Spectral Triples and Pati–Salam GUT

Matilde Marcolli

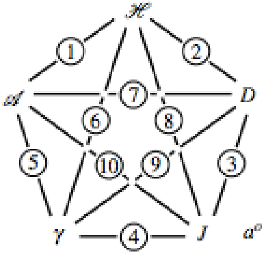
Ma148b Spring 2016
Topics in Mathematical Physics

References

- A.H. Chamseddine, A. Connes, W. van Suijlekom, *Beyond the Spectral Standard Model: Emergence of Pati-Salam Unification*, JHEP 1311 (2013) 132
- A.H. Chamseddine, A. Connes, W. van Suijlekom, *Grand Unification in the Spectral Pati-Salam Model*, arXiv:1507.08161

Real even spectral triples

1. $\mathscr{A} : \mathscr{H} \to \mathscr{H}$
2. $D^* = D$, comp. res.
3. $JD = \varepsilon' DJ$
4. $J\gamma = \varepsilon'' \gamma J$
5. $[\gamma, \mathscr{A}] = 0$



6. $\gamma^* = \gamma, \gamma^2 = \mathrm{id}_{\mathscr{H}}$
7. $[a, D] \in \mathscr{B}(\mathscr{H})$
8. $J^2 = \varepsilon\, \mathrm{id}_{\mathscr{H}}$
9. $\gamma D = -D\gamma$
10. $a^o = Ja^*J^*, [a, b^o] = 0$

All previously discussed NCG particle physics models based on datum of a real even spectral triple

$$(\mathcal{A}, \mathcal{H}, D, J, \gamma)$$

possibly with additional R-symmetry for SUSY case

Order one condition

• In particular very strong constraints on possible Dirac operators come from imposing the order one condition

$$[[D, a], b^0] = 0$$

with $b^0 = J b^* J^{-1}$

• natural question: what kind of models arise without the order one condition?

• NCG particle physics models without order one condition give GUT models: Pati–Salam

• there are other significant examples of noncommutative spaces without order one condition: quantum groups like $SU_q(2)$

Order one condition as a symmetry breaking mechanism

- A.H. Chamseddine, A. Connes, M. Marcolli, *Gravity and the Standard Model with Neutrino Mixing*, Adv. Theor. Math. Phys., Vol.11 (2007) 991–1090

• shown that imposing $[[D, a], b^0] = 0$ breaks down the L/R symmetry of the algebra $\mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$ to the SM algebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

- A.Chamseddine, A.Connes, *Why the Standard Model*, J.Geom.Phys. 58 (2008) 38–47

• shown that same argument applies with initial choice of algebra

$$\mathbb{H}_L \oplus \mathbb{H}_R \oplus M_4(\mathbb{C})$$

order one condition breaks it down to $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$

Inner fluctuations without order one condition

- usual argument for conjugation of fluctuated Dirac operator D_A by unitary $U = hJuJ^{-1}$ as gauge transformation

$$A \mapsto A^u = u[D, u^*] + uAu^*$$

only works if $[JuJ^{-1}, A] = 0$ for $A = \sum_j a_j[D, b_j]$ which requires order one condition

- without order one condition: general form of inner fluctuations

$$D' = D + A_{(1)} + \tilde{A}_{(1)} + A_{(2)}$$

$$A_{(1)} = \sum_j a_j[D, b_j]$$

$$\tilde{A}_{(1)} = \sum_j \hat{a}_j[D, \hat{b}_j], \quad \hat{a}_j = Ja_jJ^{-1}, \quad \hat{b}_j = Jb_jJ^{-1}$$

$$A_{(2)} = \sum_j \hat{a}_j[A_{(1)}, \hat{b}_j] = \sum_{j,k} \hat{a}_j a_k[[D, b_k], \hat{b}_j]$$

