A motivic approach to Potts models

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Based on joint work with Paolo Aluffi

Some references:


+ other references listed later
Potts Models: Statistical Mechanics

\( G = \) finite graph
\( \mathcal{A} = \) set of possible spin states at a vertex, \( \# \mathcal{A} = q \)

- State: assignment of a spin state to each vertex of \( G \)
- Energy: sum over edges, zero if endpoint spins not aligned, \(-J_e\) if aligned (same) spins
- Edge variables: \( t_e = e^{\beta J_e} - 1 \), with \( \beta \) thermodynamic parameter (inverse temperature)
- Physical values: \( t_e \geq 0 \) ferromagnetic case \((J_e \geq 0)\) and \(-1 \leq t_e \leq 0\) antiferromagnetic case \(-\infty \leq J_e \leq 0\).

Partition function

\[
Z_G(q, t) = \sum_{\sigma : V(G) \to \mathcal{A}} \prod_{e \in E(G)} (1 + t_e \delta_{\sigma(v), \sigma(w)})
\]

sum over all maps of vertices to spin states, and \( \partial(e) = \{v, w\} \)
Example: 2D lattice, with $q = 4$, near critical temperature
Multivariable Tutte polynomial (Fortuin–Kasteleyn)

\[ Z_G(q, t) = \sum_{G' \subseteq G} q^{k(G')} \prod_{e \in E(G')} t_e \]

\( k(G') = b_0(G') \) connected components, sum over all subgraphs \( G' \subseteq G \) with \( V(G') = V(G) \). Now \( q \) a variable.

Deletion-contraction

\[ Z_G(q, t) = Z_{G \setminus e}(q, \hat{t}) + t_e Z_{G/e}(q, \hat{t}) \]

\( \hat{t} = \) edge variables with \( t_e \) removed
(includes case of bridges and looping edges)
The problem of phase transitions
Zeros of the partition function ⇒ Phase transitions

- Ferromagnetic case: finite graphs have no physical phase transitions $t_e \geq 0$, only virtual phase transitions $t_e < 0$
- Antiferromagnetic case: $-1 \leq t_e \leq 0$, results on zero-free regions for certain graphs (Jackson–Sokal)

Families of graphs $G_\infty = \bigcup_n G_n$, ferromagnetic case, no phase transitions for fixed $G_n$, but in the limit?

Complex zeros of $Z_{G_n}(q, t)$ approaching points in the positive quadrant: estimate how the locus of (complex/real) zeros of $Z_{G_n}(q, t)$ changes in a family $G_n$

There is an extensive literature using analytic methods ...

... why a motivic approach?
A parallel story: Quantum Field Theory
Euclidean scalar field theory on a $D$-dimensional spacetime

$$\mathcal{L}(\phi) = \frac{1}{2}(\partial \phi)^2 + \frac{m^2}{2} \phi^2 + \mathcal{L}_{\text{int}}(\phi)$$

with polynomial interaction term $\mathcal{L}_{\text{int}}(\phi)$: action functional

$$S(\phi) = \int \mathcal{L}(\phi) d^D x$$

Path integrals (expectation values of observables $\mathcal{O}(\phi)$)

$$\langle \mathcal{O} \rangle = \frac{\int \mathcal{O}(\phi) e^{\frac{i}{\hbar} S(\phi)} D[\phi]}{\int e^{\frac{i}{\hbar} S(\phi)} D[\phi]}$$

ill defined infinite dimensional integrals ... but computed by perturbative expansion in Feynman graphs
Feynman graphs and Feynman rules (Euclidean)

- Internal lines $\Rightarrow$ propagator = quadratic form $q_i$
  \[
  \frac{1}{q_1 \cdots q_n}, \quad q_i(k_i) = k_i^2 + m^2
  \]

- Vertices: conservation (valences = monomials in $\mathcal{L}$)
  \[
  \sum_{e_i \in E(G): s(e_i) = \nu} k_i = 0
  \]

