Neural Codes and Neural Rings: Topology and Algebraic Geometry

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Ma191b: Geometry of Neuroscience

References for this lecture:

- Curto, Carina; Itskov, Vladimir; Veliz-Cuba, Alan; Youngs, Nora, *The neural ring: an algebraic tool for analyzing the intrinsic structure of neural codes*, Bull. Math. Biol. 75 (2013), no. 9, 1571–1611.
- Nora Youngs, *The Neural Ring: using Algebraic Geometry to analyze Neural Codes*, arXiv:1409.2544
- Yuri Manin, *Neural codes and homotopy types: mathematical models of place field recognition*, Mosc. Math. J. 15 (2015), no. 4, 741–748
- Carina Curto, Nora Youngs, *Neural ring homomorphisms and maps between neural codes*, arXiv:1511.00255
- Elizabeth Gross, Nida Kazi Obatake, Nora Youngs, *Neural ideals and stimulus space visualization*, arXiv:1607.00697
- Yuri Manin, *Error-correcting codes and neural networks*, Selecta Math. (N.S.) 24 (2018), no. 1, 521–530

Basic setting

- set of neurons $[n] = \{1, \dots, n\}$
- *neural code* $\mathcal{C} \subset \mathbb{F}_2^n$ with $\mathbb{F}_2 = \{0, 1\}$
- *codewords* (or "codes") $\mathcal{C} \ni c = (c_1, \dots, c_n)$ describe activation state of neurons
- support $\mathrm{supp}(c) = \{i \in [n] \,:\, c_i = 1\}$

$$\mathrm{supp}(\mathcal{C}) = \cup_{c \in \mathcal{C}} \mathrm{supp}(c) \subset 2^{[n]}$$

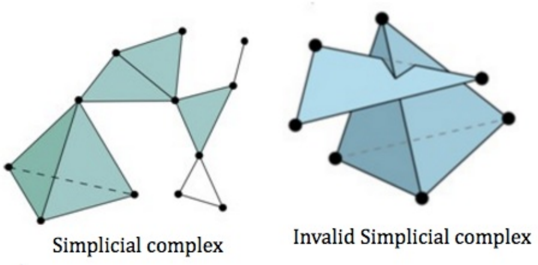
  $2^{[n]}$ = set of all subsets of $[n]$
- neglect information about timing and rate of neural activity: focus on combinatorial neural code

Simplicial complex of the code

- $\Delta \subset 2^{[n]}$ simplicial complex if when $\sigma \in \Delta$ and $\tau \subset \sigma$ then also $\tau \in \Delta$
- neural code $\mathcal{C}$ simplicial if $\mathrm{supp}(\mathcal{C})$ simplicial complex
- if not, define simplicial complex of the neural code $\mathcal{C}$ as

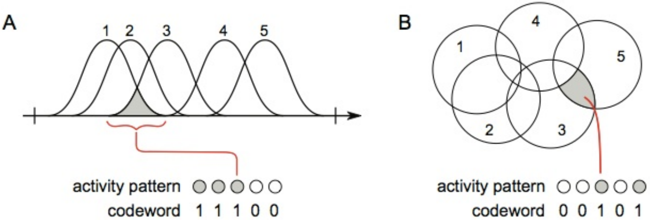
$$\Delta(\mathcal{C}) = \{\sigma \subset [n] \, : \, \sigma \subseteq \mathrm{supp}(c), \quad \text{for some } c \in \mathcal{C}\}$$

smallest simplicial complex containing $\mathrm{supp}(\mathcal{C})$



Receptive fields

- patterns of neuron activity
- maps $f_i : X \to \mathbb{R}_+$ from space X of stimuli: average firing rate of i-th neuron in $[n]$ in response to stimulus $x \in X$
- open sets $U_i = \{x \in X \ : \ f(x) > 0\}$ (receptive fields) usually assume *convex*
- *place field* of a neuron $i \in [n]$: preferred convex region of the stimulus space where it has a high firing rate (orientation-selective neurons: tuning curves, preference for particular angle, intervals on a circle)
- code words from receptive fields overlap



Convex Receptive Field Code

- stimulis space X; set of neurons $[n] = \{1, \ldots, n\}$; receptive fields $f_i : X \to \mathbb{R}_+$, with convex sets $U_i = \{f_i > 0\}$
- collection of (convex) open sets $\mathcal{U} = \{U_1, \ldots, U_n\}$
- *receptive field code*

$$\mathcal{C}(\mathcal{U}) = \{c \in \mathbb{F}_2^n \,:\, \left(\cap_{i\in \mathrm{supp}(c)} U_i\right) \smallsetminus \left(\cup_{j\notin \mathrm{supp}(c)} U_j\right) \neq \emptyset\}$$

  all binary codewords corresponding to stimuli in X
- with convention: intersection over $\emptyset$ is X and union over $\emptyset$ is $\emptyset$
- if $\cup_{i\in[n]} U_i \subsetneq X$: there are points of stimulus space not covered by receptive field (word $c = (0, 0, \ldots, 0)$ in $\mathcal{C}$); if $\cap_{i\in[n]} U_i \neq \emptyset$ word $c = (1, 1, \ldots, 1) \in \mathcal{C}$ points where all neurons activated

Main Question

• if know the code $\mathcal{C} = \mathcal{C}(\mathcal{U})$ without knowing X and $\mathcal{U}$ what can you learn about the geometry of X? (to what extent X is reconstructible from $\mathcal{C}(\mathcal{U})$)

• Step One: given a code $\mathcal{C} \subset \mathbb{F}_2^n$ with $m = \#\mathcal{C}$ (number of code words) there exists an $X \subseteq \mathbb{R}^d$ and a collection of (not necessarily convex) open sets $\mathcal{U} = \{U_1, \dots, U_n\}$ with $U_i \subset X$ such that $\mathcal{C} = \mathcal{C}(\mathcal{U})$

- list code words $c_i = (c_{i,1}, \dots, c_{i,n}) \in \mathcal{C}$, $i = 1, \dots, m$
- for each code word c_i choose a point $x_{c_i} \in \mathbb{R}^d$ and an open neighborhood $\mathcal{N}_i \ni x_{c_i}$ such that $\mathcal{N}_i \cap \mathcal{N}_j = \emptyset$ for $i \neq j$
- take $\mathcal{U} = \{U_1, \dots, U_n\}$ and $X = \cup_{j=1}^m \mathcal{N}_j$ with

$$U_j = \bigcup_{c_k \,:\, j \in \mathrm{supp}(c_k)} \mathcal{N}_k$$

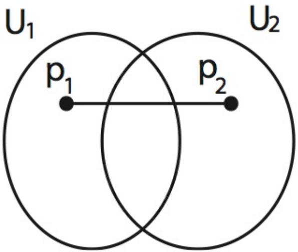
- if zero code word in $\mathcal{C}$ then $\mathcal{N}_0 = X \smallsetminus \cup_j U_j$ is set of outside points not captured by code
- by construction $\mathcal{C} = \mathcal{C}(\mathcal{U})$

