A motivic approach to Potts models

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Based on joint work with Paolo Aluffi

Some references:

- P. Aluffi, M.M., *A motivic approach to phase transitions in Potts models*, J. Geom. Phys., Vol.63 (2013) 6–31
- M.M. *Feynman motives*, World Scientific, 2010.

+ other references listed later

Potts Models: Statistical Mechanics

G = finite graph

$\mathfrak{A}$ = set of possible spin states at a vertex, $\#\mathfrak{A} = q$

- State: assignment of a spin state to each vertex of G
- Energy: sum over edges, zero if endpoint spins not aligned,

$-J_e$ if aligned (same) spins

- Edge variables: $t_e = e^{\beta J_e} - 1$, with β thermodynamic parameter

(inverse temperature)

- Physical values: $t_e \geq 0$ ferromagnetic case ($J_e \geq 0$) and

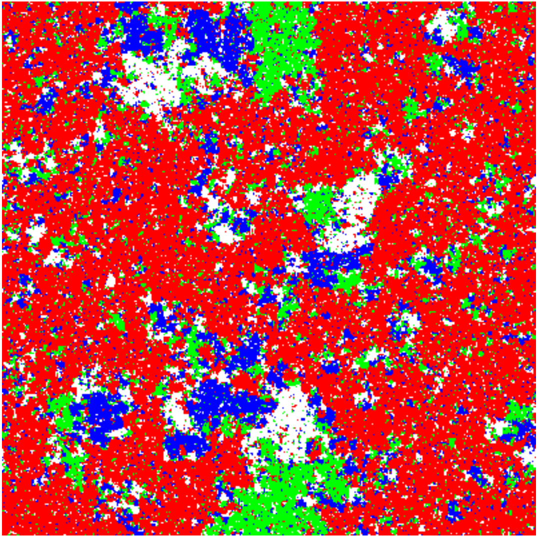
$-1 \leq t_e \leq 0$ antiferromagnetic case $-\infty \leq J_e \leq 0$.

Partition function

$$Z_G(q,t) = \sum_{\sigma: V(G) \to \mathfrak{A}} \prod_{e \in E(G)} (1 + t_e \delta_{\sigma(v),\sigma(w)})$$

sum over all maps of vertices to spin states, and $\partial(e) = \{v, w\}$

Example: 2D lattice, with $q = 4$, near critical temperature



Multivariable Tutte polynomial (Fortuin–Kasteleyn)

$$Z_G(q,t) = \sum_{G' \subseteq G} q^{k(G')} \prod_{e \in E(G')} t_e$$

$k(G') = b_0(G')$ connected components, sum over all subgraphs $G' \subseteq G$ with $V(G') = V(G)$. Now q a variable.

Deletion-contraction

$$Z_G(q,t) = Z_{G \smallsetminus e}(q,\hat{t}) + t_e Z_{G/e}(q,\hat{t})$$

$\hat{t}$ = edge variables with t_e removed
(includes case of bridges and looping edges)

The problem of phase transitions

Zeros of the partition function $\Rightarrow$ Phase transitions

• Ferromagnetic case: finite graphs have no physical phase transitions $t_e \geq 0$, only virtual phase transitions $t_e < 0$
• Antiferromagnetic case: $-1 \leq t_e \leq 0$, results on zero-free regions for certain graphs (Jackson–Sokal)

Families of graphs $G_\infty = \cup_n G_n$, ferromagnetic case, no phase transitions for fixed G_n, but in the limit?

Complex zeros of $Z_{G_n}(q, t)$ approaching points in the positive quadrant: *estimate how the locus of (complex/real) zeros of $Z_{G_n}(q, t)$ changes in a family G_n*

There is an extensive literature using analytic methods ...
... why a motivic approach?

A parallel story: Quantum Field Theory

Euclidean scalar field theory on a D-dimensional spacetime

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial\phi)^2 + \frac{m^2}{2}\phi^2 + \mathcal{L}_{int}(\phi)$$

with polynomial interaction term $\mathcal{L}_{int}(\phi)$: action functional

$$S(\phi) = \int \mathcal{L}(\phi) d^D x$$

Path integrals (expectation values of observables $\mathcal{O}(\phi)$)

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{O}(\phi)\, e^{\frac{i}{\hbar}S(\phi)}\, D[\phi]}{\int e^{\frac{i}{\hbar}S(\phi)}\, D[\phi]}$$

ill defined infinite dimensional integrals ... but computed by perturbative expansion in Feynman graphs

Feynman graphs and Feynman rules (Euclidean)