- Integration over $k_i$, internal edges
  \[
  U(G) = \int \frac{\delta\left(\sum_{i=1}^{n} \epsilon_{v,i} k_i + \sum_{j=1}^{N} \epsilon_{v,j} p_j\right)}{q_1 \cdots q_n} \ d^D k_1 \cdots d^D k_n
  \]

$n = \#E_{int}(G), \ N = \#E_{ext}(G)$

$\epsilon_{e,v} = \begin{cases} 
  +1 & t(e) = \nu \\
  -1 & s(e) = \nu \\
  0 & \text{otherwise},
\end{cases}$
Parametric form of Feynman integrals (Schwinger parameters)

\[ U(G) = \frac{\Gamma(n - D \ell/2)}{(4\pi)^{\ell D/2}} \int_{\sigma_n} \frac{P_G(t, p)^{-n+D\ell/2} \omega_n}{\Psi_G(t)^{-n+D(\ell+1)/2}} \]

massless case: polynomial \( P_G \) (cut sets and external momenta), polynomial \( \Psi_G \)

\[ \Psi_G(t) = \sum_T \prod_{e \notin T} t_e \]

sum over spanning trees (connected \( G \))

integral over simplex \( \sigma_n = \{ t \in \mathbb{R}_+^n | \sum_i t_i = 1 \} \) with vol form \( \omega_n \)
Observations

- Modulo regularization and renormalization, $U(G)$ is a period of the algebraic variety $\mathbb{A}^n \setminus X_G$, complement of the hypersurface

$$X_G = \{ t = (t_e) \in \mathbb{A}^n \mid \Psi_G(t) = 0 \}$$

- The polynomial $\Psi_G(t)$ satisfies deletion–contraction

$$\Psi_G(t) = t_e \Psi_{G \setminus e}(\hat{t}) + \Psi_{G/e}(\hat{t})$$

(e neither bridge nor looping edge)

- Related polynomial

$$\Phi_G(t) = \sum_{T} \prod_{e \in T} t_e$$

$T =$ spanning trees (maximal spanning forests); $\Psi_G$ obtained dividing by $\prod_{e \in E(G)} t_e$ and changing variables $t_e \mapsto 1/t_e$ (Cremona transformation)
Motivic complexity and the Grothendieck ring

- What kind of numbers are the residues of Feynman graphs? Periods of motives, depend on what kind of motives: mixed Tate motives $\Rightarrow$ multiple zeta values

- Estimate the “motivic complexity” through classes $[X_G]$ in the Grothendieck ring $K_0(\mathcal{V})$ generated by isomorphism classes $[X]$ of smooth (quasi)projective varieties with relations
  - $[X] = [Y] + [X \setminus Y]$: inclusion-exclusion, $Y \subset X$ closed
  - $[X \times Y] = [X][Y]$: product structure

- (Belkale–Brosnan): $[X_G]$ generate localization of $K_0(\mathcal{V})$ at $\mathbb{L}^n - \mathbb{L}$
Deletion-contraction for $[X_G]$

$$[A^n \setminus X_G] = L [A^{n-1} \setminus (X_G \setminus e \cap X_G/e)] - [A^{n-1} \setminus X_G \setminus e]$$

e neither bridge nor looping edge;

$$[A^n \setminus X_G] = L [A^{n-1} \setminus X_G/e] = L [A^{n-1} \setminus X_G \setminus e]$$ for bridges;

$$[A^n \setminus X_G] = (L - 1)[A^{n-1} \setminus X_G/e] = (L - 1)[A^{n-1} \setminus X_G \setminus e]$$ for looping edges

$L = [A^1]$ Lefschetz motive

Note: algebro-geometric term $X_G \setminus e \cap X_G/e$ difficult to control: can be motivically more complicated than $X_G \setminus e$ and $X_G/e$
Some Consequences

P. Aluffi, M. Marcolli, *Feynman motives and deletion-contraction relations*, arXiv:0907.3225

Some operations that enlarge the graph have a “controlled effect” on the Grothendieck class $[X_G]$