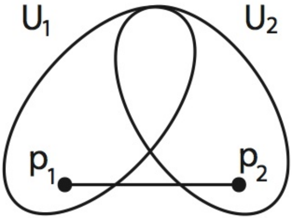
Caveat

• can always find a $(X, \mathcal{U})$ given $\mathcal{C}$ so that $\mathcal{C} = \mathcal{C}(\mathcal{U})$ but not always with U_i *convex*

• Example: $\mathcal{C} = \mathbb{F}_2^3 \setminus \{(1,1,1),(0,0,1)\}$ cannot be realized by a $\mathcal{U} = \{U_1, U_2, U_3\}$ with U_i convex

- suppose possible: $U_i \subset \mathbb{R}^d$ convex and $\mathcal{C} = \mathcal{C}(\mathcal{U})$
- know that $U_1 \cap U_2 \neq \emptyset$ because $(1,1,0) \in \mathcal{C}$
- know that $(U_1 \cap U_3) \setminus U_2 \neq \emptyset$ because $(1,0,1) \in \mathcal{C}$
- know that $(U_2 \cap U_3) \setminus U_1 \neq \emptyset$ because $(0,1,1) \in \mathcal{C}$
- take points $p_1 \in (U_1 \cap U_3) \setminus U_2$ and $p_2 \in (U_2 \cap U_3) \setminus U_1$ both in U_3 convex, so segment $\ell = tp_1 + (1-t)p_2$, $t \in [0,1]$ in U_3
- if ℓ passes through $U_1 \cap U_2$ then $U_1 \cap U_2 \cap U_3 \neq \emptyset$ but $(1,1,1) \notin \mathcal{C}$ (contradiction)
- or ℓ does not intersect $U_1 \cap U_2$ but then ℓ intersects the complement of $U_1 \cup U_2$ (see fig) this would imply $(0,0,1) \in \mathcal{C}$ (contradiction)





the two cases of the previous example

Constraints on the Stimulus Space

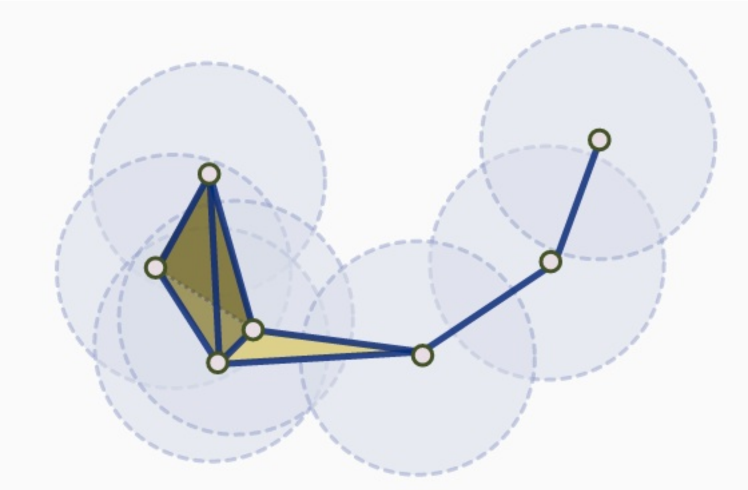
• Codes $\mathcal{C}$ that can be realized as $\mathcal{C} = \mathcal{C}(\mathcal{U})$ with U_i convex put strong constraints on the geometry of the stimulus space X

two types of constraints

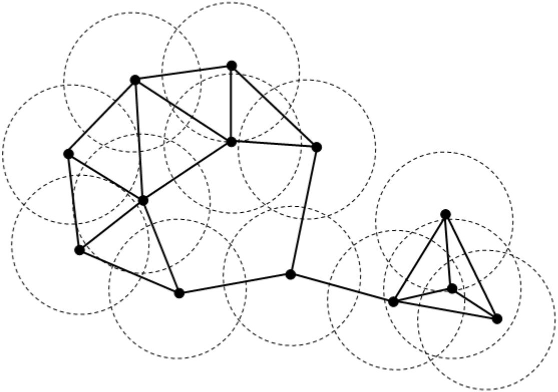
1. constraints from the simplicial complex $\Delta(\mathcal{C})$
2. other constraints from $\mathcal{C}$ not captured by $\Delta(\mathcal{C})$

Simplicial nerve of an open covering

- $\mathcal{U} = \{U_1, \ldots, U_n\}$ convex open sets in $\mathbb{R}^d$ with $d < n$
- *nerve* $\mathcal{N}(\mathcal{U})$ simplicial complex: $\sigma = \{i_1, \ldots, i_k\} \in 2^{[n]}$ is in $\mathcal{N}(\mathcal{U})$ iff $U_{i_1} \cap \cdots \cap U_{i_k} \neq \emptyset$
- $\mathcal{N}(\mathcal{U}) = \Delta(\mathcal{C}(\mathcal{U}))$



convex open sets U_i and simplicial nerve $\mathcal{N}(\mathcal{U})$

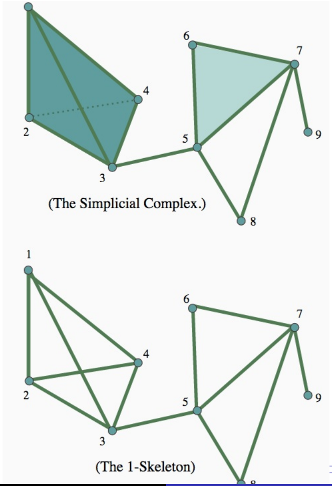


another example of convex open sets U_i and simplicial nerve $\mathcal{N}(\mathcal{U})$

The complex $\mathcal{N}(\mathcal{U})$ is also known as the Čech complex of the collection $\mathcal{U} = \{U_1, \ldots, U_n\}$ of convex open sets

• Topological fact (Helly's theorem): convex $U_1, \ldots, U_k \subset \mathbb{R}^d$ with $d < k$: if intersection of every $d+1$ of the U_i nonempty then also $\cap_{i=1}^k U_i \neq \emptyset$

Consequence: the nerve $\mathcal{N}(\mathcal{U})$ completely determined by its d-skeleton (largest n-complex with that given d-skeleton)



Nerve Theorem

• Allen Hatcher *Algebraic topology*, Cambridge University Press, 2002 (Corollary 4G.3)

• Homotopy types: The homotopy type of $X(\mathcal{U}) = \cup_{i=1}^n U_i$ is the same as the homotopy type of the nerve $\mathcal{N}(\mathcal{U})$

• Consequence: $X(\mathcal{U})$ and $\mathcal{N}(\mathcal{U})$ have the same homology and homotopy groups (but not necessarily the same dimension)

• Note: the space $X(\mathcal{U})$ may not capture all of the stimulus space X if the U_i are not an open covering of X, that is, if $X \smallsetminus X(\mathcal{U}) \neq \emptyset$

Homology groups

• very useful topological invariants, computationally tractable

• simplicial complex $\mathcal{N} \subset 2^{[n]}$; groups of k-chains $C_k = C_k(\mathcal{N})$ abelian group spanned by k-dimensional simplices of $\mathcal{N}$

