Motives in Quantum Field Theory

Matilde Marcolli

Beijing, August 2013

Motives of algebraic varieties (Grothendieck) Universal cohomology theory for algebraic varieties (with realizations)

- *Pure motives*: smooth projective varieties with correspondences

$$\mathrm{Hom}((X,p,m),(Y,q,n)) = q\,\mathrm{Corr}^{m-n}_{/\sim,\mathbb{Q}}(X,Y)\,p$$

Algebraic cycles mod equivalence (rational, homological, numerical), composition

$$\mathrm{Corr}(X,Y)\times\mathrm{Corr}(Y,Z)\to\mathrm{Corr}(X,Z)$$

$$(\pi_{X,Z})_*(\pi^*_{X,Y}(\alpha)\bullet\pi^*_{Y,Z}(\beta))$$

intersection product in $X\times Y\times Z$; with projectors $p^2=p$ and $q^2=q$ and Tate twists $\mathbb{Q}(m)$ with $\mathbb{Q}(1)=\mathbb{L}^{-1}$

Numerical pure motives: $\mathcal{M}_{num,\mathbb{Q}}(k)$ semi-simple abelian category (Jannsen)

• *Mixed motives*: varieties that are possibly singular or not projective (much more complicated theory!) Triangulated category $\mathcal{DM}$ (Voevodsky , Levine, Hanamura)

$$\mathfrak{m}(Y) \to \mathfrak{m}(X) \to \mathfrak{m}(X \smallsetminus Y) \to \mathfrak{m}(Y)[1]$$

$$\mathfrak{m}(X \times \mathbb{A}^1) = \mathfrak{m}(X)(-1)[2]$$

• *Mixed Tate motives* $\mathcal{DMT} \subset \mathcal{DM}$ generated by the $\mathbb{Q}(m)$

Over a number field: t-structure, abelian category of mixed Tate motives (vanishing result, M.Levine)

Quantum Field Theory perturbative (massless) scalar field theory

$$S(\phi) = \int \mathcal{L}(\phi) d^D x = S_0(\phi) + S_{int}(\phi)$$

in D dimensions, with Lagrangian density

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \mathcal{L}_{int}(\phi)$$

Perturbative expansion: Feynman rules and Feynman diagrams

$$S_{eff}(\phi) = S_0(\phi) + \sum_{\Gamma} \frac{\Gamma(\phi)}{\#\mathrm{Aut}(\Gamma)} \quad \text{(1PI graphs)}$$

$$\Gamma(\phi) = \frac{1}{N!} \int_{\sum_i p_i = 0} \hat{\phi}(p_1) \cdots \hat{\phi}(p_N) U(\Gamma(p_1, \dots, p_N)) dp_1 \cdots dp_N$$

$$U(\Gamma(p_1, \dots, p_N)) = \int I_\Gamma(k_1, \dots, k_\ell, p_1, \dots, p_N) d^D k_1 \cdots d^D k_\ell$$

$\ell = b_1(\Gamma)$ loops

Feynman rules for $I_\Gamma(k_1, \ldots, k_\ell, p_1, \ldots, p_N)$:

- Internal lines $\Rightarrow$ propagator = quadratic form q_i

$$\frac{1}{q_1 \cdots q_n}, \quad q_i(k_i) = k_i^2 + m^2$$

- Vertices: conservation (valences = monomials in $\mathcal{L}$)

$$\sum_{e_i \in E(\Gamma): s(e_i) = v} k_i = 0$$

- Integration over k_i, internal edges

$$U(\Gamma) = \int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j)}{q_1 \cdots q_n} \, d^D k_1 \cdots d^D k_n$$

$n = \# E_{int}(\Gamma)$, $N = \# E_{ext}(\Gamma)$

$$\epsilon_{e,v} = \begin{cases} +1 & t(e) = v \\ -1 & s(e) = v \\ 0 & \text{otherwise,} \end{cases}$$

Formal properties reduce combinatorics to 1PI graphs:

• Connected graphs: $\Gamma = \cup_{v\in T}\Gamma_v$

$$U(\Gamma_1 \amalg \Gamma_2, p) = U(\Gamma_1, p_1)U(\Gamma_2, p_2)$$

• 1PI graphs:

$$U(\Gamma, p) = \prod_{v\in T} U(\Gamma_v, p_v)\frac{\delta((p_v)_e - (p_{v'})_e)}{q_e((p_v)_e)}$$

Note: formal properties can be used to construct abstract "algebro-geometric Feynman rules" (Chern classes; Grothendieck ring)

P. Aluffi, M.M. *Algebro-geometric Feynman rules*, arXiv:0811.2514

Parametric Feynman integrals

- Schwinger parameters $q_1^{-k_1} \cdots q_n^{-k_n} =$

$$\frac{1}{\Gamma(k_1)\cdots\Gamma(k_n)} \int_0^\infty \cdots \int_0^\infty e^{-(s_1 q_1 + \cdots + s_n q_n)}\, s_1^{k_1-1} \cdots s_n^{k_n-1}\, ds_1 \cdots ds_n.$$

- Feynman trick

$$\frac{1}{q_1 \cdots q_n} = (n-1)! \int \frac{\delta(1-\sum_{i=1}^n t_i)}{(t_1 q_1 + \cdots + t_n q_n)^n}\, dt_1 \cdots dt_n$$

then change of variables $k_i = u_i + \sum_{k=1}^{\ell} \eta_{ik} x_k$

$$\eta_{ik} = \begin{cases} \pm 1 & \text{edge } \pm e_i \in \text{ loop } \ell_k \\ 0 & \text{otherwise} \end{cases}$$

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2} V_\Gamma(t,p)^{n-D\ell/2}}$$

$\sigma_n = \{t \in \mathbb{R}_+^n \,|\, \sum_i t_i = 1\}$, vol form ω_n

Graph polynomials

$$\Psi_\Gamma(t) = \det M_\Gamma(t) = \sum_T \prod_{e \notin T} t_e \quad \text{with} \quad (M_\Gamma)_{kr}(t) = \sum_{i=0}^{n} t_i \eta_{ik} \eta_{ir}$$