- Internal lines $\Rightarrow$ propagator = quadratic form q_i

$$\frac{1}{q_1 \cdots q_n}, \quad q_i(k_i) = k_i^2 + m^2$$

- Vertices: conservation (valences = monomials in $\mathcal{L}$)

$$\sum_{e_i \in E(G): s(e_i) = v} k_i = 0$$

- Integration over k_i, internal edges

$$U(G) = \int \frac{\delta(\sum_{i=1}^n \epsilon_{v,i} k_i + \sum_{j=1}^N \epsilon_{v,j} p_j)}{q_1 \cdots q_n} \, d^D k_1 \cdots d^D k_n$$

$n = \# E_{int}(G)$, $N = \# E_{ext}(G)$

$$\epsilon_{e,v} = \begin{cases} +1 & t(e) = v \\ -1 & s(e) = v \\ 0 & \text{otherwise,} \end{cases}$$

Parametric form of Feynman integrals (Schwinger parameters)

$$U(G) = \frac{\Gamma(n - D\ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_G(t,p)^{-n+D\ell/2}\omega_n}{\Psi_G(t)^{-n+D(\ell+1)/2}}$$

massless case: polynomial P_G (cut sets and external momenta), polynomial Ψ_G

$$\Psi_G(t) = \sum_T \prod_{e \notin T} t_e$$

sum over spanning trees (connected G)
integral over simplex $\sigma_n = \{t \in \mathbb{R}^n_+ | \sum_i t_i = 1\}$ with vol form ω_n

Observations

• Modulo regularization and renormalization, $U(G)$ is a period of the algebraic variety $\mathbb{A}^n \smallsetminus X_G$, complement of the hypersurface

$$X_G = \{t = (t_e) \in \mathbb{A}^n \,|\, \Psi_G(t) = 0\}$$

• The polynomial $\Psi_G(t)$ satisfies deletion–contraction

$$\Psi_G(t) = t_e \Psi_{G \smallsetminus e}(\hat{t}) + \Psi_{G/e}(\hat{t})$$

(e neither bridge nor looping edge)

• Related polynomial

$$\Phi_G(t) = \sum_T \prod_{e \in T} t_e$$

T = spanning trees (maximal spanning forests); Ψ_G obtained dividing by $\prod_{e \in E(G)} t_e$ and changing variables $t_e \mapsto 1/t_e$ (Cremona transformation)

Motivic complexity and the Grothendieck ring

∘ What kind of numbers are the residues of Feynman graphs?
periods of motives, depend on what kind of motives:
mixed Tate motives ⇒ multiple zeta values

∘ Estimate the "motivic complexity" through classes $[X_G]$ in the Grothendieck ring

$K_0(\mathcal{V})$ generated by isomorphism classes $[X]$ of smooth (quasi)projective varieties with relations

- $[X] = [Y] + [X \smallsetminus Y]$: inclusion-exclusion, $Y \subset X$ closed
- $[X \times Y] = [X][Y]$: product structure

∘ (Belkale–Brosnan):
$[X_G]$ generate localization of $K_0(\mathcal{V})$ at $\mathbb{L}^n - \mathbb{L}$

Deletion-contraction for $[X_G]$

$$[\mathbb{A}^n \smallsetminus X_G] = \mathbb{L}\,[\mathbb{A}^{n-1} \smallsetminus (X_{G\smallsetminus e} \cap X_{G/e})] - [\mathbb{A}^{n-1} \smallsetminus X_{G\smallsetminus e}]$$

e neither bridge nor looping edge;

$[\mathbb{A}^n \smallsetminus X_G] = \mathbb{L}\,[\mathbb{A}^{n-1} \smallsetminus X_{G/e}] = \mathbb{L}\,[\mathbb{A}^{n-1} \smallsetminus X_{G\smallsetminus e}]$ for bridges;

$[\mathbb{A}^n \smallsetminus X_G] = (\mathbb{L}-1)[\mathbb{A}^{n-1} \smallsetminus X_{G/e}] = (\mathbb{L}-1)[\mathbb{A}^{n-1} \smallsetminus X_{G\smallsetminus e}]$ for looping edges

$\mathbb{L} = [\mathbb{A}^1]$ Lefschetz motive

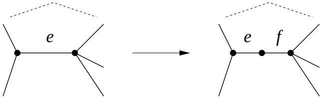
Note: algebro-geometric term $X_{G\smallsetminus e} \cap X_{G/e}$ difficult to control: can be motivically more complicated than $X_{G\smallsetminus e}$ and $X_{G/e}$

Some Consequences

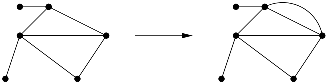
P. Aluffi, M. Marcolli, *Feynman motives and deletion-contraction relations*, arXiv:0907.3225

Some operations that enlarge the graph have a "controlled effect" on the Grothendieck class $[X_G]$

- splitting edges



- doubling edges



Obtain generating series for the classes $[X_{G_n}]$ in such families

Key: cancellations of "difficult term" in deletion-contraction in $K_0(\mathcal{V})$ in good cases
Notation: $\mathbb{U}(G) = [\mathbb{A}^n \smallsetminus X_G]$