• boundary maps on simplicial complexes $\partial_k : C_k \to C_{k-1}$

$$\partial_{k-1} \circ \partial_k = 0$$

usually stated as $\partial^2 = 0$

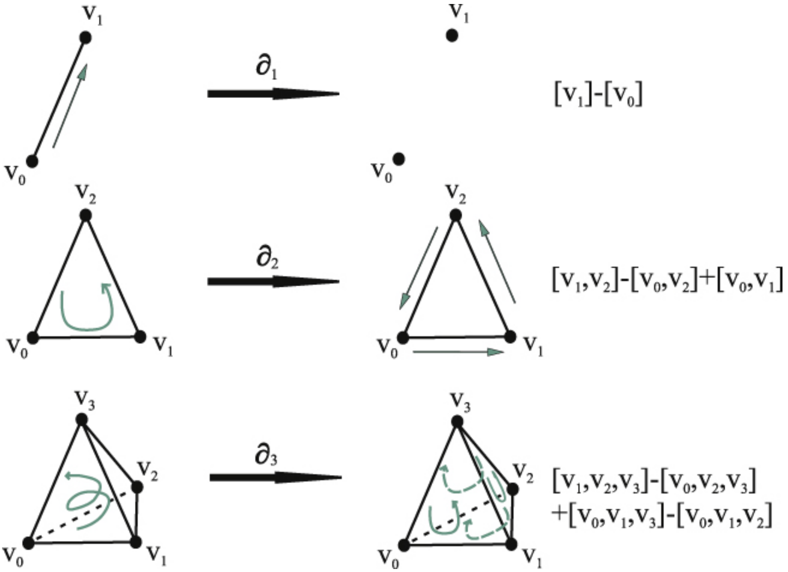
• cycles $Z_k = \mathrm{Ker}(\partial_k) \subset C_k$ and boundaries $B_{k+1} = \mathrm{Range}(\partial_{k+1}) \subset C_k$

• because $\partial^2 = 0$ inclusion $B_{k+1} \subset Z_k$

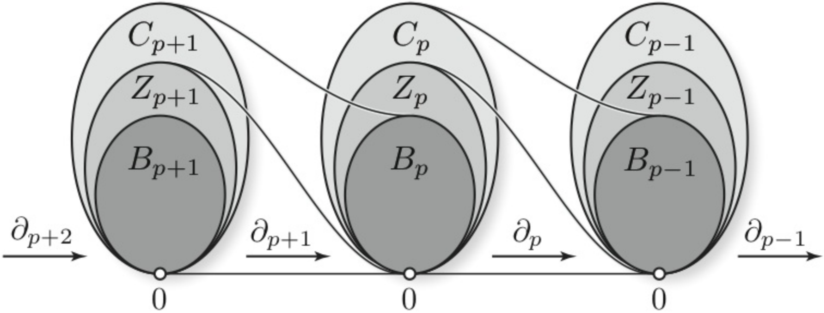
• homology groups: quotient groups

$$H_k(\mathcal{N}, \mathbb{Z}) = \frac{\mathrm{Ker}(\partial_k)}{\mathrm{Range}(\partial_{k+1})} = Z_k / B_{k+1}$$

Boundary maps



Chain complexes and Homology



$$H_p(X, \mathbb{Z}) = \mathrm{Ker}(\partial_p : C_p \to C_{p-1})/\mathrm{Im}(\partial_{p+1} : C_{p+1} \to C_p)$$

What else does $\mathcal{C}$ tells us about X?

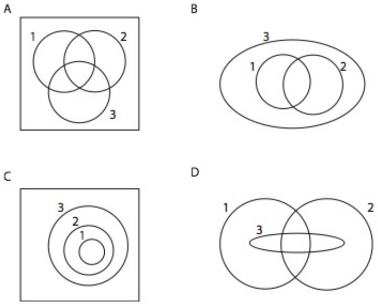


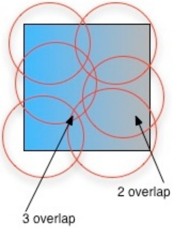
Figure 3: Four arrangements of three convex receptive fields, $\mathcal{U} = \{U_1, U_2, U_3\}$, each having $\Delta(\mathcal{C}(\mathcal{U})) = 2^{[3]}$. Square boxes denote the stimulus space X in cases where $U_1 \cup U_2 \cup U_3 \subsetneq X$. (A) $\mathcal{C}(\mathcal{U}) = 2^{[3]}$, including the all-zeros codeword 000. (B) $\mathcal{C}(\mathcal{U}) = \{111, 101, 011, 001\}$, with $X = U_3$. (C) $\mathcal{C}(\mathcal{U}) = \{111, 011, 001, 000\}$. (D) $\mathcal{C}(\mathcal{U}) = \{111, 101, 011, 110, 100, 010\}$, and $X = U_1 \cup U_2$. The minimal embedding dimension for the codes in panels A and D is $d = 2$, while for panels B and C it is $d = 1$.

all have same $\Delta(\mathcal{C}) = 2^{[3]}$ because $(1, 1, 1)$ code word for all cases

Embedding dimension

• *minimal embedding dimension d*: minimal dimension for which code $\mathcal{C}$ can be realized as $\mathcal{C}(\mathcal{U})$ with open sets $U_i \subset \mathbb{R}^d$

• *topological dimension*: minimum d such that any open covering has a refinement such that no point is in more than $d+1$ open sets of the covering



• in previous examples $\Delta(\mathcal{C}) = 2^{[3]}$ same but different *embedding dimension*

Main information carried by the code $\mathcal{C} = \mathcal{C}(\mathcal{U})$:
nontrivial inclusions

• some inclusion relations between intersections and unions always trivially satisfied: example $U_1 \cap U_2 \subset U_2 \cup U_3$ because $U_1 \cap U_2 \subset U_2$

• other inclusion relations are *specific* of the structure of the collection $\mathcal{U}$ of open sets and not always automatically satisfied: this is the *information* encoded in $\mathcal{C}(\mathcal{U})$

• all relations of the form

$$\bigcap_{i \in \sigma} U_i \subseteq \bigcup_{j \in \tau} U_j$$

for $\sigma \cap \tau = \emptyset$, including all empty intersections relations

$$\bigcap_{i \in \sigma} U_i = \emptyset$$

Problem: how to algorithmically extract this information from $\mathcal{C}$ without having to construct $\mathcal{U}$?