Massless case $m = 0$:

$$V_\Gamma(t,p) = \frac{P_\Gamma(t,p)}{\Psi_\Gamma(t)} \quad \text{and} \quad P_\Gamma(p,t) = \sum_{C \subset \Gamma} s_C \prod_{e \in C} t_e$$

cut-sets C (complement of spanning tree plus one edge)
$s_C = (\sum_{v \in V(\Gamma_1)} P_v)^2$ with $P_v = \sum_{e \in E_{ext}(\Gamma), t(e)=v} p_e$ for $\sum_{e \in E_{ext}(\Gamma)} p_e = 0$
with $\deg \Psi_\Gamma = b_1(\Gamma) = \deg P_\Gamma - 1$

$$U(\Gamma) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2} \omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

stable range $-n + D\ell/2 \geq 0$; log divergent $n = D\ell/2$:

$$\int_{\sigma_n} \frac{\omega_n}{\Psi_\Gamma(t)^{D/2}}$$

Graph hypersurfaces

Residue of $U(\Gamma)$ (up to divergent Gamma factor)

$$\int_{\sigma_n} \frac{P_\Gamma(t,p)^{-n+D\ell/2}\omega_n}{\Psi_\Gamma(t)^{-n+D(\ell+1)/2}}$$

Graph hypersurfaces $\hat{X}_\Gamma = \{t \in \mathbb{A}^n \,|\, \Psi_\Gamma(t) = 0\}$

$$X_\Gamma = \{t \in \mathbb{P}^{n-1} \,|\, \Psi_\Gamma(t) = 0\} \quad \deg = b_1(\Gamma)$$

- Relative cohomology: (range $-n + D\ell/2 \geq 0$)

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma)) \quad \text{with} \quad \Sigma_n = \{\prod_i t_i = 0\} \supset \partial\sigma_n$$

- Periods: $\int_\sigma \omega$ integrals of algebraic differential forms ω on a cycle σ defined by algebraic equations in an algebraic variety

Feynman integrals and periods

Parametric Feynman integral: algebraic differential form on cycle in algebraic variety

But... divergent: where $X_\Gamma \cap \sigma_n \neq \emptyset$, inside divisor $\Sigma_n \supset \sigma_n$ of coordinate hyperplanes

- Blowups of coordinate linear spaces defined by edges of 1PI subgraphs (toric variety $P(\Gamma)$)
- Iterated blowup $P(\Gamma)$ separates strict transform of X_Γ from non-negative real points
- Deform integration chain: monodromy problem; lift to $P(\Gamma)$
- Subtraction of divergences: Poincaré residuces and limiting mixed Hodge structure

• S. Bloch, E. Esnault, D. Kreimer, *On motives associated to graph polynomials*, arXiv:math/0510011.
• S. Bloch, D. Kreimer, *Mixed Hodge Structures and Renormalization in Physics*, arXiv:0804.4399.

Periods and motives: Constraints on numbers obtained as periods from the motive of the variety!

• Periods of mixed Tate motives are Multiple Zeta Values

$$\zeta(k_1, k_2, \ldots, k_r) = \sum_{n_1 > n_2 > \cdots > n_r \geq 1} n_1^{-k_1} n_2^{-k_2} \cdots n_r^{-k_r}$$

Conjecture proved recently:

• Francis Brown, *Mixed Tate motives over* $\mathbb{Z}$, arXiv:1102.1312.

Feynman integrals and periods: MZVs as *typical* outcome:

• D. Broadhurst, D. Kreimer, *Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops*, arXiv:hep-th/9609128

$\Rightarrow$ Conjecture (Kontsevich 1997): Motives of graph hypersurfaces are mixed Tate (or counting points over finite fields behavior)

Conjecture was first verified for all graphs up to 12 edges:

• J. Stembridge, *Counting points on varieties over finite fields related to a conjecture of Kontsevich*, 1998

But ... Conjecture is false!

- P. Belkale, P. Brosnan, *Matroids, motives, and a conjecture of Kontsevich*, arXiv:math/0012198
- Dzmitry Doryn, *On one example and one counterexample in counting rational points on graph hypersurfaces*, arXiv:1006.3533
- Francis Brown, Oliver Schnetz, *A K3 in phi4*, arXiv:1006.4064.
- Francis Brown, Dzmitry Doryn, *Framings for graph hypersurfaces*, arXiv:1301.3056

• Belkale–Brosnan: general argument shows "motives of graph hypersurfaces can be arbitrarily complicated"

• Doryn, Brown–Schnetz, Brown–Doryn: explicit counterexamples (14 edges)

Motives and the Grothendieck ring of varieties

• Difficult to determine explicitly the motive of X_Γ (singular variety!) in the triangulated category of *mixed motives*

• Simpler invariant (universal Euler characteristic for motives): class $[X_\Gamma]$ in the Grothendieck ring of varieties $K_0(\mathcal{V})$

- generators $[X]$ isomorphism classes
- $[X] = [X \smallsetminus Y] + [Y]$ for $Y \subset X$ closed
- $[X] \cdot [Y] = [X \times Y]$

Tate motives: $\mathbb{Z}[\mathbb{L}, \mathbb{L}^{-1}] \subset K_0(\mathcal{M})$
(K_0 group of category of pure motives: virtual motives)

Universal Euler characteristics:

Any additive invariant of varieties: $\chi(X) = \chi(Y)$ if $X \cong Y$

$$\chi(X) = \chi(Y) + \chi(X \smallsetminus Y), \quad Y \subset X$$

$$\chi(X \times Y) = \chi(X)\chi(Y)$$

values in a commutative ring $\mathcal{R}$ is same thing as a ring homomorphism

$$\chi : K_0(\mathcal{V}) \to \mathcal{R}$$

Examples:

- Topological Euler characteristic
- Couting points over finite fields
- Gillet–Soulé motivic $\chi_{mot}(X)$:

$$\chi_{mot} : K_0(\mathcal{V})[\mathbb{L}^{-1}] \to K_0(\mathcal{M}), \quad \chi_{mot}(X) = [(X, id, 0)]$$

for X smooth projective; complex $\chi_{mot}(X) = W^{\cdot}(X)$

Universality: a dichotomy

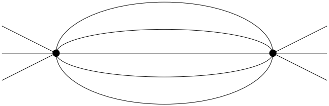
- After localization (Belkale-Brosnan): the graph hypersurfaces X_Γ *generate* the Grothendieck ring *localized* at $\mathbb{L}^n - \mathbb{L}$, $n > 1$
- Stable birational equivalence: the graph hypersurfaces span $\mathbb{Z}$ inside $\mathbb{Z}[SB] = K_0(\mathcal{V})|_{\mathbb{L}=0}$

• P. Aluffi, M.M. *Graph hypersurfaces and a dichotomy in the Grothendieck ring*, arXiv:1005.4470

Graph hypersurfaces: computing in the Grothendieck ring

• P. Aluffi, M.M. *Feynman motives of banana graphs*, arXiv:0807.1690

Example: *banana graphs* $\Psi_\Gamma(t) = t_1 \cdots t_n(\frac{1}{t_1} + \cdots + \frac{1}{t_n})$



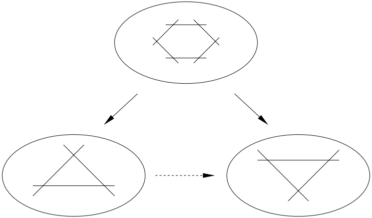
$$[X_{\Gamma_n}] = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \frac{(\mathbb{L}-1)^n - (-1)^n}{\mathbb{L}} - n\,(\mathbb{L}-1)^{n-2}$$

where $\mathbb{L} = [\mathbb{A}^1]$ Lefschetz motive and $\mathbb{T} = [\mathbb{G}_m] = [\mathbb{A}^1] - [\mathbb{A}^0]$
$X_{\Gamma^\vee} = \mathcal{L}$ hyperplane in $\mathbb{P}^{n-1}$
$\Gamma^\vee =$ dual graph $=$ polygon

Method: Dual graph and Cremona transformation

$$\mathcal{C} : (t_1 : \cdots : t_n) \mapsto (\frac{1}{t_1} : \cdots : \frac{1}{t_n})$$

outside $\mathcal{S}_n$ singularities locus of $\Sigma_n = \{\prod_i t_i = 0\}$, ideal $I_{\mathcal{S}_n} = (t_1 \cdots t_{n-1}, t_1 \cdots t_{n-2} t_n, \cdots , t_1 t_3 \cdots t_n)$



$$\Psi_\Gamma(t_1, \ldots, t_n) = (\prod_e t_e) \Psi_{\Gamma^\vee}(t_1^{-1}, \ldots, t_n^{-1})$$

$$\mathcal{C}(X_\Gamma \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n)) = X_{\Gamma^\vee} \cap (\mathbb{P}^{n-1} \smallsetminus \Sigma_n)$$

isomorphism of X_Γ and $X_{\Gamma^\vee}$ outside of Σ_n

For banana graph case obtain:

$$[\mathcal{L} \smallsetminus \Sigma_n] = [\mathcal{L}] - [\mathcal{L} \cap \Sigma_n] = \frac{\mathbb{T}^{n-1} - (-1)^{n-1}}{\mathbb{T} + 1}$$

$$X_{\Gamma_n} \cap \Sigma_n = \mathcal{S}_n \quad \text{with} \quad [\mathcal{S}_n] = [\Sigma_n] - n\mathbb{T}^{n-2}$$

$$[X_{\Gamma_n}] = [X_{\Gamma_n} \cap \Sigma_n] + [X_{\Gamma_n} \smallsetminus \Sigma_n]$$

Using Cremona transformation: $[X_{\Gamma_n}] = [\mathcal{S}_n] + [\mathcal{L} \smallsetminus \Sigma_n]$
$\Rightarrow \chi(X_{\Gamma_n}) = n + (-1)^n$

Sum over graphs

Even when non-planar: can transform by Cremona
(new hypersurface, not of dual graph)
$\Rightarrow$ graphs by removing edges from complete graph: fixed vertices

$$S_N = \sum_{\#V(\Gamma)=N} [X_\Gamma] \frac{N!}{\#\mathrm{Aut}(\Gamma)} \in \mathbb{Z}[\mathbb{L}],$$

Tate motive (though $[X_\Gamma]$ individually need not be)

- Spencer Bloch, *Motives associated to sums of graphs*, arXiv:0810.1313

Suggests that although individual graphs need not give mixed Tate contribution, the sum over graphs in Feynman amplitudes (fixed loops, not vertices) may be mixed Tate

Feynman rules in algebraic geometry

$U(\Gamma) \in \mathcal{R}$ (comm. ring $\mathcal{R}$, finite graph Γ)

$$U(\Gamma) = U(\Gamma_1) \cdots U(\Gamma_k) \quad \text{for } \Gamma = \Gamma_1 \amalg \cdots \amalg \Gamma_k$$

$$U(\Gamma) = U(L)^{\# E(T)} \prod_{v \in V(T)} U(\Gamma_v)$$

non-1PI: $\Gamma = \cup_{v \in V(T)} \Gamma_v$

Inverse propagator: $U(L)$ for $L =$ single edge

Algebro-geometric Feynman rules: $\Gamma = \Gamma_1 \amalg \Gamma_2$

$$\mathbb{A}^{n_1+n_2} \smallsetminus \hat{X}_\Gamma = (\mathbb{A}^{n_1} \smallsetminus \hat{X}_{\Gamma_1}) \times (\mathbb{A}^{n_2} \smallsetminus \hat{X}_{\Gamma_2})$$

$$\Psi_\Gamma(t_1, \ldots, t_n) = \Psi_{\Gamma_1}(t_1, \ldots, t_{n_1}) \Psi_{\Gamma_2}(t_{n_1+1}, \ldots, t_{n_1+n_2})$$

