Algebraic Renormalization

Renormalization 2 step procedure

1) Regularization

2) Renormalization

Typically (1): replace divergent integral by a function of parameters w/ singularity/poles at special value of parameters (= original integral)

while (2): "pole subtraction" procedure that takes into account combinatorics of graphs

subdivergences nested inside larger graphs

$$\gamma \subset \Gamma$$

Example:

$$\delta(p_1+p_2)\,\frac{1}{p_1^2+m^2}\,\frac{1}{p_2^2+m^2}\int \frac{1}{k^2+m^2}\,\frac{1}{(p_1+k)^2+m^2}\,d^Dk$$

Schwinger parameters

$$\frac{1}{(k^2+m^2)}\cdot\frac{1}{(p+k)^2+m^2} = \int_{s>0,\,t>0} e^{\overbrace{-s(k^2+m^2)-t((p+k)^2+m^2)}^{-Q(k)\text{ quadr form}}}\,ds\,dt$$

Other general fact $\displaystyle\int e^{-\lambda q^2}\,d^Dq = \pi^{D/2}\,\lambda^{-D/2}$

$$-q(k) = -\lambda\left((k+xp)^2 + ((x-x^2)p^2+m^2)\right)$$

$$s = (1-x)\lambda \qquad t = x\lambda$$

$$\Rightarrow \int_0^1\int_0^\infty e^{-(\lambda(x-x^2)p^2+\lambda m^2)} \int e^{-\lambda q^2} d^Dq \; \lambda\, d\lambda\, dx$$

$$= \pi^{D/2} \int_0^1 \int_0^\infty e^{-(\lambda(x-x^2)p^2+\lambda m_i^2)} \underbrace{\cancel{\int e^{-\lambda q^2} d^Dq}}_{\lambda^{-D/2}\,\lambda\, d\lambda}\, dx$$

$$= \pi^{D/2}\,\Gamma\!\left(2-\frac{D}{2}\right) \int_0^1 ((x-x^2)p^2+m^2)^{D/2-2}\, dx$$

divergent

can compute this explicitly (not divergent)

poles at $D \in 4+2\mathbb{N}$

but now everything makes sense for $D \in \mathbb{C}$

$$D \mapsto D-z \in \mathbb{C}$$

"Dimensional Regularization" (more complicated for more difficult graphs with subdivergences)

<u>Def</u>: $\int e^{-A(t)(q)}\, d^Dq \quad \mathbb{R}^L$

quadr. form

$$A(t)(q) = \sum t_j F_j(q,0)^2$$

$$= \pi^{LD/2} \det(A(t))^{-D/2}$$

<u>Example of Regularization</u>

$\rightsquigarrow$ Dim Reg : replace $U(\Gamma, p_1, \dots, p_N) = \int \frac{d^Dk_1 \cdots d^Dk_n}{q_{e_1}(k_1)\cdots q_{e_n}(k_n)}$

with $U^z(\Gamma, p_1, \dots, p_N)$ a ~~[illegible]~~ $\in \mathbb{C}[z^{-1}, z]]$

Laurent series in z

~~[illegible]~~

How to perform renormalization then? ③

$U(\Gamma)$ (forget explicit dependence of ext momenta)
unrenormalized but keep dep. on regularization parameter z

BPHZ preparation:

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma)\, U(\Gamma/\gamma)$$

all terms depend on z

subgraphs (multiconn.) s.t. Γ/γ still Feynm. graph of same theory

$C(\gamma)$ defined inductively

$$C(\Gamma) = -T(\bar{R}(\Gamma))$$ polar part (as series in z)

$$R(\Gamma) = (1-T)(\bar{R}(\Gamma))$$
renormalized

$R(\Gamma)\big|_{z=0}$ renorm. Feynm. amplitude
no longer divergent at $z=0$

- How to better interpret this mathematically?