- doubling edges

$$\mathbb{U}(G_{2e}) = (\mathbb{T} - 1)\mathbb{U}(G) + \mathbb{T}\,\mathbb{U}(G \smallsetminus e) + (\mathbb{T} + 1)\mathbb{U}(G/e)$$

neither bridge nor looping edge; $\mathbb{U}(G_{2e}) = \mathbb{T}^2\mathbb{U}(G \smallsetminus e)$ looping edge; $\mathbb{U}(G_{2e}) = \mathbb{T}(\mathbb{T} + 1)\mathbb{U}(G \smallsetminus e)$ bridge
$\mathbb{T} = \mathbb{L} - 1 = [\mathbb{G}_m]$ class of the multiplicative group

- splitting an edge $\Rightarrow$ multiply the class by $\mathbb{T} + 1$

Example: can control classes $[X_G]$ of G chains of polygons, in mixed Tate part $\mathbb{Z}[\mathbb{L}]$ of $K_0(\mathcal{V})$.

A lot more is now known for classes $[X_G]$ of graph hypersurfaces
Incomplete list of some recent results:

- P. Aluffi, M.M., *Algebro-geometric Feynman rules*, arXiv:0811.2514
- P. Aluffi, M.M., *Feynman motives and deletion-contraction relations*, arXiv:0907.3225
- O. Schnetz, *Quantum field theory over* $\mathbb{F}_q$, arXiv:0909.0905
- D. Doryn, *On one example and one counterexample in counting rational points on graph hypersurfaces*, arXiv:1006.3533
- F. Brown, O. Schnetz, *A K3 in* ϕ^4, arXiv:1006.4064
- P. Aluffi, *Chern classes of graph hypersurfaces and deletion-contraction*, arXiv:1106.1447
- F. Brown, D. Doryn, *Framings for graph hypersurfaces*, arXiv:1301.3056

Graph polynomials and Potts models

Φ_G is a limiting case of the Multivariable Tutte polynomial:

• Take $\mathcal{P}_G(q,t)$ the homogeneous polynomial leading term of $Z_G(q,t)$ (in $(q,t) \in \mathbb{A}^{n+1}$ variables)

• This is the contribution of subgraphs that are forests with $V(G') = V(G)$ (spanning)

$$\mathcal{P}_G(q,t) = \sum_{G' \subseteq G,\, b_1(G')=0,\, \#V(G')=N} q^{k(G')} \prod_{e \in E(G')} t_e$$

• The locus $\mathcal{P}_G(q,t) = 0$ is the *tangent cone* at zero of the affine hypersurface defined by $Z_G(q,t) = 0$

• $\mathcal{P}_G(q,t) = 0$ has a component $H = \{q = 0\}$ with multiplicity $b_0(G)$ and another component $\mathcal{Q}_G(q,t) = 0$ that intersects H in the locus $\Phi_G(t) = 0$

Extending to Potts models the motivic approach: Some goals

• Measure topological complexity of locus of real zeros of $Z_{G_n}(q,t)$ in terms of Hodge numbers (motivic): Petrovsky–Oleinik inequality

• Interpret some Gibbs averages (like local magnetization)

$$\langle \mathcal{O} \rangle = \frac{\sum_\sigma \mathcal{O}(\sigma) e^{-\beta H_\sigma}}{\sum_\sigma e^{-\beta H_\sigma}} = \frac{\sum_\sigma \mathcal{O}(\sigma) p(q,t,\sigma)}{Z_G(q,t)}$$

as periods of motives (when averaging over some sets parameters), control the behavior over family G_n of graphs

• Behavior of zeros of $Z_{G_n}(q,t)$, over families of graphs G_n with some "construction method"

Families of graphs: polygons, linked polygons, banana graphs, trees, chains of polygons

Difficulty: easily lose control of the algebro-geometric term in the deletion-contraction and recursion formula for more complicated graphs (lattices, zig-zag graphs)

Notation:

• $\mathcal{Z}_G$ hypersurface in $\mathbb{A}^{\#E(G)+1}$ defined by $Z_G(q,t) = 0$ (Potts model hypersurface)

• $[\mathcal{Z}_G]$ class in the Grothendieck ring

• $\{\mathcal{Z}_G\} = \mathbb{L}^{\#E(G)+1} - [\mathcal{Z}_G] = [\mathbb{A}^{\#E(G)+1} \smallsetminus \mathcal{Z}_G]$ class of the complement

Algebro-geometric deletion-contraction for Potts models:

$$\{\mathcal{Z}_G\} = \mathbb{L}\{\mathcal{Z}_{G/e} \cap \mathcal{Z}_{G \smallsetminus e}\} - \{\mathcal{Z}_{G/e}\}$$

(includes cases of bridges and looping edges)