In projective space not product but *join*:

$$\mathbb{P}^{n_1+n_2-1} \smallsetminus X_\Gamma \to (\mathbb{P}^{n_1-1} \smallsetminus X_{\Gamma_1}) \times (\mathbb{P}^{n_2-1} \smallsetminus X_{\Gamma_2})$$

$\mathbb{G}_m$-bundle (assume Γ_i not a forest)

Ring of immersed conical varieties $\mathcal{F}$

$V \subset \mathbb{A}^N$ N not fixed, homogeneous ideals (conical), $[V]$ up to linear changes of coordinates (less than up to isomorphism)

$$[V \cup W] = [V] + [W] - [V \cap W]$$

$$[V] \cdot [W] = [V \times W]$$

embedded version of Grothendieck ring

- Mod by isomorphisms $\Rightarrow$ maps to $K_0(\mathcal{V})$
- Maps to polynomial invariant

$$I_{CSM} : \mathcal{F} \to \mathbb{Z}[T]$$

not factoring through Grothendieck group
(characteristic classes of singular varieties)

Algebro-geometric Feynman rules: homomorphisms

$$I : \mathcal{F} \to \mathcal{R}, \qquad \mathbb{U}(\Gamma) := I([\mathbb{A}^n]) - I([\hat{X}_\Gamma])$$

$\Rightarrow I([\mathbb{A}^n \smallsetminus \hat{X}_\Gamma])$ Feynman rule with

$$\mathbb{U}(L) = I([\mathbb{A}^1])$$

Inverse propagator $=$ affine line $[\mathbb{A}^1] \Rightarrow$ Lefschetz motive $\mathbb{L}$

- Universal algebro-geometric Feynman rule

$$\mathbb{U}(\Gamma) = [\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] \in \mathcal{F}$$

Motivic $=$ factors through $K_0(\mathcal{V})$

$$[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = (\mathbb{L} - 1)[\mathbb{P}^{n-1} \smallsetminus X_\Gamma] \in K_0(\mathcal{V})$$

(if Γ not a forest) since $[\hat{X}_\Gamma] = (\mathbb{L} - 1)[X_\Gamma] + 1$ affine cone

Euler characteristics and Feynman rules

• *Is the Euler characteristic as Feynman rule?*

• *Not* in projective case: for $\Gamma = \Gamma_1 \amalg \Gamma_2$ complement $\mathbb{P}^{n-1} \smallsetminus X_\Gamma$ is $\mathbb{G}_m$-bundle over

$$(\mathbb{P}^{n_1-1} \smallsetminus X_{\Gamma_1}) \times (\mathbb{P}^{n_2-1} \smallsetminus X_{\Gamma_2})$$

so Euler characteristic $\chi(\mathbb{P}^{n-1} \smallsetminus X_\Gamma) = 0$

• *Trivial* in affine case: $\chi(\mathbb{A}^n \smallsetminus \hat{X}_\Gamma) = 0$

• Question (Bloch): is there a modification of the Euler characteristic χ_{new}?

$$\chi_{new}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma) = \chi_{new}(\mathbb{P}^{n_1-1} \smallsetminus X_{\Gamma_1})\,\chi_{new}(\mathbb{P}^{n_2-1} \smallsetminus X_{\Gamma_2})$$

Characteristic classes of singular varieties

• *Nonsingular case:* Chern class $c(V) = c(TV) \cap [V]$ with $\int c(TV) \cap [V] = \chi(V)$ deg of zero dim component (Poincaré–Hopf)

• *Singular case:* $c_{CSM}(X)$ Chern–Schwartz–MacPherson class (M.H. Schwartz: radial vector fields; MacPherson: functoriality)

- Constructible functions $\mathbb{F}(X)$ functor: $f_*(1_W) = \chi(W \cap f^{-1}(p))$
- Natural transformation to homology (Chow):

$c_*(1_X) = c(TX) \cap [X]$ for smooth, and MacPherson formula with Mather classes and local Euler obstructions for general case

Hypersurfaces with isolated singularities $\Rightarrow$ Milnor numbers

...but X_Γ non-isolated singularities

Properties of c_{CSM}

• Inclusion-exclusion (not isomorphism-invariant)

$$c_{CSM}(X) = c_{CSM}(Y) + c_{CSM}(X \smallsetminus Y)$$

• classes $c_{CSM}(X_\Gamma)$ in ambient $\mathbb{P}^{n-1}$

equivalent to knowing the Euler characteristics of iterated hyperplane sections (Aluffi)

Example: banana graphs: $\chi(X_{\Gamma_n})$ = top deg term

$$c_{CSM}(X_{\Gamma_n}) = ((1+H)^n - (1-H)^{n-1} - nH - H^n) \cdot [\mathbb{P}^{n-1}]$$

Feynman rules from CSM classes

$$c_*(1_{\hat{X}}) = a_0[\mathbb{P}^0] + a_1[\mathbb{P}^1] + \cdots + a_N[\mathbb{P}^N] \in A(\mathbb{P}^N)$$

natural transformation from constructible function $1_{\hat{X}}$ for $\hat{X} \subset \mathbb{A}^N$ loc closed in $\mathbb{P}^N$ to Chow group $A(\mathbb{P}^N)$
Define:

$$G_{\hat{X}}(T) := a_0 + a_1 T + \cdots + a_N T^N$$

independent of N, stops at $\dim \hat{X}$; invariant under coordinate changes, with

$$G_{\hat{X}\cup\hat{Y}}(T) = G_{\hat{X}}(T) + G_{\hat{Y}}(T) - G_{\hat{X}\cap\hat{Y}}(T)$$

from inclusion-exclusion of CSM classes
So it defines a map

$$I_{CSM}([\hat{X}]) = G_{\hat{X}}(T), \quad I_{CSM} : \mathcal{F} \to \mathbb{Z}[T]$$

