Noncommutative motives and their applications

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The classical theory of pure motives (Grothendieck)

• $\mathscr{V}_k$ category of smooth projective varieties over a field k; morphisms of varieties

• (Pure) Motives over k: linearization and idempotent completion (+ inverting the Lefschetz motive)

• Correspondences: $\mathrm{Corr}_{\sim,F}(X,Y)$: F-linear combinations of algebraic cycles $Z \subset X \times Y$ of codimension = dim X

• composition of correspondences:

$$\mathrm{Corr}(X,Y) \times \mathrm{Corr}(Y,Z) \to \mathrm{Corr}(X,Z)$$

$$(\pi_{X,Z})_*(\pi_{X,Y}^*(\alpha) \bullet \pi_{Y,Z}^*(\beta))$$

intersection product in $X \times Y \times Z$

• Equivalence relations on cycles: rational (or "algebraic"), homological, numerical

- $\alpha \sim_{rat} 0$ if $\exists\, \beta$ correspondence in $X \times \mathbb{P}^1$ with $\alpha = \beta(0) - \beta(\infty)$ (moving lemma; Chow groups; Chow motives)
- $\alpha \sim_{hom} 0$: vanishing under a chosen Weil cohomology functor H^*
- $\alpha \sim_{num} 0$: trivial intersection number with every other cycle

The category of motives has different properties depending on the choice of the equivalence relation on correspondences

Effective motives   Category $\mathrm{Mot}^{eff}_{\sim,F}(k)$:

• Objects: (X,p) smooth projective variety X and idempotent $p^2 = p$ in $\mathrm{Corr}_{\sim,F}(X,X)$

• Morphisms:

$$\mathrm{Hom}_{\mathrm{Mot}^{eff}_{\sim,F}(k)}((X,p),(Y,q)) = q\mathrm{Corr}_{\sim,F}(X,Y)p$$

• tensor structure $(X,p)\otimes(Y,q) = (X\times Y, p\times q)$

• notation $h(X)$ or $M(X)$ for the motive (X, id)

Tate motives

• $\mathbb{L}$ Lefschetz motive: $h(\mathbb{P}^1) = 1\oplus\mathbb{L}$ with $1 = h(Spec(k))$.

• formal inverse $\mathbb{L}^{-1}$ = Tate motive; notation $\mathbb{Q}(1)$

Motives Category $\mathrm{Mot}_\sim(k)$

• Objects: $(X,p,m) := (X,p)\otimes \mathbb{L}^{-m} = (X,p)\otimes \mathbb{Q}(m)$

• Morphisms:

$$\mathrm{Hom}_{\mathrm{Mot}_\sim(k)}((X,p,m),(Y,q,n)) = q\mathrm{Corr}^{m-n}_{\sim,F}(X,Y)p$$

shifts the codimension of cycles (Tate twist)

• Chow motives; homological motives; numerical motives

Jannsen's semi-simplicity result

Theorem (Jannsen 1991): TFAE

• $\mathrm{Mot}_{\sim,F}(k)$ is a semi-simple *abelian* category

• $\mathrm{Corr}^{\dim X}_{\sim,F}(X,X)$ is a finite-dimensional semi-simple F-algebra, for each X

• The equivalence relation is numerical: $\sim = \sim_{num}$

Weil cohomologies $H^* : \mathscr{V}_k^{op} \to VecGr_F$

• Künneth formula: $H^*(X \times Y) = H^*(X) \otimes H^*(Y)$
• dim $H^2(\mathbb{P}^1) = 1$; Tate twist: $V(r) = V \otimes H^2(\mathbb{P}^1)^{\otimes -r}$
• trace map (Poincaré duality) $tr : H^{2d}(X)(d) \to F$
• cycle map $\gamma_n : \mathscr{Z}^n(X)_F \to H^{2n}(X)(n)$ (algebraic cycles to cohomology classes)

Examples: deRham, Betti, ℓ-adic étale

Grothendieck's idea of motives: universal cohomology theory for algebraic varieties lying behind all realizations via Weil cohomologies

Also recall: Grothendieck's standard conjectures of type C and D

• (Künneth) C: The Künneth components of the diagonal Δ_X are algebraic

• (Hom=Num) D Homological and numerical equivalence coincide

(Also B: Lefschetz involution algebraic; I Hodge involution pos def quadratic form on alg cycles with homological eq)

Motivic Galois groups

More structure than abelian category: Tannakian category $\mathrm{Rep}_F(G)$ fin dim lin reps of an affine group scheme G

- F-linear, abelian, tensor category (*symmetric monoidal*) $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$
- functorial isomorphisms:

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \stackrel{\simeq}{\to} (X \otimes Y) \otimes Z$$

$$c_{X,Y} : X \otimes Y \stackrel{\simeq}{\to} Y \otimes X \quad \text{with} \quad c_{X,Y} \circ c_{Y,X} = 1_{X \otimes Y}$$

$$\ell_X : X \otimes 1 \stackrel{\simeq}{\to} X, \quad r_X : 1 \otimes X \stackrel{\simeq}{\to} X$$

- *Rigid*: duality $\vee : \mathscr{C} \to \mathscr{C}^{op}$ with $\epsilon : X \otimes X^\vee \to 1$ and $\eta : 1 \to X^\vee \otimes X$

$$X \simeq X \otimes 1 \stackrel{1_X \otimes \eta}{\to} X \otimes X^\vee \otimes X \stackrel{\epsilon \otimes 1_X}{\to} 1 \otimes X \simeq X$$

composition is identity

• categorical trace (Euler characteristic)
$tr(f) = \epsilon \circ c_{X^\vee \otimes X} \circ (1_{X^\vee} \otimes f) \circ \eta$; $\dim X = tr(1_X)$

• Tannakian: as above (and with $\mathrm{End}(1) = F$) and *fiber functor*
$\omega : \mathscr{C} \to Vect(K)$
$K =$ extension of F; ω exact faithful tensor functor; *neutral Tannakian* if $K = F$

• equivalence $\mathscr{C} \simeq \mathrm{Rep}_F(G)$, affine group scheme
$G = \mathrm{Gal}(\mathscr{C}) = \underline{Aut}^{\otimes}(\omega)$

• Deligne's characterization (char 0): Tannakian iff $tr(1_X)$ non-negative for all X

Tannakian categories and standard conjectures

In the case of $\mathrm{Mot}_{\sim_{num}}(k)$, when Tannakian?