• key method: Algebraic Geometry (ideals and varieties)

• Rings and ideals: R commutative ring with unit, $I \subset R$ ideal (additive subgroup; for $a \in I$ and for all $b \in R$ product $ab \in I$)

• set S generators of $I = \langle S \rangle$

$$I = \{r_1 a_1 + \cdots + r_n a_n \; : \; r_i \in R, \, a_i \in S, \, n \in \mathbb{N}\}$$

• *prime ideal*: $\wp \subsetneq R$ and if $ab \in \wp$ then $a \in \wp$ or $b \in \wp$

• *maximal ideal*: $\mathfrak{m} \subsetneq R$ and if I ideal $\mathfrak{m} \subset I \subset R$ then either $\mathfrak{m} = I$ or $I = R$ (geometrically maximal ideals correspond to points)

• *radical ideal*: $r^n \in I$ implies $r \in I$ for all n

• prime decomposition: radical $I = \wp_1 \cap \cdots \cap \wp_n$ with $\wp_i$ prime ideals

Affine Algebraic Varieties

• polynomial ring $R = K[x_1, \ldots, x_n]$ over a field K; $I \subset R$ ideal $\Rightarrow$ variety $V(I)$

$$V(I) = \{v \in K^n \,:\, f(v) = 0, \forall f \in I\}$$

• ideals $I \subseteq J \Rightarrow$ varieties $V(J) \subseteq V(I)$

• *spectrum* of a ring R: set of prime ideals

$$\mathrm{Spec}(R) = \{\wp \subset R \,:\, \wp \text{ prime ideal } \}$$

• modeling n neurons with binary states on/off, so $K = \mathbb{F}_2 = \{0, 1\}$ and $v = (v_1, \ldots, v_n) \in \mathbb{F}_2^n$ a possible state of the set of neurons

Neural Ring

- given a binary code $\mathcal{C} \subset \mathbb{F}_2^n$ (neural code)
- ideal $I = I_{\mathcal{C}} \subset \mathbb{F}_2[x_1, \ldots, x_n]$ of polynomials vanishing on codewords

$$I_{\mathcal{C}} = \{f \in \mathbb{F}_2[x_1, \ldots, x_n] \,:\, f(c) = 0, \, \forall c \in \mathcal{C}\}$$

- quotient ring (neural ring)

$$R_{\mathcal{C}} = \mathbb{F}_2[x_1, \ldots, x_n]/I_{\mathcal{C}}$$

• Note: working over $\mathbb{F}_2$ so $2 \equiv 0$, with relations $x_i(1 - x_i)$, so in $R_{\mathcal{C}}$ all elements idempotent $y^2 = y$ (cross terms vanish): Boolean ring isomorphic to $\mathbb{F}_2^{\#\mathcal{C}}$, but useful to keep the explicit coordinate functions x_i that measure the activity of the i-th neuron

Neural Ring Spectrum

• maximal ideals in polynomial ring $\mathbb{F}_2[x_1, \ldots, x_n]$ correspond to points $v \in \mathbb{F}_2^n$, namely

$$\mathfrak{m}_v = \langle x_1 - v_1, \ldots, x_n - v_n \rangle$$

• in a Boolean ring prime ideal spectrum and maximal ideal spectrum coincide

• for the neural ring $R_{\mathcal{C}}$ spectrum

$$\mathrm{Spec}(R_{\mathcal{C}}) = \{\bar{\mathfrak{m}}_v \,:\, v \in \mathcal{C} \subset \mathbb{F}_2^n\}$$

where $\bar{\mathfrak{m}}_v$ image in quotient ring of maximal ideal $\mathfrak{m}_v$ in $\mathbb{F}_2[x_1, \ldots, x_n]$

• so spectrum of the neural ring recovers the code words of $\mathcal{C}$

Neural ideal

• in general difficult to provide explicit generators for the ideal $I_{\mathcal{C}}$ (problem for practical computational purposes)

• another closely related (more tractable) ideal: neural ideal $J_{\mathcal{C}}$

• given $v \in \mathbb{F}_2^n$ (a possible state of a system of n neurons) take function

$$\rho_v = \prod_{i=1}^{n} (1 - v_i - x_i) = \prod_{i \in \mathrm{supp}(v)} x_i \prod_{j \notin \mathrm{supp}(v)} (1 - x_j)$$

$\rho_v \in \mathbb{F}_2[x_1, \ldots, x_n]$

• binary code $\mathcal{C} \subset \mathbb{F}_2^n \Rightarrow$ ideal $J_{\mathcal{C}}$

$$J_{\mathcal{C}} = \langle \rho_v \, : \, v \notin \mathcal{C} \rangle$$

when $\mathcal{C} = \mathbb{F}_2^n$ have $J_{\mathcal{C}} = 0$ trivial ideal

- ideal of Boolean relations $\mathcal{B} = \mathcal{B}_n$

$$\mathcal{B} = \langle x_i(1 - x_i) \, : \, i \in [n] \rangle$$

- relation between ideals $I_{\mathcal{C}}$ and $J_{\mathcal{C}}$

$$I_{\mathcal{C}} = J_{\mathcal{C}} + \mathcal{B} = \langle \rho_v, \; x_i(1 - x_i) \, : \, v \notin \mathcal{C}, \, i \in [n] \rangle$$

Neural Ring Relations

- Notation: given $\mathcal{U} = \{U_1, \ldots, U_n\}$ open sets and $\sigma \subset [n]$

$$U_\sigma := \cap_{i\in\sigma} U_i, \quad x_\sigma := \prod_{i\in\sigma} x_i, \quad (1 - x_\tau) := \prod_{j\in\tau} (1 - x_j)$$

- interpret coordinates x_i as functions on X:

$$x_i(p) = \begin{cases} 1 & p \in U_i \\ 0 & p \notin U_i \end{cases}$$

- inclusions and relations: $U_\sigma \subset U_i \cup U_j$, then $x_\sigma = 1$ implies either $x_i = 1$ or $x_j = 1$ so relation

$$x_\sigma (1 - x_i)(1 - x_j)$$

- all inclusion $U_\sigma \subseteq \cup_{i\in\tau} U_i$ correspond to relations $x_\sigma \prod_{i\in\tau}(1 - x_i)$
- ideal $I_{\mathcal{C}(\mathcal{U})}$ generated by them (relations defining $R_{\mathcal{C}}$)

$$I_{\mathcal{C}(\mathcal{U})} = \langle x_\sigma \prod_{i\in\tau} (1 - x_i) \; : \; U_\sigma \subseteq \cup_{i\in\tau} U_i \rangle$$

Canonical Form pseudomonomial relations

• subsets $\sigma, \tau \subset [n]$: if $\sigma \cap \tau \neq \emptyset$ then $x_\sigma(1 - x_\tau) \in \mathcal{B}$, if $\sigma \cap \tau = \emptyset$ then $x_\sigma(1 - x_\tau) \in J_\mathcal{C}$

• functions of the form $f(x) = x_\sigma(1 - x_\tau)$ with $\sigma \cap \tau = \emptyset$ *pseudomonomial*; ideal J generated by such: *pseudomonomial ideal*

• *minimal pseudomonomial*: $f \in J$ pseudomonomial, no other pseudomonomial g with $\deg(g) < \deg(f)$ and $f = gh$ for some $h \in \mathbb{F}_2[x_1, \dots, x_n]$

• *canonical form* of pseudomonomial ideal $J = \langle f_1, \dots, f_\ell \rangle$ with f_k all the minimal pseudomonomials in J

• ideal $J_\mathcal{C} = \langle \rho_v \, : \, v \notin \mathcal{C} \rangle$ is pseudomonomial (not $I_\mathcal{C}$ because of Boolean relations)

Canonical Form of Neural Ring $J_{\mathcal{C}}$: $CF(J_{\mathcal{C}})$

• given a binary code $\mathcal{C} \subset \mathbb{F}_2^n$ suppose realized as $\mathcal{C} = \mathcal{C}(\mathcal{U})$ with $\mathcal{U} = \{U_1, \ldots, U_n\}$ in X (not necessarily convex)