Hopf algebra of Feynman graphs (ignore ext. structure)

as an algebra: commutative algebra on Γ 1PI Feyn. graphs of the given QFT

as a coalgebra:

$$\Delta(\Gamma) = \sum_{\substack{\gamma \in \Gamma \\ \text{s.t. } \Gamma/\gamma \text{ also 1PI Feyn.}}} \gamma \otimes \Gamma/\gamma$$

$$= \underbrace{1 \otimes \Gamma + \Gamma \otimes 1}_{\text{primitive part}} + \underbrace{\sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma}_{\text{nonprim. part}}$$

terms of lower degree

Grading by $\#E_{int}(\Gamma)$

$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}_n$ graded

$\mathcal{H}_n$ span of Γ's $\#E_{int}(\Gamma) = n$

$\mathcal{H}_0 = \mathbb{R}$ (or $\mathbb{C}$ or $\mathbb{Q}$: field $\mathbb{K}$ of char $= 0$)
connected

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antipode: defined inductively

$$S(X) = -X + \sum S(X')X''$$

when $\Delta(X) = 1\otimes X + X\otimes 1 + \sum X'\otimes X''$

$$\begin{cases} (\Delta\otimes id)\Delta = (id\otimes\Delta)\Delta \\ (id\otimes\varepsilon)\Delta = id = (\varepsilon\otimes id)\Delta \\ m(id\otimes S)\Delta = m(S\otimes id)\Delta = 1\cdot\varepsilon \end{cases}$$

unit $1: k \to H$ (to H_0)

counit $\varepsilon: H \to k$ (augmentation)

k-alg. morphisms

Δ also $\Delta: H \to H\otimes_k H$

S k-alg antihom

A commutative Hopf alg. is dual to an affine grp. scheme

for all A commut. alg. over k

$$G(A) = \mathrm{Hom}_{Alg_k}(H, A) \quad \text{is a group}$$

$\phi: H \to A$ homom of commut. algebras

(lin maps with $\phi(xy) = \phi(x)\phi(y)$ $\phi(1) = 1$)

product in $G(A)$

$$(\phi_1 * \phi_2)(X) = \langle \phi_1\otimes\phi_2, \underset{1\otimes X + X\otimes 1 + \sum X'\otimes X''}{\overset{\shortparallel}{\Delta(X)}}\rangle$$

$$= \phi_2(X) + \phi_1(X) + \sum \phi_1(X')\phi_2(X'') \quad \text{(convol. prod)}$$

inverse = composition w/ antipode $\phi \mapsto \phi\circ S$

Unit (counit of H: $\varepsilon: H \to k \to A$)

In particular the unrenormalized Feynman rule

$U^z(\Gamma) \in \mathbb{C}[z^{-1}, z]] = A$

$U \in \mathrm{Hom}_{Alg_k}(\mathcal{H}, A)$ just means if $\Gamma = \Gamma_1 \sqcup \Gamma_2$

$U(\Gamma) = U(\Gamma_1) \cdot U(\Gamma_2)$

morphism of commutative algebras

<u>but</u> with one additional property: in $\mathbb{C}[z^{-1}, z]]$ have operator (linear) T of projection onto polar part

What formal alg. properties of $(\mathbb{C}[z^{-1}, z]], T)$ need?

(R, T) Rota-Baxter algebra of weight $-1 = \lambda$

R commutative algebra $/k$

$T: R \to R$ linear map s.t.

$T(x)\, T(y) = T(x\, T(y)) + T(T(x)\, y) + \lambda\, T(xy)$

Properties of RB alg w/ $\lambda = -1$:

- $R_+ = (1-T)R$ $\quad R_- =$ unitization of TR

are <u>subalgebras</u> (not just subvector spaces)

<u>e.g.</u> $(1-T)(a)\,(1-T)(b) = (1-T)(ab) + (1-T)(T(a)b + a\,T(b))$

by RB relation

BPHZ factorization (Birkhoff factorization)

$\exists!\ \phi = (\phi_- \circ S) * \phi_+$

$\phi_\pm : \mathcal{H} \to R_\pm$ algebra homomorphism

normalized by $\varepsilon_- \circ \phi_- = \varepsilon$

augmentation in R_- (unital)

Construction of $\phi_\pm$: inductively over degrees in $\mathcal{H}$

$$\phi_-(X) = -T\Big(\phi(X) + \sum \phi_-(X')\,\phi(X'')\Big)$$

(requires ϕ_- previously constructed for lower deg. terms

$$\phi_+(X) = (1-T)\underbrace{\Big(\phi(X) + \sum \phi_-(X')\,\phi(X'')\Big)}_{\tilde{\phi}(X)}$$

BPHZ preparation

indeed algebra homomorphisms

e.g.