- problem: $tr(1_X) = \chi(X)$ Euler characteristic can be negative

- $\mathrm{Mot}^{\dagger}_{\sim_{num}}(k)$ category $\mathrm{Mot}_{\sim_{num}}(k)$ with modified commutativity constraint $c_{X,Y}$ by the Koszul sign rule
(corrects for signs in the Euler characteristic)

- (Jannsen) if standard conjecture C (Künneth) holds then $\mathrm{Mot}^{\dagger}_{\sim_{num}}(k)$ is Tannakian

- If conjecture D also holds then H^* fiber functor

Motives and Noncommutative motives

• Motives (pure): smooth projective algebraic varieties X
cohomology theories $H_{dR}, H_{Betti}, H_{etale}, \dots$
universal cohomology theory: motives $\Rightarrow$ realizations

• NC Motives (pure): smooth proper dg-categories $\mathscr{A}$
homological invariants: K-theory, Hochschild and cyclic cohomology
universal homological invariant: NC motives

dg-categories

$\mathscr{A}$ category whose morphism sets $\mathscr{A}(x,y)$ are complexes of k-modules ($k =$ base ring or field) with composition satisfying Leibniz rule

$$d(f\circ g) = df\circ g + (-1)^{\deg(f)} f\circ dg$$

$\mathtt{dgcat} =$ category of (small) dg-categories with dg-functors (preserving dg-structure)

From varieties to dg-categories

$$X \Rightarrow \mathscr{D}^{dg}_{perf}(X)$$

dg-category of perfect complexes

H^0 gives derived category $\mathscr{D}_{perf}(X)$ of perfect complexes of $\mathscr{O}_X$-modules

(loc quasi-isom to finite complexes of loc free sheaves of fin rank)

saturated dg-categories (Kontsevich)

- smooth dgcat: perfect as a bimodule over itself
- proper dgcat: if the complexes $\mathscr{A}(x, y)$ are perfect
- saturated = smooth + proper

smooth projective variety $X \Rightarrow$ smooth proper dgcat $\mathscr{D}^{dg}_{perf}(X)$
(but also smooth proper dgcat not from smooth proj varieties)

derived Morita equivalences

• $\mathscr{A}^{op}$ same objects and morphisms $\mathscr{A}^{op}(x,y) = \mathscr{A}(y,x)$; right dg $\mathscr{A}$-module: dg-functor $\mathscr{A}^{op} \to \mathscr{C}_{dg}(k)$ (dg-cat of complexes of k-modules); $\mathscr{C}(\mathscr{A})$ cat of $\mathscr{A}$-modules; $\mathscr{D}(\mathscr{A})$ (derived cat of $\mathscr{A}$) localization of $\mathscr{C}(\mathscr{A})$ w/ resp to quasi-isom

• functor $F : \mathscr{A} \to \mathscr{B}$ is derived Morita equivalence iff induced functor $\mathscr{D}(\mathscr{B}) \to \mathscr{D}(\mathscr{A})$ (restriction of scalars) is an equivalence of triangulated categories

• cohomological invariants (K-theory, Hochschild and cyclic cohomologies) are derived Morita invariant: send derived Morita equivalences to isomorphisms symmetric monoidal category $\mathtt{Hmo}$

• homotopy category: dg-categories up to derived Morita equivalences

• $\otimes$ extends from k-algebras to dg-categories

• can be derived with respect to derived Morita equivalences (gives symmetric monoidal structure on $\mathtt{Hmo}$)

• saturated dg-categories = dualizable objects in $\mathtt{Hmo}$ (Cisinski–Tabuada)

Further refinement: $\mathtt{Hmo}_0$

- all cohomological invariants listed are "additive invariants":

$$E : \mathtt{dgcat} \to A, \quad E(\mathscr{A}) \oplus E(\mathscr{B}) = E(|M|)$$

where A additive category and $|M|$ dg-category $Obj(|M|) = Obj(\mathscr{A}) \cup Obj(\mathscr{B})$ morphisms $\mathscr{A}(x,y)$, $\mathscr{B}(x,y)$, $X(x,y)$ with X a $\mathscr{A}$–$\mathscr{B}$ bimodule

- $\mathtt{Hmo}_0$: objects dg-categories, morphisms $K_0\mathrm{rep}(\mathscr{A},\mathscr{B})$ with $\mathrm{rep}(\mathscr{A},\mathscr{B}) \subset \mathscr{D}(\mathscr{A}^{op} \otimes^{\mathbb{L}} \mathscr{B})$ full triang subcat of $\mathscr{A}$–$\mathscr{B}$ bimodules X with $X(a,-) \in \mathscr{D}_{perf}(\mathscr{B})$; composition = (derived) tensor product of bimodules

- (Tabuada) $\mathscr{U}_A : \mathtt{dgcat} \to \mathtt{Hmo}_0$, id on objects, sends dg-functor to class in Grothendieck group of associated bimodule ($\mathscr{U}_A$ characterized by a universal property)

- all additive invariants factor through $\mathtt{Hmo}_0$

noncommutative Chow motives (Kontsevich) $\mathrm{NChow}_F(k)$

• $\mathtt{Hmo}_{0;F}$ = the F-linearization of additive category $\mathtt{Hmo}_0$

• $\mathtt{Hmo}_{0;F}^{\natural}$ = idempotent completion of $\mathtt{Hmo}_{0;F}$

• $\mathrm{NChow}_F(k)$ = idempotent complete full subcategory gen by saturated dg-categories

$\mathrm{NChow}_F(k)$:

- Objects: $(\mathscr{A}, e)$ smooth proper dg-categories (and idempotents)
- Morphisms $K_0(\mathscr{A}^{op} \otimes_k^{\mathbb{L}} \mathscr{B})_F$ (correspondences)
- Composition: induced by derived tensor product of bimodules

relation to commutative Chow motives (Tabuada):

$$\mathrm{Chow}_{\mathbb{Q}}(k)/_{-\otimes\mathbb{Q}(1)} \hookrightarrow \mathrm{NChow}_{\mathbb{Q}}(k)$$

commutative motives embed as noncommutative motives after moding out by the Tate motives

orbit category $\mathrm{Chow}_{\mathbb{Q}}(k)/_{-\otimes\mathbb{Q}(1)}$

$(\mathscr{C}, \otimes, \mathbf{1})$ additive, $F-$ *linear*, rigid symmetric monoidal;
$\mathscr{O} \in \mathrm{Obj}(\mathscr{C})$ $\otimes$-invertible object:
orbit category $\mathscr{C}/_{-\otimes\mathscr{O}}$ same objects and morphisms

$$\mathrm{Hom}_{\mathscr{C}/_{-\otimes\mathscr{O}}}(X, Y) = \oplus_{j\in\mathbb{Z}} \mathrm{Hom}_{\mathscr{C}}(X, Y \otimes \mathscr{O}^{\otimes j})$$

Numerical noncommutative motives

M.M., G.Tabuada, *Noncommutative motives, numerical equivalence, and semi-simplicity*, arXiv:1105.2950, American J. Math. (to appear)

$(\mathscr{A}, e)$ and $(\mathscr{B}, e')$ objects in $\mathrm{NChow}_F(k)$ and correspondences

$$\underline{X} = e \circ [\sum_i a_i X_i] \circ e', \qquad \underline{Y} = e' \circ [\sum_j b_j Y_j] \circ e$$

X_i and Y_j bimodules

$\Rightarrow$ intersection number:

$$\langle \underline{X}, \underline{Y} \rangle = \sum_{ij} [HH(\mathscr{A}; X_i \otimes^{\mathbb{L}}_{\mathscr{B}} Y_j)] \in K_0(k)_F$$

with $[HH(\mathscr{A}; X_i \otimes^{\mathbb{L}}_{\mathscr{B}} Y_j)]$ class in $K_0(k)_F$ of Hochschild homology complex of $\mathscr{A}$ with coefficients in the $\mathscr{A}$–$\mathscr{A}$ bimodule $X_i \otimes^{\mathbb{L}}_{\mathscr{B}} Y_j$

numerically trivial: $\underline{X}$ if $\langle \underline{X}, \underline{Y} \rangle = 0$ for all $\underline{Y}$

• $\otimes$-ideal $\mathscr{N}$ in the category $\mathrm{NChow}_F(k)$

• $\mathscr{N}$ largest $\otimes$-ideal strictly contained in $\mathrm{NChow}_F(k)$

numerical motives: $\mathrm{NNum}_F(k)$

$$\mathrm{NNum}_F(k) = \mathrm{NChow}_F(k)/\mathscr{N}$$

Thm: abelian semisimple (M.M., G.Tabuada, arXiv:1105.2950)

• $\mathrm{NNum}_F(k)$ is abelian semisimple

analog of Jannsen's result for commutative numerical pure motives

What about Tannakian structures and motivic Galois groups?