• some $\sigma \subseteq [n]$ minimal for a property P if P satisfied by σ and not satisfied by any $\tau \subsetneq \sigma$

• canonical form $CF(J_{\mathcal{C}})$ of $J_{\mathcal{C}}$ three types of relations:

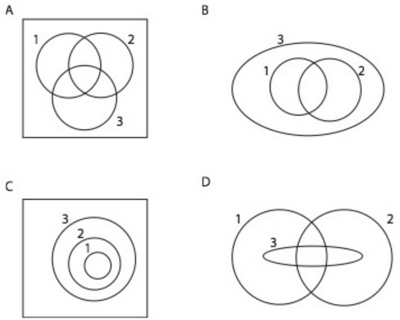
1. x_σ with σ minimal for $U_\sigma = \emptyset$
2. $x_\sigma(1 - x_\tau)$ with $\sigma \cap \tau =$, $U_\sigma \neq \emptyset \cup_{i\in\tau} U_i \neq X$, and σ, τ minimal for $U_\sigma \subseteq \cup_{i\in\tau} U_i$
3. $(1 - x_\tau)$ with τ minimal for $X \subseteq \cup_{i\in\tau} U_i$

• minimal embedding dimension

$$d \geq \max_{\sigma : x_\sigma \in CF(J_{\mathcal{C}})} \#\sigma - 1$$

• there are efficient algorithms to compute $CF(J_{\mathcal{C}})$ given $\mathcal{C}$ (without passing through $\mathcal{U}$)

Example



A. $CF(J_{\mathcal{C}}) = \{0\}$. There are no relations here because $\mathcal{C} = 2^{[3]}$.

B. $CF(J_{\mathcal{C}}) = \{1 - x_3\}$. This Type 3 relation reflects the fact that $X = U_3$.

C. $CF(J_{\mathcal{C}}) = \{x_1(1 - x_2), x_2(1 - x_3), x_1(1 - x_3)\}$. These Type 2 relations correspond to $U_1 \subset U_2$, $U_2 \subset U_3$, and $U_1 \subset U_3$. Note that the first two of these receptive field relationships imply the third; correspondingly, the third canonical form relation satisfies: $x_1(1 - x_3) = (1 - x_3) \cdot [x_1(1 - x_2)] + x_1 \cdot [x_2(1 - x_3)]$.

D. $CF(J_{\mathcal{C}}) = \{(1 - x_1)(1 - x_2)\}$. This Type 3 relation reflects $X = U_1 \cup U_2$, and implies $U_3 \subset U_1 \cup U_2$.

How good as codes are neural codes?

- how does one evaluate properties of codes in coding theory?
- codes and code parameters, bounds
- neural codes as error correcting codes
- neural mechanisms passing from bad to good codes (Chaudhuri–Fiete)
- expander graphs and codes
- Hopfield equations and hyperplane arrangements
- Hopfield networks
- expander codes and Hopfield networks

When is a code a good code?

- view error correcting codes as an optimization problem
  - optimize encoding: more choice of code words make for better encoding
  - optimize decoding: sparse code words make for better decoding (better error correction: only one true code word near a corrupted one)

- alphabet $A = \mathbb{F}_2$ (for binary codes), code $C \subset \mathbb{F}_2^n$ (length n of code words), $x = (x_1, \ldots, x_n) \in C$ code words

- *unstructured*: don't necessarily require that the code is linear ($C \subset \mathbb{F}_2^n$ not necessarily an $\mathbb{F}_2$-vector space)

Code parameters

- $k = k(C) := \log_2 \# C$ and $[k] = [k(C)]$ integer part of $k(C)$
$$2^{[k]} \leq \# C = 2^k < 2^{[k]+1}$$
- *Hamming distance*: $x = (a_i)$ and $y = (b_i)$ in C
$$d((a_i),(b_i)) := \#\{i \in (1,\dots,n) \,|\, a_i \neq b_i\}$$
- *Minimal distance* $d = d(C)$ of the code
$$d(C) := \min\{d(a,b) \,|\, a,b \in C, a \neq b\}$$
- code parameters:
  - $R = k/n =$ *transmission rate* of the code
  - $\delta = d/n =$ *relative minimum distance* of the code

Small R: fewer code words, easier decoding, but longer encoding signal; small δ: too many code words close to received one, more difficult decoding.

- Optimization problem: increase both R and δ... how good can codes be?

Bounds in the space of code parameters

• code points $(R(C), \delta(C))$ in square $[0,1]^2$

• there is a tension between optimizing R and δ, which can be seen in several bounds

• singleton bound: $R + \delta \leq 1$

• typical random codes (Shannon Random Code Ensemble: code words and letter generated uniformly and randomly as i.i.d. random variables) tend to *accumulate in the region below the Gilbert–Varshamov curve*

• Gilbert–Varshamov curve: $R = \frac{1}{2}(1 - H_2(\delta))$ with q-ary entropy

$$H_q(\delta) = \delta \log_q(q-1) - \delta \log_q \delta - (1-\delta) \log_q(1-\delta)$$

• this comes from looking at the asymptotic behavior of volumes of balls in the Hamming distance when the code length $n \to \infty$, governed by the function $H_q(\delta)$

Volume estimate:

$$q^{(H_q(\delta)-o(1))n} \leq Vol_q(n, d = n\delta) = \sum_{j=0}^{d} \binom{n}{j} (q-1)^j \leq q^{H_q(\delta)n}$$

Gives probability of parameter δ for SRCE meets the GV bound with probability exponentially (in n) near 1: expectation

$$E \sim \binom{q^k}{2} Vol_q(n, d) q^{-n} \sim q^{n(H_q(\delta)-1+2R)+o(n)}$$

Asymptotic bound

• there is another curve in the space of code parameters: the asymptotic bound, existence was proved by Manin using spoiling operations on codes

- Yu.I.Manin, *What is the maximum number of points on a curve over* $\mathbb{F}_2$*?* J. Fac. Sci. Tokyo, IA, Vol. 28 (1981), 715–720.