Semigroup of inner perturbations

$$\mathrm{Pert}(\mathcal{A}) = \{\sum_j a_j \otimes b_j^{op} \in \mathcal{A}\otimes\mathcal{A}^{op} \,:\, \sum_j a_j b_j = 1,\ \sum_j a_j \otimes b_j^{op} = \sum_j b_j^* \otimes a_j^{op}\}$$

• acting on Dirac operator D by

$$\sum_j a_j \otimes b_j^{op} : D \mapsto \sum_j a_j D b_j$$

• semigroup structure implies that inner fluctuations of inner fluctuations are still inner fluctuations, even without order one condition

Finite spectral triple without order one

• still assume order zero condition $[a, b^0] = 0$ (bimodule)

• still assume KO-dimension $= 6$ (as for SM)

• these two requirements imply that center of the complexified algebra

$$Z(\mathcal{A}_{\mathbb{C}}) = \mathbb{C} \oplus \mathbb{C}$$

• dimension of the Hilbert space square of an integer $\Rightarrow$ $\mathcal{A}_{\mathbb{C}} = M_k(\mathbb{C}) \oplus M_k(\mathbb{C})$

• imposing a symplectic symmetry on first algebra gives $k = 2a$ and algebra $M_a(\mathbb{H})$

• chirality operator on $M_a(\mathbb{H})$ requires a even

• realistic physical assumption $k = 4$

$$\mathcal{A} = \mathbb{H}_R \oplus \mathbb{H}_L \oplus M_4(\mathbb{C})$$

Gauge group

• inner automorphisms of algebra $\mathcal{A}$: Pati–Salam type left-right model

$$SU(2)_R \times SU(2)_L \times SU(4)$$

• $SU(4)$ color group has lepton number as 4-th color

Fermions

• dimension of finite Hilbert space

$$384 = 2^7 \times 3$$

3 for generations, $\mathbf{2}_L$ and $\mathbf{2}_R$ for $SU(2)_L$ and $SU(2)_R$ and $16 = 1 + 15$ for $SU(4)$, with further doubling for matter/antimatter

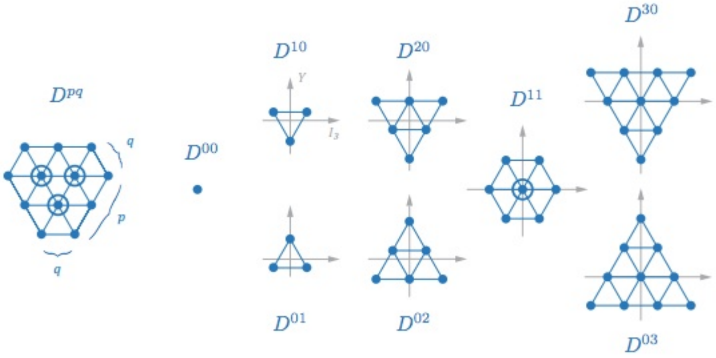
$SU(N)$ representations

• $SU(N)$: Lie algebra $N^2 - 1$ dimensional, basis t_a; there are $N - 1$ Casimir operators (center of the universal enveloping algebra) that label irreducible representations

• $SU(2)$: three dim Lie, σ_a (Pauli matrices), one Casimir operator $J^2 = \sigma_a \sigma_a$, eigenvalues $j(j+1)$ with $j \in \frac{1}{2}\mathbb{Z}$, irreducible representations labelled by $p = 2j \in \mathbb{Z}_+$, dimension $D^p = p + 1$ (angular momentum)

• $SU(3)$: eight dim Lie, λ_a (Gell-Mann matrices), two Casimir operators $\lambda_a\lambda_a$ and $f_{abc}\lambda_a\lambda_b\lambda_c$: irreducible representations labelled by these with two quantum numbers $p, q \in \mathbb{Z}_+$, dimensions

$$D^{pq} = \frac{1}{2}(p+1)(q+1)(p+q+2)$$



• $SU(4)$: three Casimir operators and irreducible representations parameterized by three quantum numbers $p, q, r \in \mathbb{Z}_+$, dimensions

$$D^{pqr} = \frac{1}{12}(p+1)(q+1)(p+q+2)(q+r+2)(p+q+r+3)$$

$D^{000} = 1$, $D^{100} = D^{001} = 4$, $D^{010} = 8$, $D^{200} = D^{002} = 10$, $D^{101} = 15$

Higgs fields

• assume that the unperturbed Dirac operator D satisfies order one condition when restricted to the SM subalgebra $\mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$ (consistency with SM)