By checking cases: from deletion-contraction of $Z_G(q,t)$

$$Z_G(q,t) = Z_{G\smallsetminus e}(q,\hat{t}) + t_e Z_{G/e}(q,\hat{t})$$

- If $Z_{G/e}(q,\hat{t}) \neq 0$ then $Z_G(q,t) \neq 0$ if $t_e \neq -Z_{G\smallsetminus e}(q,\hat{t})/Z_{G/e}(q,\hat{t})$: a $\mathbb{G}_m$ of t_e's gives class

$$(\mathbb{L}-1)\{\mathcal{Z}_{G/e}\}$$

- If $Z_{G/e}(q,\hat{t}) = 0$ then $Z_G(q,t) \neq 0$ means $Z_{G\smallsetminus e}(q,\hat{t}) \neq 0$: gives $\mathbb{A}^1$ of t_e's for each $(q,\hat{t})$ with $Z_{G/e}(q,\hat{t}) = 0$ and $Z_{G\smallsetminus e}(q,\hat{t}) \neq 0$, so class

$$\mathbb{L}\,[\mathcal{Z}_{G/e} \smallsetminus (\mathcal{Z}_{G/e} \cap \mathcal{Z}_{G\smallsetminus e})]$$

- Adding these

$$\{\mathcal{Z}_G\} = (\mathbb{L}-1)\{\mathcal{Z}_{G/e}\} + \mathbb{L}(\{\mathcal{Z}_{G/e} \cap \mathcal{Z}_{G\smallsetminus e}\} - \{\mathcal{Z}_{G/e}\})$$

$$= \mathbb{L}\{\mathcal{Z}_{G/e} \cap \mathcal{Z}_{G\smallsetminus e}\} - \{\mathcal{Z}_{G/e}\}$$

where by inclusion-exclusion
$[\mathcal{Z}_{G/e}] - [\mathcal{Z}_{G/e} \cap \mathcal{Z}_{G\smallsetminus e}] = \{\mathcal{Z}_{G/e} \cap \mathcal{Z}_{G\smallsetminus e}\} - \{\mathcal{Z}_{G/e}\}$

Simple properties of $\{\mathcal{Z}_G\}$

- G = single vertex: $\{\mathcal{Z}_G\} = \mathbb{L} - 1$
- G = single edge, one or two vertices: $\{\mathcal{Z}_G\} = (\mathbb{L} - 1)^2$
- $G' = G_1 \cup_v G_2$ (two graphs joined at a vertex)

and G'' disjoint union

$$Z_{G'} = \frac{1}{q} Z_{G_1} Z_{G_2} \Rightarrow \{\mathcal{Z}_{G'}\} = \{\mathcal{Z}_{G''}\}$$

but $\{\mathcal{Z}_{G''}\}$ not simply product because one variable q in common

- joining to graphs with an edge: $(\mathbb{L} - 1)\{\mathcal{Z}_{G''}\}$

Splitting edges: generating function

graph G with chose edge e: ${}^0G = G/e$, ${}^1G = G$, ${}^kG$ egde e replaced by chain of k edges

- First step: case of ${}^2G$

$$\{\mathcal{Z}_{{}^2G}\} = \mathbb{L}((\mathbb{L}-2)\{\mathcal{Z}_{G\smallsetminus e}\cap\mathcal{Z}_{G/e}\} + \{\mathcal{Z}_{G\smallsetminus e}\} + \{Y^e_G\}) - \{\mathcal{Z}_{{}^1G}\}$$

where Y^e_G ideal of $(1+t_e)Z'$ with Z'

$$\sum_{A\subseteq E(G)} q^{k(A)} \prod_{a\in A} t_a$$

sum on all subgraphs connecting the endpoints of e in some way other than e

- Description of $\{\mathcal{Z}_{G\smallsetminus e}\cap\mathcal{Z}_{G/e}\}$

$$\mathbb{L}\{\mathcal{Z}_{G\smallsetminus e}\cap\mathcal{Z}_{G/e}\} = \{\mathcal{Z}_{G/e}\} + \{\mathcal{Z}_G\} = \{\mathcal{Z}_{{}^0G}\} + \{Z_{{}^1G}\}$$

- This gives

$$\{\mathcal{Z}_{{}^2G}\} = (\mathbb{T}-2)\{\mathcal{Z}_{{}^1G}\} + (\mathbb{T}-1)\{\mathcal{Z}_{{}^0G}\} + (\mathbb{T}+1)(\{\mathcal{Z}_{G\smallsetminus e}\} + \{Y^e_G\})$$

- multiple splitting (e last added edge)

$$\{\mathcal{Z}_{m\,G\smallsetminus e}\} + \{Y^e_{m\,G}\} = \mathbb{T}^{m-1}(\{\mathcal{Z}_{G\smallsetminus e}\} + \{Y^e_G\})$$