• no explicit expression for the asymptotic bound $R = \alpha_q(\delta)$ (in fact question about the computability of this function because of relation to Kolmogorov complexity)

- Yu.I.Manin, *A computability challenge: asymptotic bounds and isolated error-correcting codes*, arXiv:1107.4246
- Yu.I. Manin, M. Marcolli, *Kolmogorov complexity and the asymptotic bound for error-correcting codes*, J. Differential Geom. 97 (2014), no. 1, 91–108

Estimates on the asymptotic bound

• known estimates about the asymptotic bound

- Plotkin bound:

$$\alpha_q(\delta) = 0, \quad \text{for} \quad \delta \geq \frac{q-1}{q}$$

- singleton bound: $R = \alpha_q(\delta)$ lies below $R + \delta = 1$
- Hamming bound:

$$\alpha_q(\delta) \leq 1 - H_q(\frac{\delta}{2})$$

- Gilbert–Varshamov bound:

$$\alpha_q(\delta) \geq 1 - H_q(\delta)$$

• no statistical description of the asymptotic bound unlike the Gilbert–Varshamov bound and random codes

Characterization of the asymptotic bound

• $R = \alpha_q(\delta)$ separates the region below where code points are dense and have infinite multiplicity and region above where code points are isolated and have finite multiplicity

- Yuri I. Manin, Matilde Marcolli, *Error-correcting codes and phase transitions*, Mathematics in Computer Science (2011) 5:133—170

Good codes: *codes near or above the asymptotic bound*

• main source of good codes: algebro-geometric codes

- M.A. Tsfasman, S.G. Vladut, Th. Zink, *Modular curves, Shimura curves, and Goppa codes, better than Varshamov-Gilbert bound*, Math. Nachr. 109 (1982) 21–28

• other unexpected sources of codes above the asymptotic bound: linguistics

- K.Shu, M.Marcolli, *Syntactic structures and code parameters*, Math. Comput. Sci. 11 (2017), no. 1, 79–90.
- M.Marcolli, *Syntactic parameters and a coding theory perspective on entropy and complexity of language families*, Entropy 18 (2016) no. 4, paper 110

Neural codes as error-correcting codes

- C. Curto, V. Itskov, K. Morrison, Z. Roth, J.L. Walker, *Combinatorial neural codes from a mathematical coding theory perspective*, Neural Comput. 25 (2013), no. 7, 1891–1925.

- comparison between error-correcting properties of neural codes and random codes of the same length and size
- probability of a neuron failing to fire in response to a stimulus is greater that the probability of a neuron firing in the absence of a stimulus
- so *binary asymmetric channel* (BAC) false-positive probability $0 < p < 1/2$ of 0 flipping to 1 and false-negative probability $0 < q < 1/2$ opposite flip, with $p \leq q$
- also assume error probability $p \leq s$ smaller than the *sparsity* s of the neural code (with $w(c) = \#\{i : c_i = 1\}$ weight)

$$s = s(C) := \frac{1}{\#C} \sum_{c \in C} \frac{w(c)}{n}$$

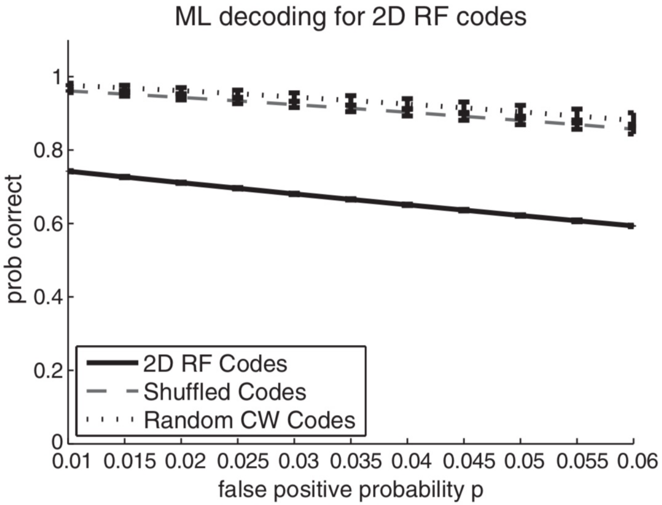
- overall probability of error in transmission $p(1-s)+qs$ (for words transmitted with equal probabilities)
- *maximum likelihood* decoder: $\mathbb{P}(r|c)$ probability of r received if c transmitted

$$c_{ML} = \mathrm{argmax}_{c\in C}\, \mathbb{P}(r|c)$$

for symmetric $p=q$ decoding equivalent to usual minimization of Hamming distance

$$c_{ML}^{p=q} = \mathrm{argmax}_{c\in C}\, d_H(c,r)$$

• observed performance for 2-dimensional receptor field neural codes compared to random codes



- neural codes have *high redundancy*: low values of the $R(C)$ transmission rate parameter $R(C) = n^{-1} \log_2 \#C$
- but also low values of the relative minimum distance $\delta(C)$: consider a pair $U_i \cap U_j \neq \emptyset$ of overlapping receptor fields open sets: code words c, c' with $c_i = 1$ and $c'_j = 1$ and there is a code word $\hat{c}$ with both $\hat{c}_i = \hat{c}_j = 1$, so achieve min of Hamming distance
- below the Gilbert-Varsamov line (where typical behavior of random codes is located)
- if an error-tolerance threshold introduced between original code word and decoded one, then error-correction more similar to random codes case

Most neural codes are *not* good codes, but "rely on" good codes

- Yuri I. Manin, *Error-correcting codes and neural networks*, Selecta Math. (N.S.) 24 (2018), no. 1, 521–530.

• models of encoding of stimulus space via error-correcting codes

• computing code parameters of typical neural codes corresponding to visual stimuli gives codes with very low positioning in the space of code parameters (not good codes): combinatorics of covering can make minimal Hamming distance between code words small

• auxiliary codes involved in the formation of "place maps" in the brain; experiments show these show signatures of "criticality"

- M. Nonnenmacher, Ch. Behrens, Ph. Berens, M. Bethge, J. Macke, *Signatures of criticality arise in simple neural models with correlations*, arXiv:1603.00097
- G. Tkacik, T. Mora, O. Marre, D. Amodei, S. Palmer, M. Berry, W. Bialekl, *Thermodynamics and signatures of criticality in a network of neurons*, Proc. Nat. Ac. Sci, 112(37):11, 2015, 508–517

• in this paper Manin makes a proposal that the "criticality" behavior of neural codes involved in place field maps is related to their position as "good codes" near the asymptotic bound

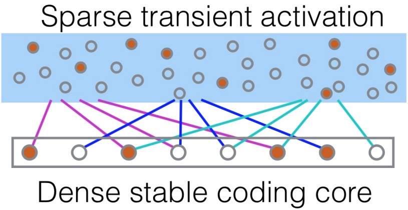
• the idea is based on a characterization of the asymptotic bound for error-correcting codes via Kolmogorov complexity and another characterization as a phase transition obtained in

- Yu.I. Manin, M. Marcolli, *Kolmogorov complexity and the asymptotic bound for error-correcting codes*, J. Differential Geom. 97 (2014), no. 1, 91–108
- Yuri I. Manin, Matilde Marcolli, *Error-correcting codes and phase transitions*, Mathematics in Computer Science (2011) 5:133—170

• in turn criticality can be characterized as behavior of a system near a phase transition and codes weighted by a probability distribution based on Kolmogorov complexity have a phase transition at the asymptotic bound bad to good neural codes via expaner graphs (Rishidev Chaudhuri and Ila Fiete)

- R. Chaudhuri, I. Fiete, *Bipartite expander Hopfield networks as self-decoding high-capacity error correcting codes*, 33rd Conference on Neural Information Processing Systems (NeurIPS 2019), Vancouver, Canada.