Not easy to see:

$$I_{CSM}([\hat{X}]) = G_{\hat{X}}(T), \quad I_{CSM} : \mathcal{F} \to \mathbb{Z}[T]$$

is a *ring homomorphism*

$$G_{\hat{X}\times\hat{Y}}(T) = G_{\hat{X}}(T) \cdot G_{\hat{Y}}(T)$$

need CSM classes of joins $J(X, Y) \subset \mathbb{P}^{m+n-1}$

$$(sx_1 : \cdots : sx_m : ty_1 : \cdots : ty_n), \quad (s : t) \in \mathbb{P}^1$$

$\hat{X} \times \hat{Y}$ affine cone over $J(X, Y)$:

$$c_*(1_{J(X,Y)}) = ((f(H) + H^m)(g(H) + H^n) - H^{m+n}) \cap [\mathbb{P}^{m+n-1}]$$

$$c_*(1_X) = H^n f(H) \cap [\mathbb{P}^{n+m-1}], \; c_*(1_Y) = H^m g(H) \cap [\mathbb{P}^{n+m-1}]$$

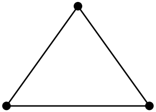
CSM Feynman rule:

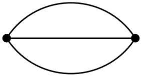
$$\mathbb{U}_{CSM}(\Gamma) = C_\Gamma(T) = I_{CSM}([\mathbb{A}^n]) - I_{CSM}([\hat{X}_\Gamma])$$

- it is algebro geometric but not motivic:

$$C_{\Gamma_1}(T) = T(T+1)^2 \quad C_{\Gamma_2}(T) = T(T^2+T+1)$$

$$[\mathbb{A}^n \smallsetminus \hat{X}_{\Gamma_i}] = [\mathbb{A}^3] - [\mathbb{A}^2] \in K_0(\mathcal{V})$$





Properties of $C_\Gamma(T)$:

- $C_\Gamma(T)$ monic of deg n
- Γ = forest $\Rightarrow C_\Gamma(T) = (T+1)^n$
- Inverse propagator $\mathbb{U}_{CSM}(L) = T + 1$
- Coeff of T^{n-1} is $n - b_1(\Gamma)$
- $C'_\Gamma(0) = \chi(\mathbb{P}^{n-1} \smallsetminus X_\Gamma)$

$\Rightarrow$ it is a modification of $\chi(\mathbb{P}^{n-1} \smallsetminus \hat{X}_\Gamma)$ giving Feynman rule (answer to the question of χ_{new})

Deletion–contraction relation

In general cannot compute explicitly $[X_\Gamma]$: would like relations that simplify the graph... but cannot have *true* deletion-contraction relation, else always mixed Tate... What kind of deletion-contraction?

- P. Aluffi, M.M. *Feynman motives and deletion-contraction relations*, arXiv:0907.3225

- Graph polynomials: Γ with $n \geq 2$ edges, $\deg \Psi_\Gamma = \ell > 0$

$$\Psi_\Gamma = t_e \Psi_{\Gamma \smallsetminus e} + \Psi_{\Gamma / e}$$

$$\Psi_{\Gamma \smallsetminus e} = \frac{\partial \Psi_\Gamma}{\partial t_n} \quad \text{and} \quad \Psi_{\Gamma / e} = \Psi_\Gamma |_{t_n = 0}$$

- General fact: $X = \{\psi = 0\} \subset \mathbb{P}^{n-1}$, $Y = \{F = 0\} \subset \mathbb{P}^{n-2}$

$$\psi(t_1, \ldots, t_n) = t_n F(t_1, \ldots, t_{n-1}) + G(t_1, \ldots, t_{n-1})$$

$\overline{Y}$ = cone of Y in $\mathbb{P}^{n-1}$: Projection from $(0 : \cdots : 0 : 1) \Rightarrow$ isomorphism

$$X \smallsetminus (X \cap \overline{Y}) \xrightarrow{\sim} \mathbb{P}^{n-2} \smallsetminus Y$$

Then deletion-contraction: for $\hat{X}_\Gamma \subset \mathbb{A}^n$

$$[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus (\hat{X}_{\Gamma \smallsetminus e} \cap \hat{X}_{\Gamma/e})] - [\mathbb{A}^{n-1} \smallsetminus \hat{X}_{\Gamma \smallsetminus e}]$$

if e not a bridge or a looping edge

$$[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus \hat{X}_{\Gamma \smallsetminus e}] = \mathbb{L} \cdot [\mathbb{A}^{n-1} \smallsetminus \hat{X}_{\Gamma/e}]$$

if e bridge

$$\begin{aligned} [\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] &= (\mathbb{L} - 1) \cdot [\mathbb{A}^{n-1} \smallsetminus \hat{X}_{\Gamma \smallsetminus e}] \\ &= (\mathbb{L} - 1) \cdot [\mathbb{A}^{n-1} \smallsetminus \hat{X}_{\Gamma/e}] \end{aligned}$$

if e looping edge

Note: intersection $\hat{X}_{\Gamma \smallsetminus e} \cap \hat{X}_{\Gamma/e}$ difficult to control motivically: first place where non-Tate contributions will appear