- Then recursion relation controls Y^e_G: $m \geq 0$

$$\{\mathcal{Z}_{m+3\,G}\} = (2\mathbb{T}-2)\{\mathcal{Z}_{m+2\,G}\} - (\mathbb{T}^2 - 3\mathbb{T}+1)\{\mathcal{Z}_{m+1\,G}\} - \mathbb{T}(\mathbb{T}-1)\{\mathcal{Z}_{m\,G}\}$$

- Generating function

$$\sum_{m\geq 0}\{\mathcal{Z}_{m\,G}\}\frac{s^m}{m!} = \left(e^{(\mathbb{T}-1)s} - (\mathbb{T}-1)\cdot\frac{e^{\mathbb{T}s}-e^{-s}}{\mathbb{T}+1}\right)\{Z_{0\,G}\}$$
$$+\left((\mathbb{T}-1)\cdot\frac{e^{(\mathbb{T}-1)s}-e^{-s}}{\mathbb{T}} - (\mathbb{T}-2)\cdot\frac{e^{\mathbb{T}s}-e^{-s}}{\mathbb{T}+1}\right)\{Z_{1\,G}\}$$
$$+\left(-\frac{e^{(\mathbb{T}-1)s}-e^{-s}}{\mathbb{T}} + \frac{e^{\mathbb{T}s}-e^{-s}}{\mathbb{T}+1}\right)\{Z_{2\,G}\}$$

Doubling edges: generating function (dual to edge splitting)

$$Z_{G \smallsetminus e} + (t_e + t_f + t_e t_f) Z_{G/e} = Z_{G \smallsetminus e} + (u_e u_f - 1) Z_{G/e}$$

with $u_e = 1 + t_e$, $u_f = 1 + t_f$

- If $Z_{G/e} = 0$, then $Z_{G \smallsetminus e} \neq 0$ (u_e and u_f free):

$$(\mathbb{T} + 1)^2 \cdot [\mathcal{Z}_{G/e} \smallsetminus (\mathcal{Z}_{G \smallsetminus e} \cap \mathcal{Z}_{G/e})]$$

- $Z_{G/e} \neq 0$ then $u_1 u_2 \neq 1 - \frac{Z_{G \smallsetminus e}}{Z_{G/e}}$ two possibilities:

1) $\frac{Z_{G \smallsetminus e}}{Z_{G/e}} = 1$ (then $u_1 u_2 \neq 0$): $\mathbb{L}^2 - 2\mathbb{L} + 1 = \mathbb{T}^2$

2) $\frac{Z_{G \smallsetminus e}}{Z_{G/e}} \neq 1$ (then $u_1 u_2 \neq c$ for some $c \neq 0$) For $c \neq 0$: $u_2 \neq 0$, $u_1 = c/u_2 \Rightarrow \mathbb{L} - 1$, then class of $u_1 u_2 \neq c$ is $\mathbb{L}^2 - \mathbb{L} + 1 = \mathbb{T}^2 + \mathbb{T} + 1$

So doubling an edge gives for class of the complement

$$(\mathbb{T}+1)^2\cdot[\mathcal{Z}_{G/e}\smallsetminus(\mathcal{Z}_{G\smallsetminus e}\cap\mathcal{Z}_{G/e})]+\mathbb{T}^2[(\mathbb{A}^{|E|}\smallsetminus\mathcal{Z}_{G/e})\cap(\mathcal{Z}_{G\smallsetminus e}=\mathcal{Z}_{G/e})]$$
$$+(\mathbb{T}^2+\mathbb{T}+1)[(\mathbb{A}^{|E|}\smallsetminus\mathcal{Z}_{G/e})\smallsetminus(\mathcal{Z}_{G\smallsetminus e}=\mathcal{Z}_{G/e})]$$

which simplifies to

$$\mathbb{T}\cdot\{\mathcal{Z}_G\}-(\mathbb{T}+1)\cdot\{\mathcal{Z}_{G\smallsetminus e}=\mathcal{Z}_{G/e}\}$$

So need class of complement of $Z_{G\smallsetminus e}-Z_{G/e}=0$
G' doubling edge e in G:

$$\{\mathcal{Z}_{G'}\}=\mathbb{T}\cdot\{\mathcal{Z}_G\}+(\mathbb{T}+1)\cdot\{W_G^e\}$$

with W_G^e summing over subgraphs of G/e which acquire an additional connected component in $G\smallsetminus e$

Multiple parallel edges

$G^{(m)}$ with m edges parallel to e in G

$$\{\mathcal{Z}_{G^{(m+2)}}\} = (2\mathbb{T}+1)\{\mathcal{Z}_{G^{(m+1)}}\} - \mathbb{T}(\mathbb{T}+1)\{\mathcal{Z}_{G^{(m)}}\}$$

using $\{W^e_{G'}\} = (\mathbb{T}+1)\{W^e_G\} = \{\mathcal{Z}_{G'}\} - \mathbb{T}\{\mathcal{Z}_G\}$

