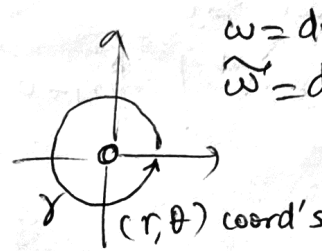
$\partial: C_k \rightarrow C_{k-1}$ (oriented boundary)

e.g. $\partial(\gamma) = x_1 - x_0$ for curves in C_1
 $\gamma(0) = x_0 \quad \gamma(1) = x_1$

$$\partial^2 = \partial \circ \partial = 0$$

homology:

$$H_k(C, \partial) = \frac{\text{Ker}(\partial: C_k \rightarrow C_{k-1})}{\text{Im}(\partial: C_{k+1} \rightarrow C_k)}$$



$$w = dr$$

$$\tilde{w} = d\theta$$

$$\int_{\gamma} \tilde{w} = 2\pi \neq 0$$

on closed curve

function

$$w = d(r)$$

$$\int_{\gamma} w = r(x_1) - r(x_0)$$

$\gamma(0) = x_0$
 $\gamma(1) = x_1$
= 0 on closed curve

$\int_{\sigma} w$ linear in σ and w
vector spaces/ab groups of w 's and σ 's

$dw = 0$ constant on homol. class

$$\int_{\gamma_1} w - \int_{\gamma_2} w = \int_S dw$$

cohomology:

C^k = formal comb. of lin. functionals on C_k
(e.g. diff k -forms Ω^k)

$d: C^k \rightarrow C^{k+1}$ differential / coboundary

$$d^2 = d \circ d = 0$$

cohomology:

$$H^k(C, d) = \frac{\text{Ker}(d: C^k \rightarrow C^{k+1})}{\text{Im}(d: C^{k-1} \rightarrow C^k)}$$

✓ comprehension (of alg. var.)

scheme of fin. type

$f: A \rightarrow B$ hom of comm. alg's B A -alg of fin. type
 dually $f: X \rightarrow Y$ map of schemes if fin. gen. as A -alg.
 of fin. type if Y covered by affine open subschemes $V_i = \text{Spec}(A_i)$ (X of fin. type over Y):
 s.t. $f^{-1}(V_i)$ covered by $U_{ij} = \text{Spec}(B_{ij})$ aff. open subschemes
 s.t. B_{ij} is an A_i -alg of fin. type

scheme generalizes alg. var. (takes into account multiplicities
 affine scheme $X = \text{Spec}(R)$ R comm. n.l.g. (or k -alg.)
prime ideals of R $x=0$ or $x^2=0$ same var. diff. scheme

$X(S) = \text{Hom}(R, S)$
comm k -alg

$X(k) = \text{Hom}(R, k)$
comm k -alg

w/ topol.
 $\& U \subset X$ open $\rightsquigarrow \mathcal{O}_X(U)$ is ring of regular functions

- complete variety (analog of compactness for mflds)

X s.t. $\forall$ var. Y proj. map $X \times Y \rightarrow Y$ is closed map
(maps closed sets to closed sets)

closed subvar's of complete are complete

- Divisors (Weil/Cartier) $\xleftarrow{\text{homol.}}$ $\xleftarrow{\text{algebraic}}$

Weil: $D = \sum_i n_i Z_i$ $n_i \in \mathbb{Z}$
effective $n_i \geq 0$

$Z_i =$ reduced & irreducible closed subvar (subscheme) of codim 1
ring w/ no nilpotents ($x^2=0 \Rightarrow x=0$) $\xrightarrow{\text{reduced}}$ $\xleftrightarrow{\text{R/I}} \text{radical}$

- normal crossings divisors: $D = \sum \mathbb{P}^1$ and loc. coords at each pt. ~~of~~ loc. coords $(x_1, \dots, x_n)$ in which like ~~vector~~ $x_i = 0$ of coord. hyperplanes $x_1 = \dots = x_n = 0$

- rational/rational map

rational map $X \dashrightarrow Y$ of alg var is morphism $U \rightarrow Y$

for $U \subset X$ non-empty dense open set

rational map: rational map w/ rational inverse i.e. isom. from $\hat{U} \xrightarrow{\cong} \hat{V}$ non-empty dense open sets

- blowup (of a smooth subvariety)

$Z \subset X$ smooth closed subvar of codim k

$Bl_Z(X) \hookrightarrow E$ exceptional divisor
 $\pi \downarrow$
 $X \hookleftarrow Z$

projectivized normal bundle

$Bl_Z(X) \setminus E$

$\cong \downarrow \pi$
 $X \setminus Z$

so $B_{\mathbb{P}^n} \otimes \mathcal{O}(1)$
brat. map $\downarrow \pi$
 X

E fibration w/ $\mathbb{P}^{k-1}$ fibers
 $\downarrow \pi|_E$
 $Z \rightarrow \mathbb{P}(N(Z))$

• additive category

$\mathcal{C}$ category preadditive if sets of morphisms $\text{Hom}_{\mathcal{C}}(X, Y)$ are abelian groups and composition of morphism is bilinear