$$\phi_-(XY) = -T(\tilde{\phi}(X)\,\tilde{\phi}(Y)) + T(T(\tilde{\phi}(X))\,\tilde{\phi}(Y)) + T(\tilde{\phi}(X)\,T(\tilde{\phi}(Y)))$$

$$\phi_-(X)\,\phi_-(Y) = T(\tilde{\phi}(X))\,T(\tilde{\phi}(Y))$$

obtained by decomposing terms in

$$\Delta(XY) = XY\otimes 1 + 1\otimes XY + \sum (XY)' \otimes (XY)''$$

in terms of X, Y, X', X'', Y', Y''

So general setting   Algebraic Renormalization   7

"algebraic Feynman rule"

$$\phi \in \mathrm{Hom}_{\mathcal{A}lg_k}(\mathcal{H}, \mathcal{R}) \quad \text{comm alg homom.}$$

$\mathcal{H} = \text{Hopf} \qquad \mathcal{R} = \text{RB } \lambda = -1$

$\phi_\pm : \mathcal{H} \to \mathcal{R}_\pm$ renormalization

$\phi_- : \mathcal{H} \to \mathcal{R}_-$ counterterms (divergences)

$\phi_+ : \mathcal{H} \to \mathcal{R}_+$ renormalized Feynman rule

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At level of forms? $Y \subset X \;(= \mathbb{P}^n)$

smooth hypersurface

$\mathcal{M}^*_{\mathbb{P}^n, Y}$ = merom. diff forms poles along Y $\{f = 0\} = Y$ defining equation

$$\omega = \sum_{p \geq 0} \frac{\alpha_p}{f^p}$$

extraction of polar part $T(\omega) = \sum_{p \geq 1} \frac{\alpha_p}{f^p}$

if $\mathcal{M}^{even}_{\mathbb{P}^n, Y}$ (commut. alg.) T is RB $\lambda = -1$

if $\mathcal{M}^*$ all deg. graded commutative version of RB

Adj. a unit to a non-unital alg. R/k

(a, r) $(1, 0)$ is unit

$k \;\; R$ $(a,r)(b,s) = (ab, as + br + sr)$

$\varepsilon : \tilde{R} \to k$ $\ker \varepsilon \cong R$

$\varepsilon(a, r) = a$

Possible setup:

- graded-commutative version of $\mathcal{H}$ CK Hopf

$$\Gamma_1 \cdot \Gamma_2 = (-1)^{\ell(\Gamma_1)\,\ell(\Gamma_2)}\, \Gamma_2 \cdot \Gamma_1$$

(think of grading by loop # instead of by n)

- for fixed # loops ℓ

$\exists\ X_\ell, Y_\ell$ $\qquad Y_\ell \subset X_\ell$ hypersurface (smooth proj) (4k general singular)

s.t. motive $m(X_\ell)$ is mixed Tate

- morphism $\phi : \mathcal{H} \to \mathcal{U}^{*}_{X_\ell, Y_\ell}$ graded alg's

- using RB on

$$\phi_+(\Gamma) = (1-T)\left(\phi(\Gamma) + \Sigma\, \phi_-(\gamma)\, \phi(\Gamma/\gamma) \right)$$

then if original problem (unrenorm $\phi(\Gamma) = \omega_\Gamma$)

$$\int_\sigma \omega_\Gamma = \int_\sigma \phi(\Gamma)$$

$\sigma \cap Y_\ell$ ~~…~~ divergences

$\omega_\Gamma = \dfrac{g\,\omega}{f^{\alpha(\Gamma)}}$

terms that survive

$\alpha(\gamma) = 0$ some $\gamma \subset \Gamma$

~~$\alpha(\Gamma/\gamma)$~~ $\alpha(\Gamma/\gamma) = 0$

$n - \frac{D(\ell+1)}{2}$

$n = \frac{D(\ell+1)}{2}$

~~…~~

$n_1 = \frac{D(\ell_1+1)}{2}$

$n_2 = \frac{D(\ell_2+1)}{2}$

$n_1 + n_2 = \frac{D(\ell_1+\ell_2)}{2} + D$

$\quad\;\; \shortparallel$

$\quad\;\; n$

in range $n > \frac{D(\ell+1)}{2}$

outside

rg. w/ f_Γ in denom ??

$$\int_\sigma \phi_+(\Gamma) \text{ is regular}$$

$\boxed{\text{BUT:}}$ lose a lot of information in passing from $\omega_\Gamma = \phi(\Gamma)$ to $\omega^+_\Gamma = \phi_+(\Gamma)$

~~…~~ very few terms in $\tilde{\phi}(\Gamma)$ survive (often none!)