- splitting edges

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- doubling edges

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Obtain generating series for the classes $[X_{G_n}]$ in such families
Key: cancellations of “difficult term” in deletion-contraction in $K_0(\mathcal{V})$ in good cases
Notation: $U(G) = [\mathbb{A}^n \setminus X_G]$

- doubling edges

$$U(G_{2e}) = (T - 1)U(G) + TU(G \setminus e) + (T + 1)U(G/e)$$

neither bridge nor looping edge; $U(G_{2e}) = T^2U(G \setminus e)$ looping edge; $U(G_{2e}) = T(T + 1)U(G \setminus e)$ bridge

$T = L - 1 = [G_m]$ class of the multiplicative group

- splitting an edge $\Rightarrow$ multiply the class by $T + 1$

Example: can control classes $[X_G]$ of $G$ chains of polygons, in mixed Tate part $\mathbb{Z}[L]$ of $K_0(\mathcal{V})$. 
A lot more is now known for classes $[X_G]$ of graph hypersurfaces

Incomplete list of some recent results:

- O. Schnetz, *Quantum field theory over $\mathbb{F}_q$*, arXiv:0909.0905
- D. Doryn, *On one example and one counterexample in counting rational points on graph hypersurfaces*, arXiv:1006.3533
Graph polynomials and Potts models

\( \Phi_G \) is a limiting case of the Multivariable Tutte polynomial:

- Take \( P_G(q, t) \) the homogeneous polynomial leading term of \( Z_G(q, t) \) (in \( (q, t) \in \mathbb{A}^{n+1} \) variables)

- This is the contribution of subgraphs that are forests with \( V(G') = V(G) \) (spanning)

\[
P_G(q, t) = \sum_{G' \subseteq G, b_1(G')=0, \#V(G')=N} q^{k(G')} \prod_{e \in E(G')} t_e
\]

- The locus \( P_G(q, t) = 0 \) is the tangent cone at zero of the affine hypersurface defined by \( Z_G(q, t) = 0 \)

- \( P_G(q, t) = 0 \) has a component \( H = \{ q = 0 \} \) with multiplicity \( b_0(G) \) and another component \( Q_G(q, t) = 0 \) that intersects \( H \) in the locus \( \Phi_G(t) = 0 \)
Extending to Potts models the motivic approach: Some goals

- Measure topological complexity of locus of real zeros of $Z_{G_n}(q, t)$ in terms of Hodge numbers (motivic): Petrovsky–Oleinik inequality
- Interpret some Gibbs averages (like local magnetization)

$$\langle O \rangle = \frac{\sum_\sigma O(\sigma) e^{-\beta H_\sigma}}{\sum\sigma e^{-\beta H_\sigma}} = \frac{\sum_\sigma O(\sigma) p(q, t, \sigma)}{Z_G(q, t)}$$

as periods of motives (when averaging over some sets parameters), control the behavior over family $G_n$ of graphs
- Behavior of zeros of $Z_{G_n}(q, t)$, over families of graphs $G_n$ with some “construction method”

Families of graphs: polygons, linked polygons, banana graphs, trees, chains of polygons

Difficulty: easily lose control of the algebro-geometric term in the deletion-contraction and recursion formula for more complicated graphs (lattices, zig-zag graphs)
Notation:
• \( Z_G \) hypersurface in \( \mathbb{A}^{\#E(G)+1} \) defined by \( Z_G(q, t) = 0 \) (Potts model hypersurface)
• \([Z_G]\) class in the Grothendieck ring
• \( \{Z_G\} = \mathbb{L}^{\#E(G)+1} - [Z_G] = [\mathbb{A}^{\#E(G)+1} \setminus Z_G] \) class of the complement

Algebro-geometric deletion-contraction for Potts models:

\[
\{Z_G\} = \mathbb{L}\{Z_G/e \cap Z_G\setminus e\} - \{Z_G/e\}
\]

