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# KONTSEVICH'S NONCOMMUTATIVE NUMERICAL MOTIVES

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Abstract. In this note we prove that Kontsevich's category $\mathrm{NC}_{\mathrm{num}}(k)_F$ of noncommutative numerical motives is equivalent to the one constructed by the authors in [14]. As a consequence, we conclude that $\mathrm{NC}_{\mathrm{num}}(k)_F$ is abelian semi-simple as conjectured by Kontsevich.

## 1. Introduction and statement of results

Over the past two decades Bondal, Drinfeld, Kaledin, Kapranov, Kontsevich, Van den Bergh, and others, have been promoting a broad noncommutative (algebraic) geometry program where "geometry" is performed directly on dg categories; see [1, 2, 3, 5, 6, 8, 10, 11, 12, 13]. Among many developments, Kontsevich introduced a rigid symmetric monoidal category $\mathrm{NC}_{\mathrm{num}}(k)_F$ of noncommutative numerical motives (over a ground field $k$ and with coefficients in a field $F$); consult §4 for details. The key ingredient in his approach is the existence of a well-behaved bilinear form on the Grothendieck group of certain smooth and proper dg categories.

Recently, the authors introduced in [14] an alternative rigid symmetric monoidal category $\mathrm{NNum}(k)_F$ of noncommutative numerical motives; consult §5. In contrast with Kontsevich's approach, the authors used Hochschild homology in order to formalize the word "counting" in the noncommutative world. Our main result is the following:

**Theorem 1.1.** *The categories* $\mathrm{NC}_{\mathrm{num}}(k)_F$ *and* $\mathrm{NNum}(k)_F$ *are equivalent.*

By combining Theorem 1.1 with [14, Thm. 1.9] we then obtain:

**Theorem 1.2.** *Assume that $k$ is a field extension of $F$ or vice-versa. Then, the category* $\mathrm{NC}_{\mathrm{num}}(k)_F$ *is abelian semi-simple.*

Assuming several (polarization) conjectures, Kontsevich conjectured Theorem 1.2 in the particular case where $F = \mathbb{Q}$ and $k$ is of characteristic zero; see [10]. We observe that Kontsevich's beautiful insight not only holds much more generally, but moreover it does not require the assumption of any (polarization) conjecture.

**Notations.** We will work over a (fixed) ground field $k$. The field of coefficients will be denoted by $F$. Let $(\mathcal{C}(k), \otimes, k)$ be the symmetric monoidal category of complexes of $k$-vector spaces. We will use *cohomological* notation, *i.e.* the differential increases the degree.

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## 2. Differential graded categories

A *differential graded (=dg) category* $\mathcal{A}$ (over $k$) is a category enriched over $\mathcal{C}(k)$, *i.e.* the morphism sets $\mathcal{A}(x,y)$ are complexes of $k$-vector spaces and the composition operation fulfills the Leibniz rule $d(f \circ g) = d(f) \circ g + (-1)^{\deg(f)} \circ d(g)$; consult Keller's ICM address [9] for further details.