Example of application: Multiplying edges

Γ_{me} obtained from Γ by replacing edge e by m parallel edges

$(\Gamma_{0e} = \Gamma \smallsetminus e,\ \Gamma_e = \Gamma)$

Generating function: $\mathbb{T} = [\mathbb{G}_m] \in K_0(\mathcal{V})$

$$\sum_{m\geq 0} \mathbb{U}(\Gamma_{me}) \frac{s^m}{m!} = \frac{e^{\mathbb{T}s} - e^{-s}}{\mathbb{T}+1} \mathbb{U}(\Gamma) + \frac{e^{\mathbb{T}s} + \mathbb{T}e^{-s}}{\mathbb{T}+1} \mathbb{U}(\Gamma \smallsetminus e) + \left(s\, e^{\mathbb{T}s} - \frac{e^{\mathbb{T}s} - e^{-s}}{\mathbb{T}+1} \right) \mathbb{U}(\Gamma/e).$$

e not bridge nor looping edge: similar for other cases

For doubling: inclusion-exclusion

$$\mathbb{U}(\Gamma_{2e}) = \mathbb{L} \cdot [\mathbb{A}^n \smallsetminus (\hat{X}_\Gamma \cap \hat{X}_{\Gamma_o})] - \mathbb{U}(\Gamma)$$

$$[\hat{X}_\Gamma \cap \hat{X}_{\Gamma_o}] = [\hat{X}_{\Gamma/e}] + (\mathbb{L}-1) \cdot [\hat{X}_{\Gamma \smallsetminus e} \cap \hat{X}_{\Gamma/e}]$$

then cancellation

$$\mathbb{U}(\Gamma_{2e}) = (\mathbb{L}-2) \cdot \mathbb{U}(\Gamma) + (\mathbb{L}-1) \cdot \mathbb{U}(\Gamma \smallsetminus e) + \mathbb{L} \cdot \mathbb{U}(\Gamma/e)$$

Example of application: Lemon graphs and chains of polygons

Λ_m = lemon graph m wedges; Γ^{Λ}_m = replacing edge e of Γ with Λ_m

Generating function: $\sum_{m\geq 0} \mathbb{U}(\Gamma^{\Lambda}_m) s^m =$

$$\frac{(1-(\mathbb{T}+1)s)\,\mathbb{U}(\Gamma) + (\mathbb{T}+1)\mathbb{T}s\,\mathbb{U}(\Gamma \smallsetminus e) + (\mathbb{T}+1)^2 s\,\mathbb{U}(\Gamma/e)}{1-\mathbb{T}(\mathbb{T}+1)s - \mathbb{T}(\mathbb{T}+1)^2 s^2}$$

e not bridge or looping edge; similar otherwise

Recursive relation:

$$\mathbb{U}(\Lambda_{m+1}) = \mathbb{T}(\mathbb{T}+1)\mathbb{U}(\Lambda_m) + \mathbb{T}(\mathbb{T}+1)^2\mathbb{U}(\Lambda_{m-1})$$

$a_m = \mathbb{U}(\Lambda_m)$ is a *divisibility sequence*: $\mathbb{U}(\Lambda_{m-1})$ divides $\mathbb{U}(\Lambda_{n-1})$ if m divides n

Determinant hypersurfaces and Schubert cells

Mixed Tate question reformulated in terms of determinant hypersurfaces and intersections of unions of Schubert cells in flag varieties

- P. Aluffi, M.M. *Parametric Feynman integrals and determinant hypersurfaces*, arXiv:0901.2107

$$\Upsilon : \mathbb{A}^n \to \mathbb{A}^{\ell^2}, \quad \Upsilon(t)_{kr} = \sum_i t_i \eta_{ik}\eta_{ir}, \quad \hat{X}_\Gamma = \Upsilon^{-1}(\hat{\mathcal{D}}_\ell)$$

determinant hypersurface $\hat{\mathcal{D}}_\ell = \{\det(x_{ij}) = 0\}$

$$[\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell] = \mathbb{L}^{\binom{\ell}{2}} \prod_{i=1}^{\ell} (\mathbb{L}^i - 1) \Rightarrow \text{ mixed Tate}$$

When Υ embedding

$$U(\Gamma) = \int_{\Upsilon(\sigma_n)} \frac{\mathcal{P}_\Gamma(x,p)^{-n+D\ell/2}\omega_\Gamma(x)}{\det(x)^{-n+(\ell+1)D/2}}$$

If $\hat{\Sigma}_\Gamma$ normal crossings divisor in $\mathbb{A}^{\ell^2}$ with $\Upsilon(\partial\sigma_n) \subset \hat{\Sigma}_\Gamma$

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_\Gamma \smallsetminus (\hat{\Sigma}_\Gamma \cap \hat{\mathcal{D}}_\ell)) \quad \text{mixed Tate motive?}$$

Combinatorial conditions for embedding $\Upsilon : \mathbb{A}^n \smallsetminus \hat{X}_\Gamma \hookrightarrow \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell$

- Closed 2-cell embedded graph $\iota : \Gamma \hookrightarrow S_g$ with $S_g \smallsetminus \Gamma$ union of open disks (faces); closure of each is a disk.
- Two faces have at most one edge in common
- Every edge in the boundary of two faces

Sufficient: Γ 3-edge-connected with closed 2-cell embedding of face width ≥ 3.

Face width: largest $k \in \mathbb{N}$, every non-contractible simple closed curve in S_g intersects Γ at least k times (∞ for planar).

Note: 2-edge-connected =1PI; 2-vertex-connected conjecturally implies face width ≥ 2

Identifying the motive $\mathfrak{m}(X,Y)$. Set $\hat{\Sigma}_\Gamma \subset \hat{\Sigma}_{\ell,g}$ $(f = \ell - 2g + 1)$

$$\hat{\Sigma}_{\ell,g} = L_1 \cup \cdots \cup L_{\binom{f}{2}}$$

$$\begin{cases} x_{ij} & = 0 \quad 1 \leq i < j \leq f-1 \\ x_{i1} + \cdots + x_{i,f-1} & = 0 \quad 1 \leq i \leq f-1 \end{cases}$$

$$\mathfrak{m}(\mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell, \hat{\Sigma}_{\ell,g} \smallsetminus (\hat{\Sigma}_{\ell,g} \cap \hat{\mathcal{D}}_\ell))$$

$\hat{\Sigma}_{\ell,g}$ = normal crossings divisor $\Upsilon_\Gamma(\partial\sigma_n) \subset \hat{\Sigma}_{\ell,g}$
depends only on $\ell = b_1(\Gamma)$ and g = min genus of S_g