• then inner fluctuations in the vertical (NC) direction coming from terms $A_{(2)}$ are composite, quadratic in those arising in the terms $A_{(1)}$

• then (with order one on SM algebra) Higgs in representations: $(2_R, 2_L, 1)$, $(2_R, 1_L, 4)$ and $(1_R, 1_L, 1 + 15)$ (Marshak–Mohapatra model)

• otherwise would have additional fundamental Higgs fields

Spectral action computation

- Dirac operator D of the finite spectral triple is a 384×384 matrix (written in an explicit tensor notation)
- Inner fluctuations computed (with order one on SM subalgebra)
- product geometry $M \times F$ of 4-dim spacetime and finite geometry
- Spectral action $\mathrm{Tr}(f(D_A/\Lambda))$

$$\mathrm{Tr}(f(D_A/\Lambda)) = \sum_{n=0}^{\infty} F_{4-n} \Lambda^{4-n} a_n$$

- a_n Seeley deWitt coefficients of heat kernel
- $F_k(u) = f(v)$ with $u = v^2$: momenta $F_4 = 2f_4$, $F_2 = 2f_2$

$$F_4 = \int_0^\infty F(u) u du = 2 \int_0^\infty f(v) v^3 dv, \;\; F_2 = \int_0^\infty F(u) du = 2 \int_0^\infty f(v) v dv$$

and $F_0 = F(0) = f_0$ with remaining terms

$$F_{-2n} = (-1)^n F^{(n)}(0) = (-1)^n \left(\frac{1}{2v} \frac{d}{dv} \right)^n f|_{v=0}$$

Seeley-deWitt coefficients

- a_0 coefficient (volume, cosmological term)

$$
\begin{aligned}
a_0 &= \frac{1}{16\pi^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\,(1) \\
&= \frac{1}{16\pi^2}\,(4)\,(32)\,(3)\int d^4x\sqrt{g} \\
&= \frac{24}{\pi^2}\int d^4x\sqrt{g}
\end{aligned}
$$

- a_2 coefficient: Einstein–Hilbert and Higgs terms

$$
a_2 = \frac{1}{16\pi^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\left(E+\frac{1}{6}R\right)
$$

$$
\begin{aligned}
a_2 &= \frac{1}{16\pi^2}\int d^4x\sqrt{g}\left((R(-96+64)-8\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I}+2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right)\right) \\
&= -\frac{2}{\pi^2}\int d^4x\sqrt{g}\left(R+\frac{1}{4}\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I}+2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right)\right).
\end{aligned}
$$

- a_4 coefficient: modified gravity terms, Yang–Mills terms, and Higgs terms

$$a_4 = \frac{1}{16\pi^2}\int d^4x\sqrt{g}\,\mathrm{Tr}\left(\frac{1}{360}\left(5R^2 - 2R_{\mu\nu}^2 + 2R_{\mu\nu\rho\sigma}^2\right)1 + \frac{1}{2}\left(E^2 + \frac{1}{3}RE + \frac{1}{6}\Omega_{\mu\nu}^2\right)\right)$$