The *opposite* dg category $\mathcal{A}^{\mathsf{op}}$ has the same objects as $\mathcal{A}$ and complexes of morphisms given by $\mathcal{A}^{\mathsf{op}}(x,y) := \mathcal{A}(y,x)$. The $k$-linear category $\mathsf{H}^0(\mathcal{A})$ has the same objects as $\mathcal{A}$ and morphisms given by $\mathsf{H}^0(\mathcal{A})(x,y) := \mathrm{H}^0\mathcal{A}(x,y)$, where $\mathrm{H}^0$ denotes $0^{\text{th}}$-cohomology. A *right dg $\mathcal{A}$-module* $M$ (or simply a $\mathcal{A}$-module) is a dg functor $M : \mathcal{A}^{\mathsf{op}} \to \mathcal{C}_{\mathsf{dg}}(k)$ with values in the dg category $\mathcal{C}_{\mathsf{dg}}(k)$ of complexes of $k$-vector spaces. We will denote by $\mathcal{C}(\mathcal{A})$ the category of $\mathcal{A}$-modules. Recall from [9, §3] that $\mathcal{C}(\mathcal{A})$ carries a *projective* model structure. Moreover, the differential graded structure of $\mathcal{C}_{\mathsf{dg}}(k)$ makes $\mathcal{C}(\mathcal{A})$ naturally into a dg category $\mathcal{C}_{\mathsf{dg}}(\mathcal{A})$. The dg category $\mathcal{C}_{\mathsf{dg}}(\mathcal{A})$ endowed with the projective model structure is a $\mathcal{C}(k)$*-model category* in the sense of [7, Def. 4.2.18]. Let $\mathcal{D}(\mathcal{A})$ be the *derived category* of $\mathcal{A}$, *i.e.* the localization of $\mathcal{C}(\mathcal{A})$ with respect to the class of weak equivalences. Its full triangulated subcategory of compact objects (*i.e.* those $\mathcal{A}$-modules $M$ such that the functor $\mathsf{Hom}_{\mathcal{D}(\mathcal{A})}(M,-)$ preserves arbitrary sums; see [15, Def. 4.2.7]) will be denoted by $\mathcal{D}_c(\mathcal{A})$.

*Notation* 2.1. We will denote by $\widehat{\mathcal{A}}_{\mathsf{pe}}$ the full dg subcategory of $\mathcal{C}_{\mathsf{dg}}(\mathcal{A})$ consisting of those cofibrant $\mathcal{A}$-modules which become compact in $\mathcal{D}(\mathcal{A})$. Since all the objects in $\mathcal{C}(\mathcal{A})$ are fibrant, and $\mathcal{C}_{\mathsf{dg}}(\mathcal{A})$ is a $\mathcal{C}(k)$-model category, we have natural isomorphisms of $k$-vector spaces

$$\mathrm{H}^i\widehat{\mathcal{A}}_{\mathsf{pe}}(M,N) \simeq \mathsf{Hom}_{\mathcal{D}_c(\mathcal{A})}(M,N[-i]) \qquad i \in \mathbb{Z}\,. \tag{2.2}$$

As any $\mathcal{A}$-module admits a (functorial) cofibrant approximation, we obtain a natural equivalence of triangulated categories $\mathsf{H}^0(\widehat{\mathcal{A}}_{\mathsf{pe}}) \simeq \mathcal{D}_c(\mathcal{A})$.

The *tensor product* $\mathcal{A} \otimes \mathcal{B}$ of two dg categories is defined as follows: the set of objects is the cartesian product of the sets of objects, and the complexes of morphisms are given by $(\mathcal{A} \otimes \mathcal{B})((x,x'),(y,y')) := \mathcal{A}(x,y) \otimes \mathcal{B}(x',y')$. A $\mathcal{A}$-$\mathcal{B}$-*bimodule* $X$ is a dg functor $X : \mathcal{A} \otimes \mathcal{B}^{\mathsf{op}} \to \mathcal{C}_{\mathsf{dg}}(k)$, or in other words a $(\mathcal{A}^{\mathsf{op}} \otimes \mathcal{B})$-module.

*Definition* 2.3 (Kontsevich [10, 11]). A dg category $\mathcal{A}$ is *smooth* if the $\mathcal{A}$-$\mathcal{A}$-bimodule

$$\mathcal{A}(-,-) : \mathcal{A} \otimes \mathcal{A}^{\mathsf{op}} \longrightarrow \mathcal{C}_{\mathsf{dg}}(k) \qquad (x,y) \mapsto \mathcal{A}(y,x)$$

belongs to $\mathcal{D}_c(\mathcal{A}^{\mathsf{op}} \otimes \mathcal{A})$, and *proper* if for each ordered pair of objects $(x,y)$ we have $\sum_i \dim \mathrm{H}^i\mathcal{A}(x,y) < \infty$.