For commutative motives this involves standard conjectures (C = Künneth and D = homological and numerical equivalence)

Questions:

- is $\mathrm{NNum}_F(k)$ (neutral) super-Tannakian?
- is there a good analog of the standard conjecture C (Künneth)?
- does this make the category Tannakian?
- is there a good analog of standard conjecture D (numerical = homological)?
- does this neutralize the Tannakian category?
- relation between motivic Galois groups for commutative and noncommutative motives?

Tannakian categories $(\mathscr{C}, \otimes, \mathbf{1})$

F-linear, abelian, rigid symmetric monoidal with $\mathrm{End}(\mathbf{1}) = F$

• Tannakian: $\exists$ K-valued *fiber functor*, K field ext of F: exact faithful $\otimes$-functor $\omega : \mathscr{C} \to \mathrm{Vect}(K)$; neutral if $K = F$

$\omega \Rightarrow$ equivalence $\mathscr{C} \simeq \mathrm{Rep}_F(\mathrm{Gal}(\mathscr{C}))$ affine group scheme (Galois group) $\quad \mathrm{Gal}(\mathscr{C}) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$

• intrinsic characterization (Deligne): F char zero, $\mathscr{C}$ Tannakian iff $\mathrm{Tr}(id_X)$ non-negative integer for each object X

super-Tannakian categories $(\mathscr{C}, \otimes, \mathbf{1})$

F-linear, abelian, rigid symmetric monoidal with $\mathrm{End}(\mathbf{1}) = F$

$s\mathrm{Vect}(K)$ super-vector spaces $\mathbb{Z}/2\mathbb{Z}$-graded

- super-Tannakian: $\exists$ K-valued *super fiber functor*, K field ext of F: exact faithful $\otimes$-functor $\omega : \mathscr{C} \to s\mathrm{Vect}(K)$; neutral if $K = F$

$\omega \Rightarrow$ equivalence $\mathscr{C} \simeq \mathrm{Rep}_F(s\mathrm{Gal}(\mathscr{C}), \epsilon)$ super-reps of affine super-group-scheme (super-Galois group)
$s\mathrm{Gal}(\mathscr{C}) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ $\quad \epsilon =$ parity automorphism

- intrinsic characterization (Deligne) F char zero, $\mathscr{C}$ super-Tannakian iff Shur finite (if F alg closed then neutral super-Tannakian iff Schur finite)

- Schur finite: symm grp S_n, idempotent $c_\lambda \in \mathbb{Q}[S_n]$ for partition λ of n (irreps of S_n), Schur functors $S_\lambda : \mathscr{C} \to \mathscr{C}$, $\quad S_\lambda(X) = c_\lambda(X^{\otimes n})$
$\mathscr{C}$ = Schur finite iff all objects X annihilated by some Schur functor $S_\lambda(X) = 0$

Main results

M.M., G.Tabuada, *Noncommutative numerical motives, Tannakian structures, and motivic Galois groups*, arXiv:1110.2438

assume either: (i) $K_0(k) = \mathbb{Z}$, F is k-algebra; (ii) k and F both field extensions of a field K

- **Thm 1**: $\mathrm{NNum}_F(k)$ is super-Tannakian; if F alg closed also neutral
- **Thm 2**: standard conjecture $C_{NC}(\mathscr{A})$: the Künneth projectors

$$\pi^{\pm}_{\mathscr{A}} : \overline{HP}_*(\mathscr{A}) \twoheadrightarrow \overline{HP}^{\pm}_*(\mathscr{A}) \hookrightarrow \overline{HP}_*(\mathscr{A})$$

are algebraic: $\pi^{\pm}_{\mathscr{A}} = \overline{HP}_*(\underline{\pi}^{\pm}_{\mathscr{A}})$ with $\underline{\pi}^{\pm}_{\mathscr{A}}$ correspondences. If k field ext of F char 0, sign conjecture implies

$$C^+(Z) \Rightarrow C_{NC}(\mathscr{D}^{dg}_{perf}(Z))$$

i.e. on commutative motives more likely to hold than sign conjecture

• **Thm 3**: k and F char 0, one extension of other: if C_{NC} holds then change of symmetry isomorphism in tensor structure gives category $\mathrm{NNum}_F^{\dagger}(k)$ Tannakian

• **Thm 4**: standard conjecture $D_{NC}(\mathscr{A})$:

$$K_0(\mathscr{A})_F/\sim_{hom} = K_0(\mathscr{A})_F/\sim_{num}$$

homological defined by periodic cyclic homology: kernel of

$$K_0(\mathscr{A})_F = \mathrm{Hom}_{\mathrm{NChow}_F(k)}(k, \mathscr{A}) \xrightarrow{\overline{HP}_*} \mathrm{Hom}_{sVect(K)}(\overline{HP}_*(k), \overline{HP}_*(\mathscr{A}))$$

when k field ext of F char 0: $D(Z) \Rightarrow D_{NC}(\mathscr{D}^{dg}_{perf}(Z))$
i.e. for commutative motives more likely to hold than D conjecture

• **Thm 5**: F ext of k char 0: if C_{NC} and D_{NC} hold then $\mathrm{NNum}_F^\dagger(k)$ is a neutral Tannakian category with periodic cyclic homology as fiber functor

• **Thm 6**: k char 0: if C, D and C_{NC}, D_{NC} hold then

$$s\mathrm{Gal}(\mathrm{NNum}_k(k) \twoheadrightarrow \mathrm{Ker}(t : s\mathrm{Gal}(\mathrm{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$$

$$\mathrm{Gal}(\mathrm{NNum}_k^\dagger(k) \twoheadrightarrow \mathrm{Ker}(t : \mathrm{Gal}(\mathrm{Num}_k^\dagger(k)) \twoheadrightarrow \mathbb{G}_m)$$

where t induced by inclusion of Tate motives in the category of (commutative) numerical motives

(using periodic cyclic homology and de Rham cohomology)

What is kernel? Ker = “truly noncommutative motives"

$$\mathrm{Gal}(\mathrm{NNum}_k^{\dagger}(k)) \twoheadrightarrow \mathrm{Ker}(t : \mathrm{Num}_k^{\dagger}(k) \to \mathbb{G}_m)$$

$$s\mathrm{Gal}(\mathrm{NNum}_k(k)) \twoheadrightarrow \mathrm{Ker}(t : s\mathrm{Gal}(\mathrm{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$$

what do they look line? examples? general properties?

Are there truly noncommutative motives? Still an open question!