$$\begin{aligned} a_4 = \frac{1}{2\pi^2}\int d^4x\sqrt{g}\,\Big[& -\frac{3}{5}C_{\mu\nu\rho\sigma}^2 + \frac{11}{30}R^*R^* + g_L^2\left(W_{\mu\nu L}^{\alpha}\right)^2 + g_R^2\left(W_{\mu\nu R}^{\alpha}\right)^2 + g^2\left(V_{\mu\nu}^{m}\right)^2 \\ & + \nabla_\mu \Sigma_{aI}^{\dot{c}K}\nabla^\mu \Sigma_{\dot{c}K}^{aI} + \frac{1}{2}\nabla_\mu H_{\dot{a}I\dot{b}J}\nabla^\mu H^{\dot{a}I\dot{b}J} + \frac{1}{12}R\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I} + 2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right) \\ & + \frac{1}{2}\left|H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{b}J}\right|^2 + 2H_{\dot{a}I\dot{c}K}\Sigma_{bJ}^{\dot{c}K}H^{\dot{a}I\dot{d}L}\Sigma_{\dot{d}L}^{bJ} + \Sigma_{aI}^{\dot{c}K}\Sigma_{\dot{c}K}^{bJ}\Sigma_{bJ}^{\dot{d}L}\Sigma_{\dot{d}L}^{aI}\Big] \end{aligned}$$

Higgs potential

$$V = \frac{F_0}{2\pi^2}\left(\frac{1}{2}\left|H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{b}J}\right|^2 + 2H_{\dot{a}I\dot{c}K}\Sigma_{bJ}^{\dot{c}K}H^{\dot{a}I\dot{d}L}\Sigma_{\dot{d}L}^{bJ} + \Sigma_{aI}^{\dot{c}K}\Sigma_{\dot{c}K}^{bJ}\Sigma_{bJ}^{\dot{d}L}\Sigma_{\dot{d}L}^{aI}\right)$$
$$-\frac{F_2}{2\pi^2}\left(H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{a}I} + 2\Sigma_{\dot{a}I}^{cK}\Sigma_{cK}^{\dot{a}I}\right).$$

- Quartic terms

$$\frac{1}{2}\left|H_{\dot{a}I\dot{c}K}H^{\dot{c}K\dot{b}J}\right|^2 = \frac{1}{2}|k^{\nu_R}|^4\left(\Delta_{\dot{a}K}\overline{\Delta}^{\dot{a}L}\Delta_{\dot{b}L}\overline{\Delta}^{\dot{b}K}\right)^2$$

$$\begin{aligned}\Sigma_{aI}^{\dot{c}K}\Sigma_{\dot{c}K}^{bJ}\Sigma_{bJ}^{\dot{d}L}\Sigma_{\dot{d}L}^{aI} = &\left(\left((k^{*\nu}-k^{*u})\,\phi_a^{\dot{c}} + (k^{*e}-k^{*d})\,\widetilde{\phi}_a^{\dot{c}}\right)\Sigma_I^K + \left(k^{*u}\phi_a^{\dot{c}} + k^{*d}\widetilde{\phi}_a^{\dot{c}}\right)\delta_I^K\right)\\
&\left(\left((k^{\nu}-k^{u})\,\phi_{\dot{c}}^{b} + (k^{e}-k^{d})\,\widetilde{\phi}_{\dot{c}}^{b}\right)\Sigma_K^J + \left(k^{u}\phi_{\dot{c}}^{b} + k^{d}\widetilde{\phi}_{\dot{c}}^{b}\right)\delta_K^J\right)\\
&\left(\left((k^{*\nu}-k^{*u})\,\phi_b^{\dot{d}} + (k^{*e}-k^{*d})\,\widetilde{\phi}_b^{\dot{d}}\right)\Sigma_J^L + \left(k^{*u}\phi_b^{\dot{d}} + k^{*d}\widetilde{\phi}_b^{\dot{d}}\right)\delta_J^L\right)\\
&\left(\left((k^{\nu}-k^{u})\,\phi_{\dot{d}}^{a} + (k^{e}-k^{d})\,\widetilde{\phi}_{\dot{d}}^{a}\right)\Sigma_L^I + \left(k^{u}\phi_{\dot{d}}^{a} + k^{d}\widetilde{\phi}_{\dot{d}}^{a}\right)\delta_L^I\right)\end{aligned}$$