- Sufficient condition: Varieties of frames mixed Tate?

$$\mathbb{F}(V_1, \ldots, V_\ell) := \{(v_1, \ldots, v_\ell) \in \mathbb{A}^{\ell^2} \smallsetminus \hat{\mathcal{D}}_\ell \mid v_k \in V_k\}$$

Varieties of frames

• Two subspaces: $(d_{12} = \dim(V_1 \cap V_2))$

$$[\mathbb{F}(V_1, V_2)] = \mathbb{L}^{d_1+d_2} - \mathbb{L}^{d_1} - \mathbb{L}^{d_2} - \mathbb{L}^{d_{12}+1} + \mathbb{L}^{d_{12}} + \mathbb{L}$$

• Three subspaces $(D = \dim(V_1 + V_2 + V_3))$

$$[\mathbb{F}(V_1, V_2, V_3)] = (\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - 1)(\mathbb{L}^{d_3} - 1)$$

$$-(\mathbb{L} - 1)((\mathbb{L}^{d_1} - \mathbb{L})(\mathbb{L}^{d_{23}} - 1) + (\mathbb{L}^{d_2} - \mathbb{L})(\mathbb{L}^{d_{13}} - 1) + (\mathbb{L}^{d_3} - \mathbb{L})(\mathbb{L}^{d_{12}} - 1)$$

$$+(\mathbb{L} - 1)^2(\mathbb{L}^{d_1+d_2+d_3-D} - \mathbb{L}^{d_{123}+1}) + (\mathbb{L} - 1)^3$$

• Higher: difficult to find suitable induction

• Other formulation: $Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$ locus of complete flags $0 \subset E_1 \subset E_2 \subset \cdots \subset E_\ell = E$, with $\dim E_i \cap V_i = d_i$ and $\dim E_i \cap V_{i+1} = e_i$: are these mixed Tate? (for all choices of d_i, e_i)

• $\mathbb{F}(V_1, \ldots, V_\ell)$ fibration over $Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$: class $[\mathbb{F}(V_1, \ldots, V_\ell)]$

$$= [Flag_{\ell,\{d_i,e_i\}}(\{V_i\})](\mathbb{L}^{d_1} - 1)(\mathbb{L}^{d_2} - \mathbb{L}^{e_1})(\mathbb{L}^{d_3} - \mathbb{L}^{e_2}) \cdots (\mathbb{L}^{d_r} - \mathbb{L}^{e_{r-1}})$$

$Flag_{\ell,\{d_i,e_i\}}(\{V_i\})$ intersection of unions of Schubert cells in flag varieties $\Rightarrow$ Kazhdan–Lusztig?

Other approach: Feynman integrals in configuration space

• Özgür Ceyhan, M.M. *Feynman integrals and motives of configuration spaces*, arXiv:1012.5485

Singularities of Feynman amplitude along diagonals

$$\Delta_e = \{(x_v)_{v\in V_\Gamma} \,|\, x_{v_1} = x_{v_2} \text{ for } \partial_\Gamma(e) = \{v_1, v_2\}\}$$

$$Conf_\Gamma(X) = X^{V_\Gamma} \smallsetminus \bigcup_{e\in E_\Gamma} \Delta_e = X^{V_\Gamma} \smallsetminus \cup_{\gamma\subset\mathcal{G}_\Gamma} \Delta_\gamma,$$

with $\mathcal{G}_\Gamma$ subgraphs induced (all edges of Γ between subset of vertices) and 2-vertex-connected

$$Conf_\Gamma(X) \hookrightarrow \prod_{\gamma\in\mathcal{G}_\Gamma} \mathrm{Bl}_{\Delta_\gamma} X^{V_\Gamma}$$

iterated blowup description (wonderful "compactifications": generalize Fulton-MacPherson)

$$\overline{Conf}_\Gamma(X) = Conf_\Gamma(X) \cup \bigcup_{\mathcal{N}\in\, \mathcal{G}-nests} X^\circ_{\mathcal{N}}$$

stratification by $\mathcal{G}$-nests of subgraphs (based on work of Li Li)

Voevodsky motive (quasi-projective smooth X)

$$m(\overline{Conf}_\Gamma(X)) = m(X^{V_\Gamma}) \oplus \bigoplus_{\mathcal{N}\in\mathcal{G}_\Gamma\text{-nests},\mu\in M_\mathcal{N}} m(X^{V_{\Gamma/\delta_\mathcal{N}(\Gamma)}})(\|\mu\|)[2\|\mu\|]$$

where $M_\mathcal{N} := \{(\mu_\gamma)_{\Delta_\gamma\in\mathcal{G}_\Gamma} : 1 \le \mu_\gamma \le r_\gamma - 1,\ \mu_\gamma \in \mathbb{Z}\}$ with $r_\gamma = r_{\gamma,\mathcal{N}} := \dim(\cap_{\gamma'\in\mathcal{N}:\gamma'\subset\gamma}\Delta_{\gamma'}) - \dim\Delta_\gamma$ and $\|\mu\| := \sum_{\Delta_\gamma\in\mathcal{G}_\Gamma}\mu_\gamma$

Class in the Grothendieck ring

$$[\overline{Conf}_\Gamma(X)] = [X]^{|V_\Gamma|} + \sum_{\mathcal{N}\in\mathcal{G}_\Gamma\text{-nests}} [X]^{|V_{\Gamma/\delta_\mathcal{N}(\Gamma)}|} \sum_{\mu\in M_\mathcal{N}} \mathbb{L}^{\|\mu\|}$$

Key ingredient: Blowup formulae

- For mixed motives:

$$m(\mathrm{Bl}_V(Y)) \cong m(Y) \oplus \bigoplus_{k=1}^{\mathrm{codim}_Y(V)-1} m(V)(k)[2k]$$

- Bittner relation in $K_0(\mathcal{V})$: exceptional divisor E

$$[\mathrm{Bl}_V(Y)] = [Y] - [V] + [E] = [Y] + [V]([\mathbb{P}^{\mathrm{codim}_Y(V)-1}] - 1)$$