Can correct so that retain some useful information? (9)

suppose $Y_\ell \subset X_\ell$ such that

all $H^*(X_\ell \setminus Y_\ell)$ realized by forms w/ log poles

$\Omega^*_{X_\ell}(\log(Y_\ell))$ e.g. Y_ℓ normal crossings divisor (Deligne)

then T RB operator becomes much simpler

$\omega = \alpha + \beta \wedge \frac{df}{f}$ α, β holom on X_ℓ

$T\omega = \beta \wedge \frac{df}{f}$ ~~[illegible]~~ now $T_x T_y = 0$

$\frac{df}{f} \wedge \frac{df}{f} = 0$

$T(xy) = x\,T(y) + T(x)\,y$ RB identity becomes simple derivation

$\Rightarrow$ BPHZ becomes simple pole subtraction

$(1-T)(\omega) = \alpha$

Moreover: can retain information on the part thrown away $\beta \wedge \frac{df}{f}$ via the Poincaré residue along Y_ℓ

$\beta = Res_{Y_\ell}(\omega)$

in this case unrenormalized (divergent) separately considering

$\int_\sigma \phi(\Gamma)$ can be replaced by $\int_\sigma \alpha$ & $\int_{\sigma \cap Y_\ell} Res(\omega)$

Have a setting X_ℓ, Y_ℓ like this? (1

if $X_\ell = \mathbb{P}^{\ell^2-1}$ and $D_\ell \subset X_\ell$ det hypersurface

New compactification of $X_\ell \setminus D_\ell$

$\overline{X_\ell \setminus D_\ell}$ where $\overline{X_\ell \setminus D_\ell} \setminus (X_\ell \setminus D_\ell)$ "boundary"

$\quad = Y_\ell$ is simple normal crossings divisor

Kausz compactification of GL_ℓ:

PGL_ℓ has a "wonderful compactif." (DeConcini-Procesi)

$X_0 = \mathbb{P}^{\ell^2-1}$ $\qquad Y_i$ = locus of matrices of rk i

$\qquad \overline{Y}_i$ closure in X_{i-1}

$X_i = Bl_{\overline{Y}_i}(X_{i-1})$ iterated blowups

$X_{\ell-1} = \overline{PGL_\ell}$ smooth w/ normal crossing div. boundary

$Y_i = PGL_i$-bdles over a prod of Grassmannians

$KGL_\ell = \overline{GL_\ell}$ closure inside $\overline{PGL_{\ell+1}}$ wonderful compactif

Kausz compactif.

also iterated blowup $X_0 = \mathbb{P}^{\ell^2}$

$X_i = Bl_{\overline{Y}_{i-1} \cup H_i}(X_{i-1})$ $\qquad Y_i$ rk i matrices in $\mathbb{A}^{\ell^2}$

Complement of GL_ℓ = normal crossings divisor Y_ℓ in KGL_ℓ $\qquad H_i$ in $\mathbb{P}^{\ell^2-1} = \mathbb{P}^{\ell^2} \setminus \mathbb{A}^{\ell^2}$

1) motive of KGl_ℓ

$\rightsquigarrow$ motives of bundles KGl_i fibers
over prods of Grassmannians (blowup loci)

& blowup formula for motives

$\rightsquigarrow$ $m(KGl_\ell)$ mixed Tate (in progress)

2) $Y_\ell \subset KGl_\ell$ normal crossings divisor

- ω_Γ $\rightsquigarrow$ Feynman amplitude reformulated in complement of det hypersurface
- pullback to Kausz compactification
- cohomologous to a form w/ log poles
- apply pole subtraction & Poincaré residue

$\hookrightarrow$ get MT periods

$m(KGl_\ell, \Sigma_{\ell,g})$

of MT motive

no worries here about $\Sigma_{\ell,g} \cap D_\ell$ anymore

and $m(Y_\ell, \Sigma_{\ell,g} \cap Y_\ell)$ any problem from here?