$$\begin{aligned}2H_{\dot{a}I\dot{c}K}\Sigma_{bJ}^{\dot{c}K}H^{\dot{a}I\dot{d}L}\Sigma_{\dot{d}L}^{bJ} = 2\,|k^{\nu_R}|^2&\left(\Delta_{\dot{a}K}\overline{\Delta}^{\dot{a}L}\Delta_{\dot{c}I}\overline{\Delta}^{\dot{d}I}\right)\\
&\left(\left((k^{*\nu}-k^{*u})\,\phi_b^{\dot{c}} + (k^{*e}-k^{*d})\,\widetilde{\phi}_b^{\dot{c}}\right)\Sigma_J^K + \left(k^{*u}\phi_b^{\dot{c}} + k^{*d}\widetilde{\phi}_b^{\dot{c}}\right)\delta_J^K\right)\\
&\left(\left((k^{\nu}-k^{u})\,\phi_{\dot{d}}^{b} + (k^{e}-k^{d})\,\widetilde{\phi}_{\dot{d}}^{b}\right)\Sigma_L^J + \left(k^{u}\phi_{\dot{d}}^{b} + k^{d}\widetilde{\phi}_{\dot{d}}^{b}\right)\delta_L^J\right).\end{aligned}$$

- also mass terms

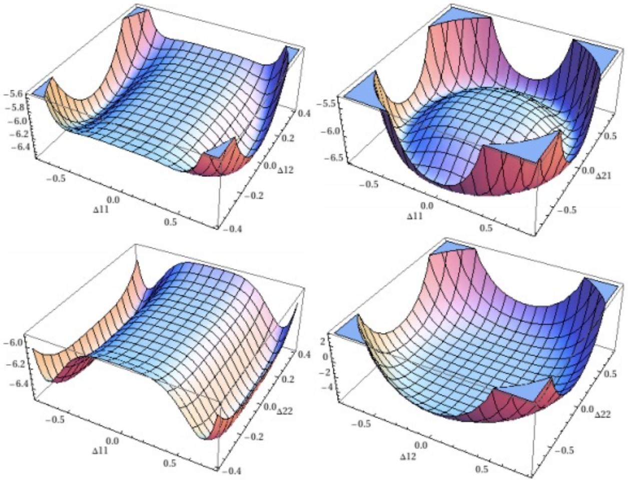


FIG. 1: The scalar potential in some of the $\Delta_{\dot{a}I}$-directions, with all other fields at their SM-vevs as in Equation (25). We have put $k^{\nu} = k^{e} = 1$ and $k^{\nu_R} = k^{u} = k^{d} = 2$. With these choices, the Standard Model vacuum corresponds to $\Delta_{\dot{1}1} = \frac{1}{\sqrt{2}}, \Sigma_{\dot{1}}^{1} = 2, \phi_{\dot{1}}^{1} = \frac{1}{2}$ and all other fields are zero. At this point the Hessian in the Δ-directions is nonnegative.

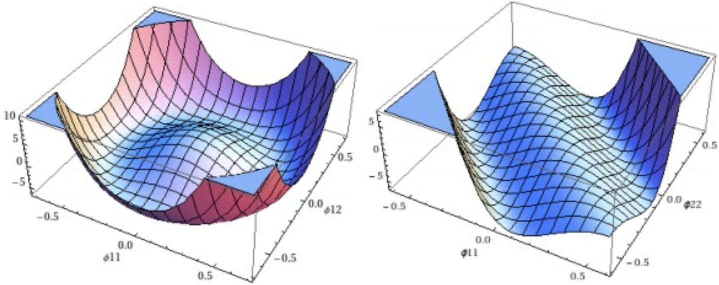


FIG. 2: The scalar potential in the ϕ_a^b-directions, after the Σ and Δ-fields have acquired their SM-vevs as in Equation (25). Again, we have put $k^\nu = k^e = 1$ and $k^{\nu_R} = k^u = k^d = 2$.