- neural mechanism from "bad codes" of open coverings of place fields to "good codes", via expander graphs
- Hopfield networks used as models for neural memory (error correction through dynamics)
- Hopfield networks with exponentially many robust stable states via expander codes



First Step: binary Hopfield networks

- binary Hopfield network: model of memory storage and retrieval in neuronal networks
- analogy with spin glass models in statistical physics
- population of N neurons S_i: each two possible states $S_i = \pm 1$ firing/not-firing (in a fixed time interval Δt)
- discrete dynamics (W_{ij} weight matrix assumed symmetric, $W_{ij} = W_{ji}$)

$$S_i(n+1) = \text{sign}\left(\sum_j W_{ij} S_j(n) \right)$$

- update rule implemented a single S_i at a time: either maintaining value or flipping

energy functional

$$\mathcal{E} = -\sum_{ij} W_{ij} S_i S_j$$

- Lyapunov function of the dynamics: decreases under flipping of a single S_i under update rule
- if $S_i(n+1) = -S_i(n)$ and $S_j(n+1) = S_j(n) = S_j$ for $j \neq i$

$$\mathcal{E}(n+1) - \mathcal{E}(n) = -2 \sum_{ji} W_{ji} S_j (S_i(n+1) - S_i(n))$$

$$= -4 \sum_{ji} W_{ji} S_j S_i(n+1) = -4 (\sum_{ji} W_{ji} S_j) \mathrm{sign}(\sum_{ji} W_{ji} S_j) < 0$$

with update rule (and symmetric W)

- so binary Hopfield dynamics flows toward energy minima

Memory storage and retrieval in binary Hopfield networks

- *idea*: choice of weight matrix W_{ij} determined by given set of patterns one wants to store in the network
- set of M patterns: binary strings $\{\pi_i^a\}_{a=1,\ldots,M,\, i=1,\ldots,N}$ satisfying $\sum_i \pi_i^a = 0$
- Hebbian learning rule: take weights of the form

$$W_{ij} = \frac{1}{N} \sum_{a=1}^{M} \pi_i^a \pi_j^a$$

- compare state S_i with one of the pattern π_i^a by measuring overlap

$$\mu^a = \frac{1}{N} \sum_i \pi_i^a S_i$$

max value $+1$ when complete match of S_i and π_i^a, min value -1 when opposite pattern (value ~ 0 low correlation)

- update rule of the binary Hopfield network

$$\sum_j W_{ij} S_j = \frac{1}{N} \sum_{j,a} \pi_i^a \pi_j^a S_j = \sum_a \pi_i^a \mu^a$$

- energy with minima at the M patters (where overlap maximum) and their opposites

$$\mathcal{E} = -N \sum_a (\mu^a)^2$$

- initializing dynamics close to one of the stored patterns will cause dynamics to reconstruct the pattern by flowing to corresponding energy minimum

- case of single pattern π_i: fixed point of dynamics

$$\mathrm{sign}(\sum_j W_{ij}\pi_j) = \mathrm{sign}(\frac{1}{N}\sum_j \pi_i\pi_j\pi_j) = \mathrm{sign}(\pi_i) = \pi_i$$

- but not always for multiple patterns:

$$\mathrm{sign}(\sum_j W_{ij}\pi_j^a) = \mathrm{sign}\left(\frac{1}{N}\sum_{j,b}\pi_i^b\pi_j^b\pi_j^a\right)$$

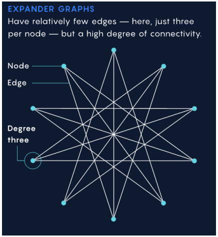
$$= \pi_i^a\ \mathrm{sign}\left(1 + \frac{1}{N}\sum_{j,b\neq a}\pi_i^b\pi_i^a\pi_j^b\pi_j^a\right)$$

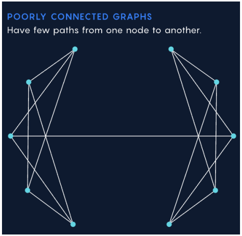
since $\sum_j \pi_j^a\pi_j^a = N$ and $\pi_i^a\pi_i^a = 1$

- continuous parameters Hopfield networks (we'll discuss later)

Second Step: Expander Graphs

- sparse graphs (few edges) but strong connectivity properties (many paths between nodes, short)
- both highly connected and sparse, exhibit pseudo-random behavior
- random walk through the graph gets lost quickly: good for stochastic and diffusion processes
- geometric property: every not "too large" subset of vertices has a "large" boundary ("expander")
- expander versus non-expander:





Expansion Constant

- two sets of vertices S, T of graph G then $E(S, T)$ set of edges of G one end-vertex in S the other in T
- $\partial S = E(S, G \smallsetminus S)$ "boundary" of the region with vertex set S
- *expansion constant* of G

$$h(G) := \min\{\frac{\# \partial S}{\# S} \,|\, S \subset V, \ \#S \leq n/2\}$$

- graph-theoretic analog of Cheeger constant for Riemannian manifolds

$$h(M) := \inf_Y \frac{V_{n-1}(X)}{\min\{V_n(M_1), V_n(M_2)\}},$$

infimum over all $(n-1)$-dimensional submanifolds $X \subset M$ that decompose $M = M_1 \cup_X M_2$

- $h(M)$ measures whether the manifold M has bottlenecks, so does $h(G)$
- related to spectrum of graph Laplacian

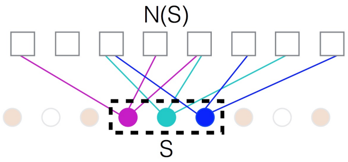
Expander Property: (γ, α)-expander graph

- $N = \#V(G)$, $\alpha > 1$, $0 < \gamma < 1$: for $S \subset V(G)$

$$\#S \leq \gamma N \implies \#\partial S \geq \alpha \# S$$

- often taken with $\gamma = 2$ as in $h(G)$ above
- a good expander graph has large expansion constant and low vertex degrees

Bipartite Expander Graphs:



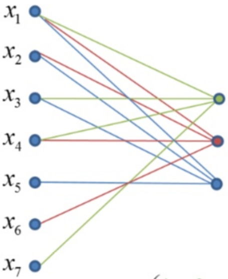
- $(\gamma, 1-\epsilon)$-expander: subsets $S \subset V(G)$ with $\deg(v) = z$ (fixed valence) and $\#S \leq \gamma N$ have

$$\#N(S) = \#\partial S > (1-\epsilon)\, z \, \#S$$

- in particular case of the (z, k)-biregular graphs $z =$input set valence, $k =$output set valence

Bipartite Graphs and Linear Codes

- binary $[n,k,d]_2$-code $C \subset \mathbb{F}_2^n$, linear if C linear subspace of $\mathbb{F}_2$-vector space $\mathbb{F}_2^n$, then $k = \dim_{\mathbb{F}_2}(C)$
- *parity-check matrix* H of C is an $(n-k) \times n$ matrix of rank $n-k$ over $\mathbb{F}_2$, with $C = \{x \in \mathbb{F}_2^n \,|\, Hx = 0\}$
- H provides description of linear code in terms of bipartite graph:



Code Constraints:

$$x_1 + x_3 + x_4 + x_7 = 0$$
$$x_1 + x_2 + x_4 + x_6 = 0$$
$$x_1 + x_2 + x_3 + x_5 = 0$$

$$H = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Tanner Codes

- start with a linear binary code $C_0 \subset \mathbb{F}_2^d$ with code parameters (R_0, δ_0) (not necessarily a good code)
- G a bipartite graph with n inputs m-outputs and right-d-regular (all output vertices same valence d)
- Tanner code $T(G, C_0) \subset \mathbb{F}_2^n$

$$T(G, C_0) = \{x = (x_v)_{v \in V_{in}(G)} \,|\, x|_{N(v')} \in C_0, \ \forall v' \in V_{out}(G)\}$$

- code parameters

$$R \geq 1 - \frac{md}{n}(1 - R_0) \quad \text{and} \quad \delta \geq \gamma$$

- δ-estimate if G is (c, d)-biregular with $c \geq 3$ and $(\gamma, c(1-\epsilon))$ expander, and $\epsilon < 1 - 1/d_0$ for $\delta_0 = d_0/d$.
- (but regularity assumptions on expander graphs not so realistic for neural codes applications)

Usual Hopfield networks

- N neurons x_i, binary values detect whether the neurons active or inactive in n-th time interval
- network with discrete dynamics

$$x_i(n+1) = \begin{cases} 1 & \text{if } \sum_j W_{ij} x_j(n) + \theta_j > 0 \\ 0 & \text{if } \sum_j W_{ij} x_j(n) + \theta_j \leq 0, \end{cases}$$

  $-\theta_j$ activation thresholds and W_{ij} weight matrix

- energy functional

$$E(x) = -\frac{1}{2} \sum_{i,j} W_{ij} x_i x_j - \sum_i \theta_i x_i$$

- discrete update rule of the dynamics: changing state of a neuron and accepting the change if it decreasing the energy

Higher Hopfield networks from linear codes

- linear code C with constraints specified by the parity check matrix H
- represented as a bipartite graph G: set of (input) nodes the n digits of the code words (code length n), set of (output) nodes the linear constraints imposed by parity check matrix (codim $n-k$)

$$\sum_{v \in N(v')} x_v = 0, \qquad \text{for } v' \in V_{out}(G)$$

- write parity check constraints multiplicatively: spins $s_j = e^{\pi i x_j} \in \{\pm 1\}$

$$\sum_{j\,:\,H_{ij}=1} x_j = 0 \Longrightarrow \prod_{j\,:\,H_{ij}=1} s_j = 1$$

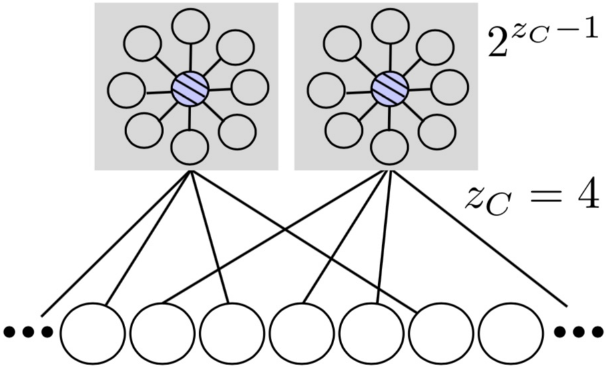
- then energy functional minimized at x_j satisfying H-constraints

$$E(s) = -\sum_i \prod_{j\,:\,H_{ij}=1} s_j$$

- the interactions $\prod_{j\,:\,H_{ij}=1} s_j$ of the higher Hopfield network N-simplex instead of graph vertex... reduce to usual Hopfield networks
- $N' = \#V_{out}(G)$ number of parity-check constraints
- assume $v' \in V_{out}(G)$ have valence $\deg(v') = z_C$
- graph G has $(\gamma, (1-\epsilon))$-expander property if for $S \subset V_{in}(G)$ with $\#S \leq \gamma N$ boundary $\partial S = N(S) \subset V_{out}(G)$ has $\#N(S) \geq c(1-\epsilon)\#S$.
- since $V_{out}(G)$ is set of linear constraints of C and $V_{in}(G)$ set of variables, the expander property means a sufficiently small subset S of variables participates in a significant part of linear constraints
- $z_C = \#N(v')$ input neurons in $V_{in}(G)$ connected to v' can take 2^{z_C} possible states $\sigma = (x_v)_{v\in N(v')}$
- fix a set $\Sigma = \{\sigma_1, \ldots, \sigma_{N'}\} = \{\sigma_{v'}\}_{v'\in V_{out}(G)}$ of such states $\sigma_{v'} = (x_{v,v'})_{v\in N(v')}$ with the property that any two σ_i, σ_j, $i \neq j$ in Σ differ in at least two digits

"Glomerulus" structure (node Hopfield network)

- obtain a "constraint node network" $G_{\nu'}$ with at most $2^{z_C - 1}$ nodes (all with shared inputs from vertices of $N(\nu')$: *glomerulus* (node Hopfield network)



Energy functional

$$E(x,y) = -(x^t U x + \theta y + \frac{1}{2} y^t W y)$$

- $x = (x_v)_{v \in V_{in}(G)}$ and $y = (y_{v'})_{v' \in V_{out}(G)}$
- $U = (U_{v,v'})$ with $v \in V_{in}(G)$ and $v' \in V_{out}(G)$

$$U_{v,v'} = \begin{cases} 1 & x_{v,v'} = 1 \\ -1 & x_{v,v'} = 0. \end{cases}$$

- $\theta = (\theta_{v'})_{v' \in V_{out}(G)}$ be given by

$$\theta_{v'} = \deg(v') - \sum_{v \in N(v')} x_{v,v'},$$

  with $\deg(v') = z_C$
- W the $(N' \times N')$-matrix with zeros on the diagonal and all other entries equal to $-(z_C - 1)$: inhibitory connection between constraint neurons
- energy minima given by $x = \sigma_{v'}$ and $y_u = \delta_{v',u}$

- if neuron within node Hopfield network $G_{v'}$ receives input matching configuration $\sigma_{v'}$ (its "preferred state") becomes active (energetically preferred configuration because of the U-term in the energy)
- suppresses other neurons in $G_{v'}$ because of inhibition W-term
- if input does not match any $\sigma_{v'}$ higher energy configuration
- minimum energy states occur when input neuron configuration satisfies all the linear constraints (and corresponding constraint node Hopfield networks active)
- total number of minimum energy states grows exponentially with size of network
- for sufficiently large N, N' sparse network (with all nodes of constraint node Hopfield networks $G_{v'}$ and all input nodes of G) and has good expander properties
- energy based decoding: flips violate parity check constraints and increase energy: get number of errors by energy value

as brain model (Chaudhuri–Fiete)

- bipartite graph: neurons as the input nodes, output nodes representing small networks of neurons interacting competitively with each other through inhibition
- size of these small networks at output nodes is bounded ($2^{z_C - 1}$) independently of sizes N, N' of vertex set of bipartite graph
- input patterns in a very high-dimensional space (possibly neocortex)
- mapped to the exponentially-many stable states of a bipartite expander Hopfield network (possibly hippocampus)
- providing memory labels for patterns that network can retrieve
- expander codes associated to these networks with good code properties: possible mechanism for passing from bad error correcting properties of neural codes to good codes