## 3. Noncommutative Chow motives

The rigid symmetric monoidal category $\mathrm{NChow}(k)_F$ of *noncommutative Chow motives* was constructed[1] in [17, 18]. It is defined as the pseudo-abelian envelope of the category whose objects are the smooth and proper dg categories, whose morphisms from $\mathcal{A}$ to $\mathcal{B}$ are given by the $F$-linearized Grothendieck group $K_0(\mathcal{A}^{\mathsf{op}} \otimes \mathcal{B})_F$, and whose composition operation is induced by the tensor product of bimodules. In

[1] In *loc. cit.* we have worked more generally over a ground commutative ring $k$.

analogy with the commutative world, the morphisms of $\mathrm{NChow}(k)_F$ are called *correspondences*. The symmetric monoidal structure is induced by the tensor product of dg categories.

## 4. Kontsevich's approach

In this section we recall and enhance Kontsevich's construction of the category $\mathrm{NC}_{\mathrm{num}}(k)_F$ of noncommutative numerical motives; consult [10]. Let $\mathcal{A}$ be a proper dg category. By construction, the dg category $\widehat{\mathcal{A}}_{\mathsf{pe}}$ is also proper and we have a natural equivalence of triangulated categories $\mathsf{H}^0(\widehat{\mathcal{A}}_{\mathsf{pe}}) \simeq \mathcal{D}_c(\mathcal{A})$. Hence, thanks to the natural isomorphisms (2.2), we can consider the following assignment

$$\mathrm{obj}\,\mathcal{D}_c(\mathcal{A}) \times \mathrm{obj}\,\mathcal{D}_c(\mathcal{A}) \longrightarrow \mathbb{Z} \qquad (M,N) \mapsto \chi(M,N)\,,$$

where $\chi(M,N)$ is the integer

$$\sum_i (-1)^i \dim \mathsf{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, N[-i])\,.$$

Recall that the Grothendieck group $K_0(\mathcal{A})$ of $\mathcal{A}$ can be defined as the Grothendieck group of the triangulated category $\mathcal{D}_c(\mathcal{A})$. A simple verification shows that the above assignment gives rise to a well-defined bilinear form $K_0(\mathcal{A}) \otimes_{\mathbb{Z}} K_0(\mathcal{A}) \to \mathbb{Z}$. By tensoring it with $F$, we then obtain

$$\chi(-,-) : K_0(\mathcal{A})_F \otimes_F K_0(\mathcal{A})_F \longrightarrow F\,. \tag{4.1}$$

The bilinear form (4.1) is in general not symmetric. Let

$$\mathrm{Ker}_L(\chi) := \{\underline{M} \in K_0(\mathcal{A})_F \,|\, \chi(\underline{M}, \underline{N}) = 0 \text{ for all } \underline{N} \in K_0(\mathcal{A})_F\}$$

$$\mathrm{Ker}_R(\chi) := \{\underline{N} \in K_0(\mathcal{A})_F \,|\, \chi(\underline{M}, \underline{N}) = 0 \text{ for all } \underline{M} \in K_0(\mathcal{A})_F\}$$

be, respectively, its left and right kernel. These $F$-linear subspaces of $K_0(\mathcal{A})_F$ are in general distinct. However, as we will prove in Theorem 4.8, they agree when we assume that $\mathcal{A}$ is moreover smooth. In order to prove this result, let us start by recalling Bondal-Kapranov's notion of a Serre functor. Let $\mathcal{T}$ be a $k$-linear Ext-*finite* triangulated category, *i.e.* $\sum_i \dim \mathsf{Hom}_{\mathcal{T}}(M, N[-i]) < \infty$ for any two objects $M$ and $N$ in $\mathcal{T}$. Following Bondal and Kapranov [1, §3], a *Serre functor* $S : \mathcal{T} \stackrel{\sim}{\to} \mathcal{T}$ is an autoequivalence together with bifunctorial isomorphisms

$$\mathsf{Hom}_{\mathcal{T}}(M,N) \simeq \mathsf{Hom}_{\mathcal{T}}(N, S(M))^*\,, \tag{4.2}$$

where $(-)^*$ stands for the $k$-duality functor. Whenever a Serre functor exists, it is unique up to isomorphism.