- Generating function:

$$\sum_{m\geq 0}\{\mathcal{Z}_{G^{(m)}}\}\frac{s^m}{m!} = ((\mathbb{T}+1)\{\mathcal{Z}_G\} - \{\mathcal{Z}_{G'}\})\, e^{\mathbb{T}s} + (\{\mathcal{Z}_{G'}\} - \mathbb{T}\{\mathcal{Z}_G\})\, e^{(\mathbb{T}+1)s}$$

Simple examples of applications:

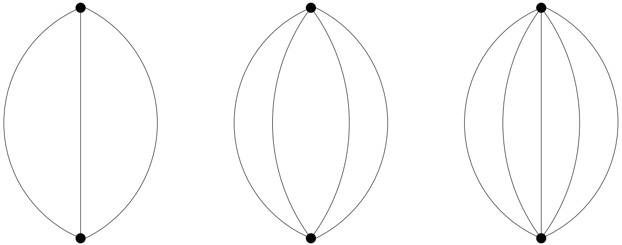
- Polygons

$G_m =$ polygon with $m+1$ sides

$$\{\mathcal{Z}_{G_m}\} = \mathbb{T}^{m+2} + \mathbb{T}(\mathbb{T}-1)(\mathbb{T}^m - (\mathbb{T}-1)^m) + (\mathbb{T}-1)\frac{(\mathbb{T}-1)^m - (-1)^m}{\mathbb{T}}$$

from the edge splitting recursion and generating function

- Banana graphs



$G^{(m)}$ = banana graph with $m+1$ edges

$$\{\mathcal{Z}_{G^{(m)}}\} = \mathbb{T}^m + (\mathbb{T}-1)(\mathbb{T}+1)^{m+1}$$

from the multiple edges recursion and generating function

Note: so far q variable: will then need q fixed

Special values of q

• $q = 0$: $\mathcal{Z}_G$ has a component $H = \{q = 0\}$ with multiplicity $b_0(G)$; remaning component, at $q = 0$ is (dual of) graph hypersurface $\Phi_G(t) = 0$

• $q = 1$:

$$Z_G(1, t) = \prod_{e \in E(G)} (1 + t_e)$$

normal crossings divisors: coordinate hyperplanes in $\mathbb{A}^n$, complement $\mathbb{T}^n = [\mathbb{G}_m]^n$

General values of q

$$\{\mathcal{Z}_{G,q}\} = (\mathbb{T}+1)\{\mathcal{Z}_{G/e,q} \cap \mathcal{Z}_{G\smallsetminus e,q}\} - \{\mathcal{Z}_{G/e,q}\}$$

- Recursions for multiple edges and splitting edges same (change initial conditions)
- Examples: polygons ${}^mG$ and bananas $G^{(m)}$

$$\{Z_{{}^mG,q}\} = \mathbb{T}^{m+1} + \mathbb{T}(\mathbb{T}^m - (\mathbb{T}-1)^m) + \frac{(\mathbb{T}-1)^m - (-1)^m}{\mathbb{T}}$$

$$\{Z_{G^{(m)},q}\} = (\mathbb{T}+1)^{m+1} - \mathbb{T}^m$$

- Behaves like a fibration $\mathcal{Z}_{G,q}$ over q with special fibers at $q = 0, 1$
- ... but, not a locally trivial fibration (explicit examples in M.M., Jessica Su, *Arithmetic of Potts model hypersurfaces*, arXiv:1112.5667)

Thermodynamic averages and periods

$$\langle F \rangle = \frac{\sum_{A\subseteq E} q^{k(A)} F(t_A) \prod_{e\in A} t_e}{\sum_{A\subseteq E} q^{k(A)} \prod_{e\in A} t_e} = \frac{1}{Z_G(q,t)} \sum_{A\subseteq E} q^{k(A)} F(t_A) \prod_{e\in A} t_e$$

$F(t_A) = F(t)|_{t_e=0, \forall e\notin A}$ observables: polynomial functions of edge variables

$$\frac{1}{Vol(\Delta)} \int_\Delta \langle F \rangle \, dv = \frac{1}{Vol(\Delta)} \int_\Delta \frac{P_{G,F}(q,t)}{Z_G(q,t)} \, dv(t)$$

with $P_{G,F}(q,t) = \sum_{A\subseteq E} q^{k(A)} F(t_A) \prod_{e\in A} t_e$

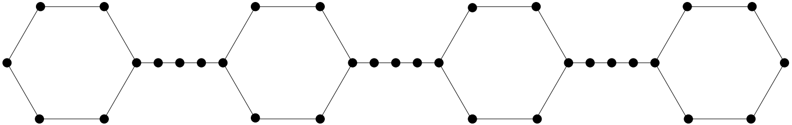
- The numbers

$$\int_\Delta \frac{P_{G,F}(q,t)}{Z_G(q,t)} \, dv(t)$$

are periods of motives: what kinds of periods?