... but the theory of NC motives can be used as a new tool to study ordinary motives

Using NC motives to study commutative motives

Example: full exceptional collections and motivic decompositions

Examples of motivic decompositions:

- Projective spaces: $h(\mathbb{P}^n) = 1 \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^n$
- Quadrics (k alg closed char 0):

$$h(Q_q)_{\mathbb{Q}} \simeq \begin{cases} 1 \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^{\otimes n} & d \text{ odd} \\ 1 \oplus \mathbb{L} \oplus \cdots \oplus \mathbb{L}^{\otimes n} \oplus \mathbb{L}^{\otimes (d/2)} & d \text{ even}. \end{cases}$$

- Fano 3-folds:

$$h(X)_{\mathbb{Q}} \simeq 1 \oplus h^1(X) \oplus \mathbb{L}^{\oplus b} \oplus (h^1(J) \otimes \mathbb{L}) \oplus (\mathbb{L}^{\otimes 2})^{\oplus b} \oplus h^5(X) \oplus \mathbb{L}^{\otimes 3},$$

$h^1(X)$ and $h^5(X)$ Picard and Albanese motives, $b = b_2(X) = b_4(X)$
J abelian variety

Full exceptional collections in the derived category $\mathscr{D}^b(X)$

A collection of objects $\{E_1, \ldots, E_n\}$ in a F-linear triangulated category $\mathscr{C}$ is *exceptional* if $\mathrm{RHom}(E_i, E_i) = F$ for all i and $\mathrm{RHom}(E_i, E_j) = 0$ for all $i > j$; it is *full* if $\mathscr{C}$ is minimal triangulated subcategory containing it.

Examples of full exceptional collections:

- Projective spaces (Beilinson): $(\mathscr{O}(-n), \ldots, \mathscr{O}(0))$
- Quadrics (Kapranov):

$$\begin{array}{rl} (\Sigma(-d), \mathscr{O}(-d+1), \ldots, \mathscr{O}(-1), \mathscr{O}) & \text{if } d \text{ is odd} \\ (\Sigma_+(-d), \Sigma_-(-d), \mathscr{O}(-d+1), \ldots, \mathscr{O}(-1), \mathscr{O}) & \text{if } d \text{ is even}, \end{array}$$

$\Sigma_\pm$ (and Σ) spinor bundles

• Toric varieties (Kawamata)

• Homogeneous space (Kuznetsov-Polishchuk)
Conjecture (KP): k alg cl char 0, parabolic subgroup $P \subset G$ of semisimple alg group then $\mathscr{D}^b(G/P)$ has full exceptional collection

• Fano 3-folds with vanishing odd cohomology (Ciolli)

• Moduli spaces of rational curves $\overline{\mathscr{M}}_{0,n}$ (Manin–Smirnov)

Note: all these cases also have motivic decompositions

Deeper reason: exceptional collections and motivic decompositions are related through the relation between commutative and NC motives

Thm 7: Full exceptional collections and motivic decompositions

if $\mathscr{D}^b(X)$ has a full exceptional collection, then $h(X)_{\mathbb{Q}}$ has a motivic decomposition

$$h(X)_{\mathbb{Q}} \simeq \mathbb{L}^{\ell_1} \oplus \cdots \oplus \mathbb{L}^{\ell_m}$$

for some $\ell_1, \ldots, \ell_m \geq 0$

(Note: works also for Deligne–Mumford stacks)

- $\mathscr{D}^b_{dg}(X)$ unique dg enhancement: $\langle E_j \rangle_{dg} \simeq \mathscr{D}^b_{dg}(k)$
- Look at corresponding elements in $\mathrm{NChow}_{\mathbb{Q}}(k)$ under universal localizing invariant $\mathscr{U} : \mathtt{dgcat}(k) \to \mathrm{NChow}_{\mathbb{Q}}(k)$

$$\oplus_{j=1}^m \mathscr{U}(\mathscr{D}^b_{dg}(k)) \stackrel{\simeq}{\to} \mathscr{U}(\mathscr{D}^b_{dg}(X))$$

from inclusions of dg categories $\langle E_j \rangle_{dg} \hookrightarrow \mathscr{D}^b_{dg}(X)$

using (Tabuada “Higher K-theory via universal invariants"): given split short exact sequence of pre-triangulated dg categories

$$\begin{tikzcd} 0 \arrow[r] & \mathscr{B} \arrow[r, "\iota_{\mathscr{B}}"'] & \mathscr{A} \arrow[l, bend right] \arrow[r] & \mathscr{C} \arrow[l, bend right, "\iota_{\mathscr{C}}"'] \arrow[r] & 0 \end{tikzcd}$$

mapped by universal localizing invariant $\mathscr{U}(-)$ to a distinguished split triangle so $\mathscr{U}(\mathscr{B}) \oplus \mathscr{U}(\mathscr{C}) \stackrel{\sim}{\to} \mathscr{U}(\mathscr{A})$
Applied to

$$\mathscr{A} := \langle E_i, \cdots, E_m \rangle_{dg}, \quad \mathscr{B} := \langle E_i \rangle_{dg}, \quad \mathscr{C} := \langle E_{i+1}, \ldots, E_m \rangle_{dg}$$

gives

$$\mathscr{U}(\mathscr{D}^b_{dg}(k)) \oplus \mathscr{U}(\langle E_{i+1}, \ldots, E_m \rangle_{dg}) \stackrel{\sim}{\to} \mathscr{U}(\langle E_i, \ldots, E_m \rangle_{dg})$$

recursively get result using $\mathscr{D}^b_{dg}(X) = \langle E_1, \ldots, E_m \rangle_{dg}$

A consequence: Hodge–Tate cohomology

Thm 8: If a smooth complex projective variety V has a full exceptional collection then it is Hodge–Tate (Hodge numbers $h^{p,q}(V) = 0$ for $p \neq q$)

Reason: motivic decomposition

Dubrovin conjecture: V smooth projective complex

(i) Quantum cohomology of V is (generically) semi-simple if and only if V is Hodge-Tate and $\mathscr{D}^b(V)$ has a full exceptional collection.

(ii) Stokes matrix of structure connection of quantum cohomology = Gram matrix of exceptional collection

$$\chi : K_0(V) \times K_0(V) \to \mathbb{Z}, \quad \sum_{n \in \mathbb{Z}} (-1)^n \dim \mathrm{Ext}^n(\mathscr{F}_1, \mathscr{F}_2)$$

First observation: Hodge-Tate hypothesis not necessary

NC-motivic approach to the Dubrovin conjecture? currently work in progress...

Jacobians of noncommutative motives

• Jacobians of curves $J(C)$: geometric model for cohomology $H^1(C)$, one of the origins of the theory of motives (Weil)

• Smooth projective X: Picard and Albanese varieties $Pic^0(X)$ and $Alb(X)$ geometric models for $H^1(X)$ and $H^{2d-1}(X)$

• Griffiths intermediate Jacobians ($F =$ Hodge filtration)

$$J_i(X) := \frac{H^{2i+1}(X,\mathbb{C})}{F^{i+1}H^{2i+1}(X,\mathbb{C}) + H^{2i+1}(X,\mathbb{Z})}$$

not algebraic but $J_i^a(X) \subseteq J_i(X)$ algebraic: image of Abel-Jacobi

$$AJ_i : CH^{i+1}(X)^0_{\mathbb{Z}} \to J_i(X)$$

with $CH^{i+1}(X)^0_{\mathbb{Z}}$ group of alg.-trivial cycles codim $i+1$ (see recent work of Charles Vial)

• Know how to go from commutative to noncommutative motives via

$$\mathrm{Chow}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)} \hookrightarrow \mathrm{NChow}(k)_{\mathbb{Q}}$$

• Question: can one go the other way? Assign functorially a "commutative part" to a noncommutative motive?