**Theorem 4.3.** *Let $\mathcal{A}$ be a smooth and proper dg category. Then, the triangulated category $\mathcal{D}_c(\mathcal{A})$ admits a Serre functor.*

*Proof.* Note first that the properness of $\widehat{\mathcal{A}}_{\mathsf{pe}}$, the equivalence of categories $\mathsf{H}^0(\widehat{\mathcal{A}}_{\mathsf{pe}}) \simeq \mathcal{D}_c(\mathcal{A})$, and the natural isomorphisms (2.2), imply that $\mathcal{D}_c(\mathcal{A})$ is Ext-finite. By combining [1, Corollary 3.5] with [3, Thm. 1.3], it suffices then to show that $\mathcal{D}_c(\mathcal{A})$ is pseudo-abelian and that it admits a strong generator; consult [3, page 2] for the notion of *strong* generator. The fact that $\mathcal{D}_c(\mathcal{A})$ is pseudo-abelian is clear from its own definition. In order to prove that it admits a strong generator, we may combine [4, Prop. 4.10] with [9, Thm. 4.12] to conclude that $\mathcal{A}$ is dg Morita equivalent to a dg algebra $A$. Hence, without loss of generality, we may replace $\mathcal{A}$ by $A$. The proof that $\mathcal{D}_c(A)$ admits a strong generator now follows from the arguments of Shklyarov on [16, page 7], which were inspired by Bondal-Van den Bergh's original proof of [3, Thm. 3.1.4]. ☐

**Lemma 4.4.** *Let $\mathcal{A}$ be a smooth and proper dg category and $M, N \in \mathcal{D}_c(\mathcal{A})$. Then, we have the following equalities*

$$\chi(M,N) = \chi(N, S(M)) = \chi(S^{-1}(N), M)\,,$$

*where $S$ is the Serre functor given by Theorem 4.3.*

*Proof.* Consider the following sequence of equalities:

$$\begin{aligned}
\chi(M,N) &= \sum_i (-1)^i \dim \mathsf{Hom}_{\mathcal{D}_c(\mathcal{A})}(M, N[-i]) \\
&= \sum_i (-1)^i \dim \mathsf{Hom}_{\mathcal{D}_c(\mathcal{A})}(N[-i], S(M)) \qquad (4.5)\\
&= \sum_i (-1)^i \dim \mathsf{Hom}_{\mathcal{D}_c(\mathcal{A})}(N, S(M)[i]) \qquad (4.6)\\
&= \chi(N, S(M))\,. \qquad (4.7)
\end{aligned}$$

Equivalence (4.5) follows from the bifunctorial isomorphisms (4.2) and from the fact that a finite dimensional $k$-vector space and its $k$-dual have the same dimension. Equivalence (4.6) follows from the fact that the suspension functor in an autoequivalence of the triangulated category $\mathcal{D}_c(\mathcal{A})$. Finally, equivalence (4.7) follows from a reordering of the finite sum which does not alter the sign of each term. This shows the equality $\chi(M,N) = \chi(N, S(M))$. The equality $\chi(M,N) = \chi(S^{-1}(N), M)$ is proven in a similar way. Simply use

$$\mathsf{Hom}_{\mathcal{T}}(M,N) \simeq \mathsf{Hom}_{\mathcal{T}}(S^{-1}(N), M)^*$$

instead of the bifunctorial isomorphisms (4.2). ☐

**Theorem 4.8.** *Let $\mathcal{A}$ be a smooth and proper dg category. Then,* $\mathrm{Ker}_L(\chi) = \mathrm{Ker}_R(\chi)$*; the resulting well-defined subspace of $K_0(\mathcal{A})_F$ will be denoted by* $\mathrm{Ker}(\chi)$.