Polygon polymer chains

${}^{(m,k)}G^N$ = joining N polygons, each $m+1$ sides by chains of $k \geq 0$ edges.

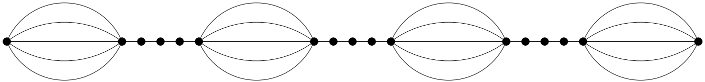


Class $\{\mathcal{Z}_{{}^{(m,k)}G^N,q}\}$ with $q \neq 0,1$:

$$\left(\mathbb{T}^{m+1} + \mathbb{T}(\mathbb{T}^m - (\mathbb{T}-1)^m) + \frac{(\mathbb{T}-1)^m - (-1)^m}{\mathbb{T}}\right)^N \mathbb{T}^{k(N-1)}$$

in mixed Tate part $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V})$

A similar case: chains of banana graphs



• ${}^kG^{(m),N}$ = connecting N banana graphs each with m parallel edges by a chain of $k \geq 0$ edges

$$\{\mathbb{Z}_{{}^kG^{(m),N},q}\} = ((\mathbb{T}+1)^{m+1} - \mathbb{T}^m)^N \mathbb{T}^{k(N-1)}$$

again in mixed Tate part $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V})$

Conclusion on thermodynamic averages:

• $[X] \in \mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V}) \Leftarrow X$ mixed Tate motive (conditionally $\Leftrightarrow$)

• (F.Brown) Periods of mixed Tate motives over $\mathbb{Z} \Leftrightarrow$ $\mathbb{Q}[(2\pi i)^{-1}]$-linear combinations of multiple zeta values

$$\zeta(n_1, \ldots, n_r) = \sum_{0<k_1<\ldots<k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}},$$

with integers $n_i \geq 1$ and $n_r \geq 2$

• Periods from thermodynamic averages are combinations of multiple zeta values for polygon chains and chains of banana graphs

Tetrahedral chains inosilicates: $\mathrm{Si}\,\mathrm{O}_3$ silicate tetrahedra
Tetrahedra in a single-chain configuration:

• Polynomial countability fails already for tetrahedron graph ($\Rightarrow$ not in $\mathbb{Z}[\mathbb{L}]$) (M.M., Jessica Su, *Arithmetic of Potts model hypersurfaces*, arXiv:1112.5667)

• Periods from thermodynamic averages can be more complicated for tetrahedral chains

Estimate topological complexity of set of virtual phase transitions

• Virtual phase transitions $\mathcal{Z}_G(\mathbb{R})$ real locus

• Physical phase transitions $\mathcal{Z}_G(\mathbb{R}) \cap \mathcal{I}$: ferromagnetic $\mathcal{I} = \{t_e \geq 0\}$, antiferromagnetic $\mathcal{I} = \{-1 \leq t_e \leq 0\}$

• Good indicators of "topological complexity": homology and cohomology, Euler characteristic

• Estimate how these behave over families of finite graphs growing to infinite graphs

• Estimates on the real locus from information on the complex geometry

Hodge numbers and the class in $K_0(\mathcal{V})$

- virtual Hodge polynomial

$$e(X)(x,y) = \sum_{p,q=0}^{d} e^{p,q}(X) x^p y^q$$

where

$$e^{p,q}(X) = \sum_{k=0}^{2d} (-1)^k h^{p,q}(H_c^k(X))$$

$h^{p,q}(H_c^k(X))$ = Hodge numbers of MHS on compact supp cohom

- ring homomorphism $e : K_0(\mathcal{V}) \to \mathbb{Z}[x,y]$
- so can read Hodge numbers of $\mathcal{Z}_G$ and $\mathbb{A}^{\#E(G)+1} \smallsetminus \mathcal{Z}_G$ from explicit formulae for $\{\mathcal{Z}_G\}$

Petrovsky–Oleinik inequalities

• original case: X complex smooth projective, $\dim X = 2p$, $X(\mathbb{R})$ real locus

$$|\chi(X(\mathbb{R})) - 1| \leq h^{p,p}(X) - 1$$

Hodge numbers control topology of real locus

• further cases with isolated singularities, $\dim X = 2p$

$$|\chi(X(\mathbb{R})) - 1| \leq \sum_{0 \leq q \leq p} h^{q,q}(H^n_0(X))$$

mixed Hodge structure on primitive cohomology

• more general cases: $X(\mathbb{R})$ algebraic set in $\mathbb{R}^n$ zeros of nonnegative polynomial even deg d: an estimate for $|\chi(X(\mathbb{R})) - 1|$ in terms of counting integral points in a polytope (related to Hodge numbers)