(includes cases of bridges and looping edges)
By checking cases: from deletion-contraction of $Z_G(q, t)$

$$Z_G(q, t) = Z_G \setminus e(q, \hat{t}) + t_eZ_G/e(q, \hat{t})$$

- If $Z_{G/e}(q, \hat{t}) \neq 0$ then $Z_G(q, t) \neq 0$ if $t_e \neq -Z_G \setminus e(q, \hat{t}) / Z_{G/e}(q, \hat{t})$: a $\mathbb{G}_m$ of $t_e$’s gives class

$$\mathbb{L} - 1 \{Z_{G/e}\}$$

- If $Z_{G/e}(q, \hat{t}) = 0$ then $Z_G(q, t) \neq 0$ means $Z_G \setminus e(q, \hat{t}) \neq 0$: gives $\mathbb{A}^1$ of $t_e$’s for each $(q, \hat{t})$ with $Z_{G/e}(q, \hat{t}) = 0$ and $Z_G \setminus e(q, \hat{t}) \neq 0$, so class

$$\mathbb{L} \left[ Z_{G/e} \setminus (Z_{G/e} \cap Z_G \setminus e) \right]$$

- Adding these

$$\{Z_G\} = (\mathbb{L} - 1)\{Z_{G/e}\} + \mathbb{L}(\{Z_{G/e} \cap Z_G \setminus e\} - \{Z_{G/e}\})$$

$$= \mathbb{L}\{Z_{G/e} \cap Z_G \setminus e\} - \{Z_{G/e}\}$$

where by inclusion-exclusion

$$[Z_{G/e}] - [Z_{G/e} \cap Z_G \setminus e] = \{Z_{G/e} \cap Z_G \setminus e\} - \{Z_{G/e}\}$$
Simple properties of \( \{\mathcal{Z}_G\} \)

- \( G = \) single vertex: \( \{\mathcal{Z}_G\} = \mathbb{I} - 1 \)
- \( G = \) single edge, one or two vertices: \( \{\mathcal{Z}_G\} = (\mathbb{I} - 1)^2 \)
- \( G' = G_1 \cup_v G_2 \) (two graphs joined at a vertex) and \( G'' \) disjoint union

\[
Z_{G'} = \frac{1}{q} Z_{G_1} Z_{G_2} \Rightarrow \{\mathcal{Z}_{G'}\} = \{\mathcal{Z}_{G''}\}
\]

but \( \{\mathcal{Z}_{G''}\} \) not simply product because one variable \( q \) in common
- joining to graphs with an edge: \( (\mathbb{I} - 1)\{\mathcal{Z}_{G''}\} \)
Splitting edges: generating function graph $G$ with chose edge $e$: $^0G = G/e$, $^1G = G$, $^kG$ egde $e$ replaced by chain of $k$ edges

• First step: case of $^2G$

$$\{Z^2_G\} = \mathbb{L}((\mathbb{L} - 2)\{Z_G\backslash e \cap Z_G/e\} + \{Z_G\backslash e\} + \{Y^e_G\}) - \{Z^1_G\}$$

where $Y^e_G$ ideal of $(1 + t_e)Z'$ with $Z'$

$$\sum_{A \subseteq E(G)} q^{k(A)} \prod_{a \in A} t_a$$

sum on all subgraphs connecting the endpoints of $e$ in some way other than $e$

• Description of $\{Z_G\backslash e \cap Z_G/e\}$

$$\mathbb{L}\{Z_G\backslash e \cap Z_G/e\} = \{Z_G/e\} + \{Z_G\} = \{Z^0_G\} + \{Z^1_G\}$$

• This gives

$$\{Z^2_G\} = (\mathbb{T} - 2)\{Z^1_G\} + (\mathbb{T} - 1)\{Z^0_G\} + (\mathbb{T} + 1)(\{Z_G\backslash e\} + \{Y^e_G\})$$
• multiple splitting \((e\ last\ added\ edge)\)

\[
\{Z_{mG}e\} + \{Y_{mG}e\} = T^{m-1}(\{Z_{G}e\} + \{Y_Ge\})
\]

• Then recursion relation controls \(Y_G^e: m \geq 0\)

\[
\{Z_{m+3G}\} = (2T-2)\{Z_{m+2G}\} - (T^2 - 3T + 1)\{Z_{m+1G}\} - T(T-1)\{Z_{mG}\}
\]