• Idea: a theory of Jacobians for NC motives

$$\mathrm{NChow}(k)_{\mathbb{Q}} \to \mathrm{Ab}(k)_{\mathbb{Q}}, \qquad N \mapsto \mathbf{J}(N)$$

$\mathbb{Q}$-linear additive Jacobian functor to category $\mathrm{Ab}(k)_{\mathbb{Q}}$ of abelian varieties up to isogeny

- Periodic cyclic homology

$$HP^{\pm} : \mathrm{NChow}(k)_{\mathbb{Q}} \to s\mathrm{Vect}(k)$$

- Piece of *HP* generated by curves

$$HP^{-}_{\mathrm{curves}}(N) := \sum_{C,\Gamma} \mathrm{Im}(HP^{-}(\mathrm{perf}(C)) \stackrel{HP^{-}(\Gamma)}{\longrightarrow} HP^{-}(N))$$

C = smooth projective curve; $\Gamma : \mathrm{perf}(C) \to N$ a morphism (correspondence) in $\mathrm{NChow}(k)_{\mathbb{Q}}$

Results (MM, G. Tabuada, arXiv:1212.1118)

Thm 9:

• k char zero, have $\mathbb{Q}$-additive linear functor

$$\mathrm{NChow}(k)_{\mathbb{Q}} \to \mathrm{Ab}(k)_{\mathbb{Q}}, \qquad N \mapsto \mathbf{J}(N)$$

• $\forall N \in \mathrm{NChow}(k)_{\mathbb{Q}}$ there is C_N smooth proj curve and $\Gamma_N : \mathrm{perf}(C_N) \to N$ with

$$H^1_{dR}(\mathbf{J}(N)) = \mathrm{Im} HP^-(\Gamma_N)$$

so $H^1_{dR}(\mathbf{J}(N)) \subseteq HP^-_{\mathrm{curves}}(N)$

• if conjecture D_{NC} holds for $\mathrm{perf}(C) \otimes N$, for smooth proj curves C,

$$H^1_{dR}(\mathbf{J}(N)) = HP^-_{\mathrm{curves}}(N)$$

• for smooth projective X let

$$NH^{2i+1}_{dR}(X) := \sum_{C,\gamma_i} \mathrm{Im}(H^1_{dR}(C) \stackrel{H_{dR}(\gamma_i)}{\to} H^{2i+1}_{dR}(X))$$

with $\gamma_i : M(C) \to M(X)(i)$ morphism in $\mathrm{Chow}(k)_{\mathbb{Q}}$

• Intersection bilinear pairing restricted to these ($0 \leq i \leq d-1$)

$$\langle -,-\rangle : NH^{2d-2i-1}_{dR}(X) \times NH^{2i+1}_{dR}(X) \to k$$

• **Thm 10:** if $k = \bar{k} \subseteq \mathbb{C}$ and X smooth projective and if pairings above are *nondegenerate* then

$$\mathbf{J}(\mathrm{perf}(X)) = \prod_{i=0}^{d-1} J^a_i(X)$$

and $H^1_{dR}(\mathbf{J}(\mathrm{perf}(X))) \otimes_k \mathbb{C} = \oplus_{i=0}^{d-1} NH^{2i+1}_{dR}(X) \otimes_k \mathbb{C}$

Sketch of argument on NC Jacobians: construction of $\mathbf{J}(N)$

- Categories of NC motives: $\mathrm{NChow}(k)_{\mathbb{Q}}$, $\mathrm{NHomo}(k)_{\mathbb{Q}}$, $\mathrm{NNum}(k)_{\mathbb{Q}}$
- $\mathrm{NNum}(k)_{\mathbb{Q}}$ is abelian semi-simple: $N = S_1 \oplus \cdots \oplus S_n$ unique finite decomposition into simple objects
- classical motives: $\mathrm{Homo}(k)_{\mathbb{Q}} \supset \{\underline{\pi}^1 M(C)\}^{\natural} = \mathrm{Ab}(k)_{\mathbb{Q}}$ and same in $\mathrm{Num}(k)_{\mathbb{Q}} \supset \{\underline{\pi}^1 M(C)\}^{\natural} = \mathrm{Ab}(k)_{\mathbb{Q}}$
- functor mapping $\mathrm{Ab}(k)_{\mathbb{Q}}$ to $\mathrm{NNum}(k)_{\mathbb{Q}}$ with image $\overline{\mathrm{Ab}}(k)_{\mathbb{Q}}$

$$\mathrm{Ab}(k)_{\mathbb{Q}} \to \mathrm{Num}(k)_{\mathbb{Q}} \to \mathrm{Num}(k)_{\mathbb{Q}}/_{-\otimes\mathbb{Q}(1)} \to \mathrm{NNum}(k)_{\mathbb{Q}}$$

- $\mathrm{Ab}(k)_{\mathbb{Q}} \simeq \overline{\mathrm{Ab}}(k)_{\mathbb{Q}}$ equivalence of categories
- $\mathscr{S}$ = simple objects of $\mathrm{NNum}(k)_{\mathbb{Q}}$ belonging to $\overline{\mathrm{Ab}}(k)_{\mathbb{Q}}$
- truncation functor $\mathrm{NNum}(k)_{\mathbb{Q}} \to \overline{\mathrm{Ab}}(k)_{\mathbb{Q}}$, with $N \mapsto \tau(N)$ only simple objects in $\mathscr{S}$ of decomposition of N

properties of functor $N \mapsto \mathbf{J}(N)$

- because $\mathrm{Ab}(k)_{\mathbb{Q}} \simeq \overline{\mathrm{Ab}}(k)_{\mathbb{Q}}$ every object in $\overline{\mathrm{Ab}}(k)_{\mathbb{Q}}$ is a direct factor of some $\underline{\pi}^1\mathrm{perf}(C)$
- so get C_N for any $N \in \mathrm{NNum}(k)_{\mathbb{Q}}$ through $\tau(N) \in \overline{\mathrm{Ab}}(k)_{\mathbb{Q}}$
- and correspondence Γ_N giving $\tau(N)$ as direct factor of $\underline{\pi}^1\mathrm{perf}(C_N)$ and this as direct factor of $\mathrm{perf}(C_N)$
- $H^1_{dR}(C_N) = HP^-(\mathrm{perf}(C_N)) = HP^-(\underline{\pi}^1\mathrm{perf}(C_N)) \stackrel{HP^-(\Gamma_N)}{\longrightarrow} HP^-(N)$
- $HP^-(\underline{\pi}^1\mathrm{perf}(C_N)) \stackrel{HP^-(\bar{\Gamma}_N)}{\to} HP^-(\tau(N))$ surjective and $HP^-(\tau(N)) \to HP^-(N)$ from $\tau(N) \hookrightarrow N$ injective $\Rightarrow$ $HP^-(\tau(N)) = \mathrm{Im}(HP^-(\bar{\Gamma}_N))$ and $H^1_{dR}(\mathbf{J}(N)) = \mathrm{Im}(HP^-(\bar{\Gamma}_N))$
- If $D_{NC}(\mathrm{perf}(C) \otimes N)$ holds then as $\mathbb{Q}$-vector spaces