Other invariants of *real* algebraic varieties

• unique *motivic* invariant that agrees with topological Euler characteristic on compact smooth real algebraic varieties and homeomorphism invariant (not homotopy invariant)

$$\chi_c(S) = \sum_k (-1)^k b_k^{BM}(S)$$

S = semi-algebraic set; b_k^{BM} = Borel–Moore Betti numbers (equivalently, ranks of $H_c^*(S)$)

• motivic = factor through Grothendieck ring $K_0(\mathcal{V}_{\mathbb{R}})$

• Note: topological Euler characteristic $\chi(\mathbb{L}) = 1$ and $\chi(\mathbb{T}) = 0$ in $K_0(\mathcal{V}_{\mathbb{C}})$, but $\chi_c(\mathbb{L}) = -1$ and $\chi_c(\mathbb{T}) = -2$ in $K_0(\mathcal{V}_{\mathbb{R}})$

Virtual Betti numbers:

• *virtual Betti numbers*: $b_k(X) = \dim H_k(X, \mathbb{Z}/2\mathbb{Z})$ of smooth real alg varieties extend uniquely to $K_0(\mathcal{V}_\mathbb{R})$ as ring homomorphism

$$\beta : K_0(\mathcal{V}_\mathbb{R}) \to \mathbb{Z}[t]$$

so that for X smooth compact

$$\beta(X, t) = \sum_k b_k(X) t^k$$

and with $\beta(X, -1) = \chi_c(X)$

• $\beta_k(X) \neq b_k^{BM}(X)$ (can be negative) but alternating sum is $\chi_c(X)$

Complex case: virtual Betti numbers and virtual Hodge polynomials

- weight k Euler characteristic

$$w_j^k(X(\mathbb{C})) = \sum_{p+q=j} h^{p,q}(H_c^k(X(\mathbb{C})))$$

- virtual Betti numbers (McCrory–Parusiński)

$$\beta_j(X(\mathbb{C})) = (-1)^j \sum_k (-1)^k w_j^k(X(\mathbb{C})).$$

- ... but in general don't have good Petrovskiĭ–Oleĭnik type estimates for $\chi_c(X(\mathbb{R}))$ in real case
- ... but can get explicit information about $\chi_c(X(\mathbb{R}))$ from explicit knowledge of class $[X]$ in the Grothendieck ring

An estimate of algorithmic complexity

- Why interested in estimating $\chi_c(X(\mathbb{R}))$?
- $\chi_c(S)$ is a lower bound for the algorithmic complexity of the (semi)algebraic set S

$$C(S) \geq \frac{1}{3}(\log_3 \chi_c(S) - n - 4)$$

for a (semi)algebraic set $S \subset \mathbb{R}^n$

Potts model: polygon chains ${}^{(m,k)}G^N$

- Euler characteristic with compact support

$$\chi_c(\mathcal{Z}_{^{(m,k)}G^N,q}(\mathbb{R})) =$$

$$(-1)^{mN+kN-k}\left((-1)^N - 2^{kN-k-N}\left(3^{m+1}+1-2^{m+3}\right)^N\right)$$

- virtual Hodge polynomial

$$e(\mathcal{Z}_{^{(m,k)}G^N,q})(\mathbb{C})(x,y) =$$

$$(xy-1)^{k(N-1)}\left(2(xy-1)^{m+1} - \frac{(-1)^m + (xy-2)^{m+1}xy}{xy-1}\right)^N$$

Potts model: chains of banana graphs ${}^k G^{(m),N}$

- Euler characteristic with compact support

$$\chi_c(\mathcal{Z}_{{}^k G^{(m),N}}(\mathbb{R})) = (-1)^{mN+kN+N-k} \left(1 - 2^{k(N-1)} (2^m+1)^N\right)$$

- virtual Hodge polynomial

$$e(\mathcal{Z}_{{}^k G^{(m),N}}(\mathbb{C}))(x,y) = (xy-1)^{k(N-1)}(xy^{m+1} - (xy-1)^m)^N$$

Other algebro-geometric aspects of Potts models

Free energy of N-state chiral Potts model from the star-triangle relations: function of "rapidity variables" on a hyperelliptic curve of genus $N-1$ (rapidity curves):

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- R.J. Baxter, *Hyperelliptic function parametrization for the chiral Potts model*, Proceedings ICM (Kyoto, 1990), Springer 1991, pp. 1305–1317.
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- B. Davies, A. Neeman, *Algebraic geometry of the three-state chiral Potts model*, Israel J. Math. 125 (2001), 253–292.
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Questions and directions

- Algebraic geometry of Potts curves: motivic aspects?
- Potts models with magnetic field; arithmetic mutivariate Tutte polynomials?
- Partition function in terms of transfer matrix: motivic aspects?
- Poincaré residues, Leray coboundaries and location of zeros?