- $\overline{Conf}_\Gamma(X)$ are mixed Tate motives if X is
- To regularize Feynman integrals: lift to blowup $\overline{Conf}_\Gamma(X)$
- Ambiguities by monodromies along exceptional divisors of the iterated blowups
- Residues of Feynman integrals and periods on hypersurface complement in $\overline{Conf}_\Gamma(X)$
- Poincaré residues: periods on intersections of divisors of the stratification

Some other recent results:

- All the original Broadhurst–Kreimer cases now proved
  Mapping to moduli space $\bar{\mathcal{M}}_{0,n}$ and using results on multiple zeta values as periods of $\bar{\mathcal{M}}_{0,n}$ (Goncharov-Manin, Brown)
  • Francis Brown, *On the periods of some Feynman integrals*, arXiv:0910.0114
- Chern classes of graph hypersurfaces
  Mixed Tate cases possible thanks to X_Γ being singular (in low codimension): Chern–Schwartz–MacPherson classes measure singularities and can be assembled into an algebro-geometric Feynman rule: deletion-contraction and recursions
  • Paolo Aluffi, *Chern classes of graph hypersurfaces and deletion-contraction relations*, arXiv:1106.1447

Regularization and renormalization

Removing divergences from Feynman integrals by adjusting bare parameters in the Lagrangian

$$\mathcal{L}_E = \frac{1}{2}(\partial\phi)^2(1-\delta Z) + \left(\frac{m^2 - \delta m^2}{2}\right)\phi^2 - \frac{g+\delta g}{6}\phi^3$$

Regularization: replace divergent integral $U(\Gamma)$ by function $U^z(\Gamma)$ with pole ($z \in \mathbb{C}^*$ in DimReg, ϵ deformation of X_Γ, etc.)
Renormalization: consistency over subgraphs (Hopf algebra structure)

- Kreimer, Connes–Kreimer, Connes–M.: Hopf algebra of Feynman graphs and BPHZ renormalization method in terms of Birkhoff factorization and differential Galois theory

- Ebrahimi-Fard, Guo, Kreimer: algebraic renormalization in terms of Rota–Baxter algebras

BPHZ renormalization method:

- Preparation:

$$\bar{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma)$$

- Counterterm: projection onto polar part

$$C(\Gamma) = -T(\bar{R}(\Gamma))$$

- Renormalized value:

$$R(\Gamma) = \bar{R}(\Gamma) + C(\Gamma)$$

$$= U(\Gamma) + C(\Gamma) + \sum_{\gamma \in \mathcal{V}(\Gamma)} C(\gamma) U(\Gamma/\gamma)$$

Connes–Kreimer Hopf algebra $\mathcal{H} = \mathcal{H}(\mathcal{T})$ (depends on theory $\mathcal{L}(\phi)$)

- Free commutative algebra in generators Γ 1PI Feynman graphs
- Grading: loop number (or internal lines)

$$\deg(\Gamma_1 \cdots \Gamma_n) = \sum_i \deg(\Gamma_i), \quad \deg(1) = 0$$

- Coproduct:

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \in \mathcal{V}(\Gamma)} \gamma \otimes \Gamma/\gamma$$

- Antipode: inductively

$$S(X) = -X - \sum S(X')X''$$

for $\Delta(X) = X \otimes 1 + 1 \otimes X + \sum X' \otimes X''$

Extended to gauge theories (van Suijlekom): Ward identities as Hopf ideals

Algebraic renormalization (Ebrahimi-Fard, Guo, Kreimer)

• Rota–Baxter algebra of weight $\lambda = -1$: $\mathcal{R}$ commutative unital algebra; $T : \mathcal{R} \to \mathcal{R}$ linear operator with

$$T(x)T(y) = T(xT(y)) + T(T(x)y) + \lambda T(xy)$$

• Example: $T =$ projection onto polar part of Laurent series

• T determines splitting $\mathcal{R}_+ = (1 - T)\mathcal{R}$, $\mathcal{R}_- =$ unitization of $T\mathcal{R}$; both $\mathcal{R}_\pm$ are algebras

• Feynman rule $\phi : \mathcal{H} \to \mathcal{R}$ commutative algebra homomorphism from CK Hopf algebra $\mathcal{H}$ to Rota–Baxter algebra $\mathcal{R}$ weight -1

$$\phi \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R})$$

• Note: ϕ does *not know* that $\mathcal{H}$ Hopf and $\mathcal{R}$ Rota-Baxter, only commutative algebras

- Birkhoff factorization $\quad \exists \phi_\pm \in \mathrm{Hom}_{\mathrm{Alg}}(\mathcal{H}, \mathcal{R}_\pm)$

$$\phi = (\phi_- \circ S) \star \phi_+$$

where $\phi_1 \star \phi_2(X) = \langle \phi_1 \otimes \phi_2, \Delta(X) \rangle$

- Connes-Kreimer inductive formula for Birkhoff factorization:

$$\phi_-(X) = -T(\phi(X) + \sum \phi_-(X')\phi(X''))$$

$$\phi_+(X) = (1 - T)(\phi(X) + \sum \phi_-(X')\phi(X''))$$

where $\Delta(X) = 1 \otimes X + X \otimes 1 + \sum X' \otimes X''$

Recent developments

• Yuri Manin proposed a use of algebraic renormalization in the context of the theory of computation and the halting problem

- Yu.I. Manin, *Renormalization and computation I: motivation and background*, arXiv:0904.4921
- Yu.I. Manin, *Renormalization and Computation II: Time Cut-off and the Halting Problem*, arXiv:0908.3430
- C.Delaney, M.M., *Dyson-Schwinger equations in the theory of computation*, arXiv:1302.5040

• Other questions: relations of the Rota–Baxter formalism to the algebro–geometric Feynman rules? Motivic version of algebraic renormalization?