$$\mathrm{Hom}_{\mathrm{NHomo}(k)_{\mathbb{Q}}}(\mathrm{perf}(C), N) = \mathrm{Hom}_{\mathrm{NNum}(k)_{\mathbb{Q}}}(\mathrm{perf}(C), N)$$

applying HP^-: morphism $HP^-(\Gamma)$ factors through $HP^-(\tau(N))$ for all C, Γ, so obtain $HP^-(\tau(N)) = HP^-_{\mathrm{curves}}(N)$

pairings

- for X smooth projective $HP^-(\mathrm{perf}(X)) = \oplus_{i=0}^{d-1} NH_{dR}^{2i+1}(X)$
- isomorphisms $NH_{dR}^{2i+1}(X) \otimes_k \mathbb{C} \simeq NH_{Betti}^{2i+1}(X) \otimes_{\mathbb{Q}} \mathbb{C}$
- pairings of $NH_{dR}^{2i+1}(X)$ nondegenerate iff pairings of $NH_{Betti}^{2i+1}(X)$ nondegenerate
- Idempotents Π_{2i+1} in $\mathrm{Homo}(k)_{\mathbb{Q}}$ with

$$\Pi_{2i+1} M(X) \simeq \underline{\pi}^1 M(J_i^{alg}(X))(-i)$$

image in $\mathrm{NNum}(k)_{\mathbb{Q}}$

- $\tau(\mathrm{perf}(X)) \simeq \oplus_{i=0}^{d-1} \underline{\pi}^1 \mathrm{perf}(J_i^{alg}(X))$ using surjection $\tau(\mathrm{perf}(X)) \twoheadrightarrow \oplus_{i=0}^{d-1} \underline{\pi}^1 \mathrm{perf}(J_i^a(X))$ and faithful $HP^- \otimes_k \mathbb{C} : \overline{\mathrm{Ab}}(k)_{\mathbb{Q}} \to s\mathrm{Vect}(\mathbb{C})$ to also get
$\dim(HP^{\pm}(\tau(\mathrm{perf}(X))) \otimes_k \mathbb{C}) \leq \dim(HP^{\pm}(\oplus_{i=0}^{d-1} \underline{\pi}^1 \mathrm{perf}(J_i^a(X))) \otimes_k \mathbb{C})$
using
$HP^{\pm}(\tau(\mathrm{perf}(X))) \otimes_k \mathbb{C} \subseteq HP^-_{\mathrm{curves}}(\mathrm{perf}(X)) \otimes_k \mathbb{C} \simeq \oplus_{i=0}^{d-1} NH_B^{2i+1}(X) \otimes_{\mathbb{Q}} \mathbb{C}$
and $HP^{\pm}(\oplus_{i=0}^{d-1} \underline{\pi}^1 \mathrm{perf}(J_i^a(X))) \otimes_k \mathbb{C} \simeq \oplus_{i=0}^{d-1} H_B^1(M(J_i^a(X))(-i)) \otimes_{\mathbb{Q}} \mathbb{C} \simeq$
$\oplus_{i=0}^{d-1} NH_B^{2i+1}(X) \otimes_{\mathbb{Q}} \mathbb{C}$

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More details on the category of NC motives:

Thm 1: Schur finiteness $\overline{HH} : \mathrm{NChow}_F(k) \to \mathscr{D}_c(F)$

F-linear symmetric monoidal functor (Hochschild homology)

$$(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HH}))^\natural \to \mathscr{D}_c(F)$$

faithful F-linear symmetric monoidal

$\mathscr{D}_c(\mathscr{A})$ = full triang subcat of compact objects in $\mathscr{D}(\mathscr{A}) \Rightarrow \mathscr{D}_c(F)$ identified with fin-dim $\mathbb{Z}$-graded F-vector spaces: Shur finite

general fact: $L : \mathscr{C}_1 \to \mathscr{C}_2$ F-linear symmetric monoidal functor: $X \in \mathscr{C}_1$ Schur finite $\Rightarrow L(X) \in \mathscr{C}_2$ Schur finite; L faithful then also converse: $L(X) \in \mathscr{C}_2$ Schur finite $\Rightarrow X \in \mathscr{C}_1$ Schur finite

conclusion: $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HH}))^\natural$ is Schur finite

also $\mathrm{Ker}(\overline{HH}) \subset \mathscr{N}$ with F-linear symmetric monoidal functor $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HH}))^\natural \to (\mathrm{NChow}_F(k)/\mathscr{N})^\natural = \mathrm{NNum}_F(k)$

$\Rightarrow \mathrm{NNum}_F(k)$ Schur finite $\Rightarrow$ super-Tannakian

Thm 2: periodic cyclic homology

mixed complex (M, b, B) with $b^2 = B^2 = Bb + bB = 0$, $\deg(b) = 1 = -\deg(B)$: *periodized*

$$\cdots \prod_{n\ even} M_n \stackrel{b+B}{\to} \prod_{n\ odd} M_n \stackrel{b+B}{\to} \prod_{n\ even} M_n \cdots$$

periodic cyclic homology (the derived cat of $\mathbb{Z}/2\mathbb{Z}$-graded complexes

$$HP : \mathtt{dgcat} \to \mathscr{D}_{\mathbb{Z}/2\mathbb{Z}}(k)$$

induces F-linear symmetric monoidal functor

$$\overline{HP}_* : \mathrm{NChow}_F(k) \to s\mathrm{Vect}(F)$$

or to $s\mathrm{Vect}(k)$ if k field ext of F

Note the issue here:

- mixed complex functor symmetric monoidal but 2-periodization not (infinite product don't commute with $\otimes$)
- *lax symmetric monoidal* with $\mathscr{D}_{\mathbb{Z}/2\mathbb{Z}}(k) \simeq S\mathrm{Vect}(k)$ (not fin dim)
- $HP : \mathtt{dgcat} \to S\mathrm{Vect}(k)$ *additive invariant*: through $\mathtt{Hmo}_0(k)$
- $\mathrm{NChow}_F(k) = (\mathtt{Hmo}_0(k)^{sp})^\natural_F$ (sp = gen by smooth proper dgcats)
- periodic cyclic hom *finite dimensional* for smooth proper dgcats + a result of Emmanouil

$\Rightarrow$ lax symmetric monoidal $\overline{HP}_* : \mathtt{Hmo}_0(k)^{sp} \to s\mathrm{Vect}(k)$ is symmetric monoidal standard conjecture C_{NC} (Künneth type)

- $C_{NC}(\mathscr{A})$: Künneth projections

$$\pi^{\pm}_{\mathscr{A}} : \overline{HP}_*(\mathscr{A}) \twoheadrightarrow \overline{HP}^{\pm}_*(\mathscr{A}) \hookrightarrow \overline{HP}_*(\mathscr{A})$$

are algebraic: $\pi^{\pm}_{\mathscr{A}} = \overline{HP}_*(\underline{\pi}^{\pm}_{\mathscr{A}})$ image of correspondences