• Generating function

\[
\sum_{m \geq 0} \{Z_{mG}\} \frac{s^m}{m!} = \left( e^{(T-1)s} - (T-1) \cdot \frac{e^{Ts} - e^{-s}}{T+1} \right) \{Z_0G\}
\]

\[
+ \left( (T-1) \cdot \frac{e^{(T-1)s} - e^{-s}}{T} - (T-2) \cdot \frac{e^{Ts} - e^{-s}}{T+1} \right) \{Z_1G\}
\]

\[
+ \left( -\frac{e^{(T-1)s} - e^{-s}}{T} + \frac{e^{Ts} - e^{-s}}{T+1} \right) \{Z_2G\}
\]
Doubling edges: generating function (dual to edge splitting)

\[ Z_{G\setminus e} + (t_e + t_f + t_e t_f)Z_{G/e} = Z_{G\setminus e} + (u_e u_f - 1)Z_{G/e} \]

with \( u_e = 1 + t_e \), \( u_f = 1 + t_f \)

• If \( Z_{G/e} = 0 \), then \( Z_{G\setminus e} \neq 0 \) (\( u_e \) and \( u_f \) free):

\[ (T + 1)^2 \cdot [Z_{G/e} \setminus (Z_{G\setminus e} \cap Z_{G/e})] \]

• \( Z_{G/e} \neq 0 \) then \( u_1 u_2 \neq 1 - \frac{Z_{G\setminus e}}{Z_{G/e}} \) two possibilities:

1) \( \frac{Z_{G\setminus e}}{Z_{G/e}} = 1 \) (then \( u_1 u_2 \neq 0 \)): \( L^2 - 2L + 1 = T^2 \)

2) \( \frac{Z_{G\setminus e}}{Z_{G/e}} \neq 1 \) (then \( u_1 u_2 \neq c \) for some \( c \neq 0 \)) For \( c \neq 0 \): \( u_2 \neq 0 \), \( u_1 = c / u_2 \Rightarrow L - 1 \), then class of \( u_1 u_2 \neq c \) is

\( L^2 - L + 1 = T^2 + T + 1 \)
So doubling an edge gives for class of the complement

\[(T+1)^2 \cdot [Z_{G/e} \setminus (Z_{G/e} \cap Z_{G/e})] + T^2 \cdot [(A^{|E| \setminus Z_{G/e}}) \cap (Z_{G/e} = Z_{G/e})] + (T^2 + T + 1) \cdot [(A^{|E| \setminus Z_{G/e}} \setminus (Z_{G/e} = Z_{G/e})]

which simplifies to

\[T \cdot \{Z_G\} - (T + 1) \cdot \{Z_{G/e} = Z_{G/e}\}\]

So need class of complement of \(Z_{G/e} \setminus Z_{G/e} = 0\)

\(G'\) doubling edge \(e\) in \(G\):

\[\{Z_{G'}\} = T \cdot \{Z_G\} + (T + 1) \cdot \{W^e_G\}\]

with \(W^e_G\) summing over subgraphs of \(G/e\) which acquire an additional connected component in \(G \setminus e\)
Multiple parallel edges
$G^{(m)}$ with $m$ edges parallel to $e$ in $G$

$$\{ \mathcal{Z}_{G^{(m+2)}} \} = (2T + 1)\{ \mathcal{Z}_{G^{(m+1)}} \} - T(T + 1)\{ \mathcal{Z}_{G^{(m)}} \}$$

using $\{ W^e_{G'} \} = (T + 1)\{ W^e_G \} = \{ \mathcal{Z}_{G'} \} - T\{ \mathcal{Z}_G \}$

• Generating function:

$$\sum_{m \geq 0} \{ \mathcal{Z}_{G^{(m)}} \} \frac{s^m}{m!} = ((T + 1)\{ \mathcal{Z}_{G} \} - \{ \mathcal{Z}_{G'} \}) e^{Ts}$$

$$+ (\{ \mathcal{Z}_{G'} \} - T\{ \mathcal{Z}_G \}) e^{(T+1)s}$$
Simple examples of applications:

- **Polygons**
  
  $G_m = \text{polygon with } m + 1 \text{ sides}$

  $$\{Z_{G_m}\} = T^{m+2} + T(T-1)(T^m - (T-1)^m) + (T-1)\frac{(T-1)^m - (-1)^m}{T}$$

  from the edge splitting recursion and generating function
Banana graphs

\[ G^{(m)} = \text{banana graph with } m + 1 \text{ edges} \]

\[ \{ \mathcal{Z}_{G^{(m)}} \} = \mathbb{T}^m + (\mathbb{T} - 1)(\mathbb{T} + 1)^{m+1} \]

from the multiple edges recursion and generating function
Note: so far $q$ variable: will then need $q$ fixed

**Special values of $q$**

- $q = 0$: $Z_G$ has a component $H = \{ q = 0 \}$ with multiplicity $b_0(G)$; remaining component, at $q = 0$ is (dual of) graph hypersurface $\Phi_G(t) = 0$
- $q = 1$:

$$Z_G(1, t) = \prod_{e \in E(G)} (1 + t_e)$$

normal crossings divisors: coordinate hyperplanes in $\mathbb{A}^n$, complement $\mathbb{T}^n = [G_m]^n$
General values of $q$

$$\{Z_{G,q}\} = (T + 1)\{Z_{G/e,q} \cap Z_{G\setminus e,q}\} - \{Z_{G/e,q}\}$$

- Recursions for multiple edges and splitting edges same (change initial conditions)
- Examples: polygons $mG$ and bananas $G^{(m)}$

$$\{Z_{mG,q}\} = T^{m+1} + T(T^m - (T - 1)^m) + \frac{(T - 1)^m - (-1)^m}{T}$$

$$\{Z_{G^{(m)},q}\} = (T + 1)^{m+1} - T^m$$

- Behaves like a fibration $Z_{G,q}$ over $q$
  with special fibers at $q = 0, 1$
- ... but, not a locally trivial fibration (explicit examples in M.M., Jessica Su, *Arithmetic of Potts model hypersurfaces*, arXiv:1112.5667)
Thermodynamic averages and periods

\[ \langle F \rangle = \frac{\sum_{A \subseteq E} q^{k(A)} F(t_A) \prod_{e \in A} t_e}{\sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} t_e} = \frac{1}{Z_G(q, t)} \sum_{A \subseteq E} q^{k(A)} F(t_A) \prod_{e \in A} t_e \]

\[ F(t_A) = F(t) \big|_{t_e = 0, \forall e \notin A} \text{ observables: polynomial functions of edge variables} \]

\[ \frac{1}{Vol(\Delta)} \int_{\Delta} \langle F \rangle \, dv = \frac{1}{Vol(\Delta)} \int_{\Delta} \frac{P_{G,F}(q, t)}{Z_G(q, t)} \, dv(t) \]

with \( P_{G,F}(q, t) = \sum_{A \subseteq E} q^{k(A)} F(t_A) \prod_{e \in A} t_e \)

- The numbers

\[ \int_{\Delta} \frac{P_{G,F}(q, t)}{Z_G(q, t)} \, dv(t) \]

are periods of motives: what kinds of periods?
Polygon polymer chains

\((m,k) G^N = \text{joining } N \text{ polygons, each } m + 1 \text{ sides by chains of } k \geq 0 \text{ edges.}\)

Class \(\{Z_{(m,k)} G^N, q\}\) with \(q \neq 0, 1:\)

\[
\left( T^{m+1} + T(T^m - (T - 1)^m) + \frac{(T - 1)^m - (-1)^m}{T} \right)^N T^k(N-1)
\]

in mixed Tate part \(\mathbb{Z}[[L]] \subset K_0(\mathcal{V})\)
A similar case: chains of banana graphs

\[ kG^{(m)}_{N} = \text{connecting } N \text{ banana graphs each with } m \text{ parallel edges by a chain of } k \geq 0 \text{ edges} \]

\[ \{ Z_{kG^{(m)}_{N,q}} \} = ((T + 1)^{m+1} - T^m)^N T^k(N-1) \]

again in mixed Tate part \( \mathbb{Z}[\mathbb{L}] \subset K_0(V) \)
Conclusion on thermodynamic averages:

- \([X] \in \mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V}) \iff X\) mixed Tate motive (conditionally \(\iff\))
- (F.Brown) Periods of mixed Tate motives over \(\mathbb{Z} \iff \mathbb{Q}[(2\pi i)^{-1}]\)-linear combinations of multiple zeta values

\[
\zeta(n_1, \ldots, n_r) = \sum_{0 < k_1 < \cdots < k_r} \frac{1}{k_1^{n_1} \cdots k_r^{n_r}},
\]

with integers \(n_i \geq 1\) and \(n_r \geq 2\)

- Periods from thermodynamic averages are combinations of multiple zeta values for polygon chains and chains of banana graphs
Tetrahedral chains inosilicates: SiO$_3$ silicate tetrahedra
Tetrahedra in a single-chain configuration:


- Periods from thermodynamic averages can be more complicated for tetrahedral chains
Estimate topological complexity of set of virtual phase transitions

- Virtual phase transitions \( \mathcal{Z}_G(\mathbb{R}) \) real locus
- Physical phase transitions \( \mathcal{Z}_G(\mathbb{R}) \cap \mathcal{I} \): ferromagnetic \( \mathcal{I} = \{ t_e \geq 0 \} \), antiferromagnetic \( \mathcal{I} = \{ -1 \leq t_e \leq 0 \} \)

- Good indicators of “topological complexity”: homology and cohomology, Euler characteristic
- Estimate how these behave over families of finite graphs growing to infinite graphs
- Estimates on the real locus from information on the complex geometry
Hodge numbers and the class in $K_0(\mathcal{V})$

- virtual Hodge polynomial

$$e(X)(x, y) = \sum_{p,q=0}^d e^{p,q}(X)x^p y^q$$

where

$$e^{p,q}(X) = \sum_{k=0}^{2d} (-1)^k h^{p,q}(H^k_c(X))$$

$h^{p,q}(H^k_c(X))$ = Hodge numbers of MHS on compact supp cohom

- ring homomorphism $e : K_0(\mathcal{V}) \to \mathbb{Z}[x, y]$

- so can read Hodge numbers of $\mathcal{Z}_G$ and $\mathbb{A}^{#E(G)+1} \setminus \mathcal{Z}_G$ from explicit formulae for $\{\mathcal{Z}_G\}$
Petrovsky–Oleinik inequalities

• original case: $X$ complex smooth projective, $\dim X = 2p$, $X(\mathbb{R})$ real locus
  \[ |\chi(X(\mathbb{R})) - 1| \leq h^{p,p}(X) - 1 \]
  Hodge numbers control topology of real locus

• further cases with isolated singularities, $\dim X = 2p$
  \[ |\chi(X(\mathbb{R})) - 1| \leq \sum_{0 \leq q \leq p} h^{q,q}(H^n_0(X)) \]
  mixed Hodge structure on primitive cohomology

• more general cases: $X(\mathbb{R})$ algebraic set in $\mathbb{R}^n$ zeros of nonnegative polynomial even deg $d$: an estimate for $|\chi(X(\mathbb{R})) - 1|$ in terms of counting integral points in a polytope (related to Hodge numbers)
Other invariants of real algebraic varieties

- unique motivic invariant that agrees with topological Euler characteristic on compact smooth real algebraic varieties and homeomorphism invariant (not homotopy invariant)

\[ \chi_c(S) = \sum_k (-1)^k b_k^{BM}(S) \]

\( S = \) semi-algebraic set; \( b_k^{BM} = \) Borel–Moore Betti numbers (equivalently, ranks of \( H^*_c(S) \))

- motivic = factor through Grothendieck ring \( K_0(\mathcal{V}_\mathbb{R}) \)