Additive preadditive and

$\exists$ object $0 \in \text{Obj}(\mathcal{C})$ s.t. $\text{Hom}_{\mathcal{C}}(0, 0) = \text{trivial group}$

$\exists$ for prod & coprod

$\forall X, X' \in \text{Obj}(\mathcal{C}) \quad \exists Y \in \text{Obj}(\mathcal{C})$ and morphisms

$$X \xrightarrow{f_1} Y \xleftarrow{f_2} X' \quad X \xleftarrow{h_1} Y \xrightarrow{h_2} X'$$

with

$$h_1 f_1 = 1_X \quad h_2 f_2 = 1_{X'}, \quad h_2 f_1 = 0 = h_1 f_2$$

$$\forall h_1 + f_2 h_2 = 1_Y$$

Given a preadditive $\mathcal{C}$
can always obtain an additive $\text{Mat}(\mathcal{C})$

$$n / \text{Obj}(\text{Mat}(\mathcal{C})) = \bigoplus_{\text{finite sum}} X_1 \quad \forall X_i \in \text{Obj}(\mathcal{C})$$

and morphisms
are matrices of morphisms in $\mathcal{C}$

$$X_1 \oplus \dots \oplus X_n \xrightarrow{M} Y_1 \oplus \dots \oplus Y_r$$

M $n \times r$ matrix w/ $M_{ij} \in \text{Hom}_{\mathcal{C}}(X_i, Y_j)$

"additive envelope" of $\mathcal{C}$

algebra structure of $\mathcal{H}$ manifests itself
in monoidal str. of category $\otimes: \text{Rep}_F G \times \text{Rep}_F G \rightarrow \text{Rep}_F G$

Tensor category: F now just commut. ring

$\mathcal{C}$ F -linear category with

- a bifunctor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$
- a unit object 1
- functorial isomorphisms

$$a_{LMN}: L \otimes (M \otimes N) \xrightarrow{\sim} (L \otimes M) \otimes N$$

$$c_{MN}: M \otimes N \xrightarrow{\sim} N \otimes M \text{ with } c_{MN}^{-1} = c_{NM}$$

$$u_M: M \otimes 1 \xrightarrow{\sim} M \quad u'_M: 1 \otimes M \xrightarrow{\sim} M$$

with compatibility conditions expressed by diagrams

Pentagon diagram:

$$\begin{array}{ccccc} (A \otimes B) \otimes C \otimes D & \xrightarrow{a_{ABC} \otimes 1_D} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{a_{A, B \otimes C, D}} & A \otimes ((B \otimes C) \otimes D) \\ & \searrow a_{A \otimes B, C, D} & & \swarrow 1_A \otimes a_{BCD} & \\ (A \otimes B) \otimes (C \otimes D) & \xrightarrow{a_{A, B, C \otimes D}} & A \otimes (B \otimes (C \otimes D)) & & \end{array}$$

Triangle diagram

$$\begin{array}{ccc} (A \otimes 1) \otimes B & \xrightarrow{a_{A, 1, B}} & A \otimes (1 \otimes B) \\ & \searrow u_{A \otimes B} & \swarrow 1_A \otimes u'_B \\ & A \otimes B & \end{array}$$

Note: monoidal category
if only with
 a & u, u' no c

Hexagonal diagrams :

(7)

$$\begin{array}{ccc}
 A \otimes (B \otimes C) \xrightarrow{c} (B \otimes C) \otimes A & & (A \otimes B) \otimes C \xrightarrow{c} C \otimes (A \otimes B) \\
 \nearrow a & \searrow a & \nearrow a^{-1} \\
 (A \otimes B) \otimes C & B \otimes (C \otimes A) & A \otimes (B \otimes C) \\
 \searrow c \otimes 1 & \nearrow 1 \otimes c & \searrow 1 \otimes c \\
 (A \otimes B) \otimes C & \xrightarrow{a} B \otimes (A \otimes C) & A \otimes (C \otimes B) \xrightarrow{a^{-1}} (A \otimes C) \otimes B \\
 & \nearrow 1 \otimes c & \nearrow c \otimes 1
 \end{array}$$

Coherence diagram :

$$\begin{array}{ccc}
 A \otimes \mathbb{1} & \xrightarrow{c} & \mathbb{1} \otimes A \\
 \searrow u & & \searrow u' \\
 & A &
 \end{array}$$

Note : without requiring

with $c_{NM} = c_{MN}^{-1}$
symmetric

is braided monoidal category
 (tensor category)

Rigid Tensor category :

Additional structure : duality

$$v: e \rightarrow e^{\text{op}}$$

s.t. $- \otimes M^v$ is left adjoint of $- \otimes M$
 and $M^v \otimes -$ is right adjoint of $M \otimes -$

i.e. there are adjunction morphisms

$$\varepsilon: M \otimes M^v \rightarrow \mathbb{1} \quad \text{and} \quad \eta: \mathbb{1} \rightarrow M \otimes M^v$$

evaluation and coevaluation

$$\varepsilon \otimes 1_M \circ 1_M \otimes \eta = 1_M \quad \& \quad 1_{M^v} \otimes \varepsilon \circ \eta \otimes 1_{M^v} = 1_{M^v}$$