- then from Keller + Hochschild-Konstant-Rosenberg have
$\overline{HP}_*(\mathscr{D}^{dg}_{perf}(Z)) = HP_*(\mathscr{D}^{dg}_{perf}(Z)) = HP_*(Z) = \oplus_{n\, even/odd} H^n_{dR}(Z)$

- hence $C^+(Z) \Rightarrow C_{NC}(\mathscr{D}^{dg}_{perf}(Z))$ with $\underline{\pi}^{\pm}_{\mathscr{D}^{dg}_{perf}(Z)}$ image of $\underline{\pi}^{\pm}_Z$ under
$\mathrm{Chow}(k) \to \mathrm{Chow}(k)/_{-\otimes\mathbb{Q}(1)} \hookrightarrow \mathrm{NChow}(k)$

classical: (using deRham as Weil cohomology) $C(Z)$ for Z correspondence, the Künneth projections $\pi^n_Z : H^*_{dR}(Z) \twoheadrightarrow H^n_{dR}(Z)$ are algebraic, $\pi^n_Z = H^*_{dR}(\underline{\pi}^n_Z)$, with $\underline{\pi}^n_Z$ correspondences

sign conjecture: $C^+(Z)$: Künneth projectors $\pi^+_Z = \sum_{n=0}^{\dim Z} \pi^{2n}_Z$ are algebraic, $\pi^+_Z = H^*_{dR}(\underline{\pi}^+_Z)$ (hence π^-_Z also)

Thm 3: Tannakian category first steps

• have F-linear symmetric monoidal and also full and essentially surjective functor: $\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*) \to \mathrm{NChow}_F(k)/\mathscr{N}$

• assuming $C_{NC}(\mathscr{A})$: have $\underline{\pi}^{\pm}_{(\mathscr{A},e)} = e \circ \underline{\pi}^{\pm}_{\mathscr{A}} \circ e$; if $\underline{X}$ trivial in $\mathrm{NChow}_F(k)/\mathscr{N}$ intersection numbers $\langle \underline{X}^n, \underline{\pi}^{\pm}_{(\mathscr{A},e)} \rangle$ vanishes ($\mathscr{N}$ is $\otimes$-ideal)

• intersection number is categorical trace of $\underline{X}^n \circ \underline{\pi}^{\pm}_{(\mathscr{A},e)}$ (M.M., G.Tabuada, 1105.2950)

$$\Rightarrow \mathrm{Tr}(\overline{HP}_*(\underline{X}^n \circ \underline{\pi}^{\pm}_{(\mathscr{A},e)}) = \mathrm{Tr}(\overline{HP}^{\pm}_*(\underline{X})^n) = 0$$

trace all n-compositions vanish $\Rightarrow$ nilpotent $\overline{HP}^{\pm}_*(\underline{X})$

• conclude: nilpotent ideal as kernel of

$$\mathrm{End}_{\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*)}(\mathscr{A}, e) \twoheadrightarrow \mathrm{End}_{\mathrm{NChow}_F(k)/\mathscr{N}}(\mathscr{A}, e)$$

• then functor $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^\natural \to \mathrm{NNum}_F(k)$ full conservative essentially surjective: (quotient by $\mathscr{N}$ full and ess surj; idempotents can be lifted along surj F-linear homom with nilpotent

Tannakian category: modification of tensor structure

• $H : \mathscr{C} \to s\mathrm{Vect}(K)$ symmetric monoidal F-linear (K ext of F) faithful, Künneth projectors $\pi_N^{\pm} = H(\underline{\pi}_N^{\pm})$ for $\underline{\pi}_N^{\pm} \in \mathrm{End}_{\mathscr{C}}(N)$ for all $N \in \mathscr{C}$ then modify symmetry isomorphism

$$c^{\dagger}_{N_1,N_2} = c_{N_1,N_2} \circ (e_{N_1} \otimes e_{N_2}) \quad \text{with } e_N = 2\underline{\pi}_N^{+} - id_N$$

• get F-linear symmetric monoidal functor $\mathscr{C}^{\dagger} \xrightarrow{H} s\mathrm{Vect}(K) \to \mathrm{Vect}(K)$

• if $P : \mathscr{C} \to \mathscr{D}$, F-linear symmetric monoidal (essentially) surjective, then $P : \mathscr{C}^{\dagger} \to \mathscr{D}^{\dagger}$ (use image of e_N to modify $\mathscr{D}$ compatibly)

• apply to functors $\overline{HP}_* : (\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^{\natural} \to s\mathrm{Vect}(K)$ and $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^{\natural} \to \mathrm{NNum}_F(k)$

$\Rightarrow$ obtain $\mathrm{NNum}_F^{\dagger}(k)$ satisfying Deligne's intrinsic characterization for Tannakian: with $\tilde{N}$ lift to $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^{\natural,\dagger}$ have

$$\mathrm{rk}(N) = \mathrm{rk}(\overline{HP}_*(\tilde{N})) = \dim(\overline{HP}_*^{+}(\tilde{N})) + \dim(\overline{HP}_*^{-}(\tilde{N})) \geq 0$$

Thm 4: Noncommutative homological motives

$$\overline{HP}_* : \mathrm{NChow}_F(k) \to s\mathrm{Vect}(K)$$

$$K_0(\mathscr{A})_F = \mathrm{Hom}_{\mathrm{NChow}_F(k)}(k, \mathscr{A}) \overset{\overline{HP}_*}{\to} \mathrm{Hom}_{s\mathrm{Vect}(K)}(\overline{HP}_*(k), \overline{HP}_*(\mathscr{A}))$$

kernel gives homological equivalence $K_0(\mathscr{A})_F$ mod $\sim_{hom}$

- $D_{NC}(\mathscr{A})$ standard conjecture:

$$K_0(\mathscr{A})_F / \sim_{hom} = K_0(\mathscr{A})_F / \sim_{num}$$

- on $\mathrm{Chow}_F(k)/_{-\otimes \mathbb{Q}(1)}$ induces homological equivalence with sH_{dR} (de Rham even/odd) $\Rightarrow \mathscr{Z}^*_{hom}(Z)_F \twoheadrightarrow K_0(\mathscr{D}^{dg}_{perf}(Z))_F / \sim_{hom}$
- classical cycles $\mathscr{Z}^*_{hom}(Z)_F \simeq \mathscr{Z}^*_{num}(Z)_F$; for numerical $\mathscr{Z}^*_{num}(Z)_F \overset{\sim}{\to} K_0(\mathscr{D}^{dg}_{perf}(Z))_F / \sim_{num}$; then get

$$D(Z) \Rightarrow D_{NC}(\mathscr{D}^{dg}_{perf}(Z))$$

Thm 5: assume C_{NC} and D_{NC} then

$$\overline{HP}_* : \mathrm{NNum}_F^\dagger(k) \to \mathrm{Vect}(F)$$

exact faithful $\otimes$-functor: fiber functor $\Rightarrow$ *neutral* Tannakian category $\mathrm{NNum}_F^\dagger(k)$

Thm 6: Motivic Galois groups

- Galois group of neutral Tannakian category $\mathrm{Gal}(\mathrm{NNum}_F^\dagger(k))$ want to compare with commutative case $\mathrm{Gal}(\mathrm{Num}_F^\dagger(k))$
- super-Galois group of super-Tannakian category $s\mathrm{Gal}(\mathrm{NNum}_F(k))$ compare with commutative motives case $s\mathrm{Gal}(\mathrm{Num}_F(k))$
- related question: what are truly noncommutative motives?