- Note: topological Euler characteristic \( \chi(\mathbb{L}) = 1 \) and \( \chi(\mathbb{T}) = 0 \) in \( K_0(\mathcal{V}_\mathbb{C}) \), but \( \chi_c(\mathbb{L}) = -1 \) and \( \chi_c(\mathbb{T}) = -2 \) in \( K_0(\mathcal{V}_\mathbb{R}) \)
Virtual Betti numbers:

- **virtual Betti numbers**: \( b_k(X) = \dim H_k(X, \mathbb{Z}/2\mathbb{Z}) \) of smooth real alg varieties extend uniquely to \( K_0(\mathcal{V}_\mathbb{R}) \) as ring homomorphism

\[
\beta : K_0(\mathcal{V}_\mathbb{R}) \to \mathbb{Z}[t]
\]

so that for \( X \) smooth compact

\[
\beta(X, t) = \sum_k b_k(X) t^k
\]

and with \( \beta(X, -1) = \chi_c(X) \)

- \( \beta_k(X) \neq b_k^{BM}(X) \) (can be negative) but alternating sum is \( \chi_c(X) \)
Complex case: virtual Betti numbers and virtual Hodge polynomials

• weight $k$ Euler characteristic

$$w_j^k(X(\mathbb{C})) = \sum_{p+q=j} h^{p,q}(H_c^k(X(\mathbb{C})))$$

• virtual Betti numbers (McCrory–Parusiński)

$$\beta_j(X(\mathbb{C})) = (-1)^j \sum_k (-1)^k w_j^k(X(\mathbb{C})).$$

• ... but in general don’t have good Petrovskii–Oleinik type estimates for $\chi_c(X(\mathbb{R}))$ in real case

• ... but can get explicit information about $\chi_c(X(\mathbb{R}))$ from explicit knowledge of class $[X]$ in the Grothendieck ring
An estimate of algorithmic complexity

- Why interested in estimating $\chi_c(X(\mathbb{R}))$?
- $\chi_c(S)$ is a lower bound for the algorithmic complexity of the (semi)algebraic set $S$

$$C(S) \geq \frac{1}{3} (\log_3 \chi_c(S) - n - 4)$$

for a (semi)algebraic set $S \subset \mathbb{R}^n$
Potts model: polygon chains \((m,k)^{G^N}\)

- Euler characteristic with compact support
  \[
  \chi_c(Z_{(m,k)}G^N,q(\mathbb{R})) = (-1)^{mN+kN-k} \left((-1)^N - 2^{kN-k-N} (3^{m+1} + 1 - 2^{m+3})^N \right)
  \]

- Virtual Hodge polynomial
  \[
  e(Z_{(m,k)}G^N,q)(\mathbb{C})(x, y) = (xy - 1)^k(N-1) \left(2(xy - 1)^{m+1} - \frac{(-1)^m + (xy - 2)^{m+1}xy}{xy - 1} \right)^N
  \]
Potts model: chains of banana graphs $^k G^{(m), N}$

- Euler characteristic with compact support

$$\chi_c(\mathcal{Z}_{k G^{(m), N}(\mathbb{R})}) = (-1)^{mN+kN+N-k} \left(1 - 2^k(N-1)(2^m + 1)^N\right)$$

- virtual Hodge polynomial

$$e(\mathcal{Z}_{k G^{(m), N}(\mathbb{C})})(x, y) = (xy - 1)^{k(N-1)}(xy^{m+1} - (xy - 1)^m)^N$$
Other algebro-geometric aspects of Potts models
Free energy of $N$-state chiral Potts model from the star-triangle relations: function of “rapidity variables” on a hyperelliptic curve of genus $N - 1$ (rapidity curves):

- V.B. Matveev, A.O. Smirnov, *Star-triangle equations and some properties of algebraic curves that are connected with the integrable chiral Potts model*, Mat. Zametki 46 (1989), no. 3, 31–39, 126
Questions and directions

- Algebraic geometry of Potts curves: motivic aspects?
- Potts models with magnetic field; arithmetic multivariate Tutte polynomials?
- Partition function in terms of transfer matrix: motivic aspects?
- Poincaré residues, Leray coboundaries and location of zeros?