Tate triples (Deligne–Milne)

• For $A = \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{G}_m$ or μ_2, Tannakian cat $\mathscr{C}$ with A-grading: A-grading on objects with $(X \otimes Y)^a = \oplus_{a=b+c} X^b \otimes Y^c$; homom $w : B \to \underline{\mathrm{Aut}}^{\otimes}(id_{\mathscr{C}})$ (weight); central hom $B \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$

• Tate triple $(\mathscr{C}, w, T)$: $\mathbb{Z}$-graded Tannakian $\mathscr{C}$ with weight w, invertible object T (Tate object) weight -2

• Tate triple $\Rightarrow$ central homom $w : \mathbb{G}_m \to \mathrm{Gal}(\mathscr{C})$ and homom $t : \mathrm{Gal}(\mathscr{C}) \to \mathbb{G}_m$ with $t \circ w = -2$.

• $H = Ker(t : \mathrm{Gal}(\mathscr{C}) \to \mathbb{G}_m)$ defines Tannakian category $\simeq \mathrm{Rep}(H)$. It is the "quotient Tannakian category" (Milne) of inclusion of subcategory gen by Tate object into $\mathscr{C}$

Galois group and orbit category

• $\mathscr{T} = (\mathscr{C}, w, T)$ Tate triple, $\mathscr{S} \subset \mathscr{C}$ gen by T, pseudo-ab envelope $(\mathscr{C}/_{-\otimes T})^\natural$ of orbit cat $\mathscr{C}/_{-\otimes T}$ is neutral Tannakian with

$$\mathrm{Gal}((\mathscr{C}/_{-\otimes T})^\natural) \simeq \mathrm{Ker}(t : \mathrm{Gal}(\mathscr{C}) \twoheadrightarrow \mathbb{G}_m)$$

• Quotient Tannakian categories with resp to a fiber functor (Milne): $\omega_0 : \mathscr{S} \to \mathrm{Vect}(F)$ then $\mathscr{C}/\omega_0$ pseudo-ab envelope of $\mathscr{C}'$ with same objects as $\mathscr{C}$ and morphisms $\mathrm{Hom}_{\mathscr{C}'}(X, Y) = \omega_0(\underline{Hom}_{\mathscr{C}}(X, Y)^H)$ with X^H largest subobject where H acts trivially

• fiber functor $\omega_0 : X \mapsto \mathrm{colim}_n \mathrm{Hom}_{\mathscr{C}}(\oplus_{r=-n}^n \mathbf{1}(r), X) \in \mathrm{Vect}(F)$
$\Rightarrow$ get $\mathscr{C}' = \mathscr{C}/_{-\otimes T}$

super-Tannakian case: super Tate triples

- Need a super-Tannakian version of Tate triples

- super Tate triple: $\mathscr{S}\mathscr{T} = (\mathscr{C}, \omega, \underline{\pi}^{\pm}, \mathscr{T}^{\dagger})$ with $\mathscr{C}$ = neutral super-Tannakian; $\omega : \mathscr{C} \to s\mathrm{Vect}(F)$ super-fiber functor; idempotent endos: $\omega(\underline{\pi}_X^{\pm}) = \pi_X^{\pm}$ Künneth proj.; neutral Tate triple $\mathscr{T}^{\dagger} = (\mathscr{C}^{\dagger}, w, T)$ with $\mathscr{C}^{\dagger}$ modified symmetry constraint from $\mathscr{C}$ using $\underline{\pi}^{\pm}$

- assuming C and D: a super Tate triple for (comm) num motives

$$(\mathrm{Num}_k(k), \overline{sH^*_{dR}}, \underline{\pi}_X^{\pm}, (\mathrm{Num}_k^{\dagger}(k), w, \mathbb{Q}(1)))$$

super-Tannakian case: orbit category

• $\mathscr{S}\mathscr{T} = (\mathscr{C}, \omega, \underline{\pi}^{\pm}, \mathscr{T}^{\dagger})$ super Tate triple; $\mathscr{S} \subset \mathscr{C}$ full neutral super-Tannakian subcat gen by T

• Assume: $\underline{\pi}^{-}_{T}(T) = 0$; for $K = \mathrm{Ker}(t : \mathrm{Gal}(\mathscr{C}^{\dagger}) \to \mathbb{G}_m)$ of Tate triple $\mathscr{T}^{\dagger}$, if $\epsilon : \mu_2 \to H$ induced $\mathbb{Z}/2\mathbb{Z}$ grading from $t \circ w = -2$; then (H, ϵ) super-affine group scheme is Ker of $s\mathrm{Gal}(\mathscr{C}) \to s\mathrm{Gal}(\mathscr{S})$ and $\mathrm{Rep}_F(H, \epsilon) = \mathrm{Rep}^{\dagger}_F(H)$.

• Conclusion: pseudoabelian envelope of $\mathscr{C}/_{-\otimes T}$ is neutral super-Tannakian and seq of exact $\otimes$-functors $\mathscr{S} \subset \mathscr{C} \to (\mathscr{C}/_{-\otimes T})^{\natural}$ gives

$$s\mathrm{Gal}((\mathscr{C}/_{-\otimes T})^{\natural}) \xrightarrow{\sim} \mathrm{Ker}(t : s\mathrm{Gal}(\mathscr{C}) \to \mathbb{G}_m)$$

• have also $(\mathscr{C}^{\dagger}/_{-\otimes T})^{\natural} \simeq (\mathscr{C}/_{-\otimes T})^{\natural,\dagger} \simeq \mathrm{Rep}^{\dagger}_F(H, \epsilon) \simeq \mathrm{Rep}_F(H)$

Then for Galois groups:

• then surjective $\mathrm{Gal}(\mathrm{NNum}^\dagger_k(k)) \twoheadrightarrow \mathrm{Gal}((\mathrm{Num}^\dagger_k(k)/_{-\otimes\mathbb{Q}(1)})^\natural)$ from embedding of subcategory and
$\mathrm{Gal}((\mathrm{Num}^\dagger_k(k)/_{-\otimes\mathbb{Q}(1)})^\natural) = \mathrm{Ker}(t : \mathrm{Num}^\dagger_k(k) \to \mathbb{G}_m)$

• for super-Tannakian: surjective (from subcategory)
$s\mathrm{Gal}(\mathrm{NNum}_k(k)) \twoheadrightarrow s\mathrm{Gal}((\mathrm{Num}_k(k)/_{-\otimes\mathbb{Q}(1)})^\natural)$ and
$s\mathrm{Gal}((\mathrm{Num}_k(k)/_{-\otimes\mathbb{Q}(1)})^\natural) \simeq \mathrm{Ker}(t : s\mathrm{Gal}(\mathrm{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$

• What is kernel? Ker = “truly noncommutative motives"

$$\mathrm{Gal}(\mathrm{NNum}^\dagger_k(k)) \twoheadrightarrow \mathrm{Ker}(t : \mathrm{Num}^\dagger_k(k) \to \mathbb{G}_m)$$

$$s\mathrm{Gal}(\mathrm{NNum}_k(k)) \twoheadrightarrow \mathrm{Ker}(t : s\mathrm{Gal}(\mathrm{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$$

what do they look line? examples? general properties?