*Proof.* We start by proving the inclusion $\mathrm{Ker}_L(\chi) \subseteq \mathrm{Ker}_R(\chi)$. Let $\underline{M}$ be an element of $\mathrm{Ker}_L(\chi)$. Since $K_0(\mathcal{A})_F$ is generated by the elements of shape $[N]$, with $N \in \mathcal{D}_c(\mathcal{A})$, it suffices then to show that $\chi([N], \underline{M}) = 0$ for every such $N$. Note that $\underline{M}$ can be written as $[a_1 M_1 + \cdots + a_n M_n]$, with $a_1, \ldots, a_n \in F$ and $M_1, \ldots, M_n \in \mathcal{D}_c(\mathcal{A})$. We have then the following equalities

$$\begin{aligned}
\chi([N], \underline{M}) &= a_1 \chi(N, M_1) + \cdots + a_n \chi(N, M_n) \\
&= a_1 \chi(M_1, S(N)) + \cdots + a_n \chi(M_n, S(N)) \qquad (4.9)\\
&= \chi(\underline{M}, [S(N)])\,,
\end{aligned}$$

where (4.9) follows from Lemma 4.4. Finally, since by hypothesis $\underline{M}$ belongs to $\mathrm{Ker}_L(\chi)$, we have $\chi(\underline{M}, [S(N)]) = 0$ and so we conclude that $\chi([N], \underline{M}) = 0$. Using the equality $\chi(M,N) = \chi(S^{-1}(N), M)$ of Lemma 4.4, the proof of the inclusion $\mathrm{Ker}_R(\chi) \subseteq \mathrm{Ker}_L(\chi)$ is similar. ☐

Let $(\mathcal{A}, e)$ and $(\mathcal{B}, e')$ be two noncommutative Chow motives. Recall that $\mathcal{A}$ and $\mathcal{B}$ are smooth and proper dg categories and that $e$ and $e'$ are idempotent elements of $K_0(\mathcal{A}^{\mathsf{op}} \otimes \mathcal{A})_F$ and $K_0(\mathcal{B}^{\mathsf{op}} \otimes \mathcal{B})_F$, respectively. Recall also that

$$\mathsf{Hom}_{\mathrm{NChow}(k)_F}((\mathcal{A}, e), (\mathcal{B}, e')) := (e \circ K_0(\mathcal{A}^{\mathsf{op}} \otimes \mathcal{B})_F \circ e')\,. \qquad (4.10)$$

Since smooth and proper dg categories are stable under tensor product (see [4, §4]), the above bilinear form (4.1) (applied to $\mathcal{A} = \mathcal{A}^{\mathsf{op}} \otimes \mathcal{B}$) restricts to a bilinear form

$$\chi(-,-) : \mathsf{Hom}_{\mathrm{NChow}(k)_F}((\mathcal{A}, e), (\mathcal{B}, e')) \otimes_F \mathsf{Hom}_{\mathrm{NChow}(k)_F}((\mathcal{A}, e), (\mathcal{B}, e')) \longrightarrow F\,.$$

By Theorem 4.8 we obtain then a well-defined kernel $\mathrm{Ker}(\chi)$. These kernels (one for each ordered pair of noncommutative Chow motives) assemble themselves in a $\otimes$-ideal $\mathcal{K}\mathrm{er}(\chi)$ of the category $\mathrm{NChow}(k)_F$.

*Definition* 4.11 (Kontsevich [10]). The category $\mathrm{NC}_{\mathrm{num}}(k)_F$ of *noncommutative numerical motives* (over $k$ and with coefficients in $F$) is the pseudo-abelian envelope of the quotient category $\mathrm{NChow}(k)_F/\mathcal{K}\mathrm{er}(\chi)$.

*Remark* 4.12. The fact that $\mathcal{K}\mathrm{er}(\chi)$ is a well-defined $\otimes$-ideal of $\mathrm{NChow}(k)_F$ will become clear(er) after the proof of Theorem 1.1.

## 5. Alternative approach

The authors introduced in [14] an alternative category $\mathrm{NNum}(k)_F$ of noncommutative numerical motives. Let $(\mathcal{A}, e)$ and $(\mathcal{B}, e')$ be two noncommutative Chow motives and $\underline{X} = (e \circ [\sum_i a_i X_i] \circ e')$ and $\underline{Y} = (e' \circ [\sum_j b_j Y_j] \circ e)$ two correspondences. Recall that $X_i$ and $Y_j$ are bimodules and that the sums are indexed by a finite set. The *intersection number* $\langle \underline{X} \cdot \underline{Y} \rangle$ of $\underline{X}$ with $\underline{Y}$ is given by the formula

$$\sum_{i,j,n} (-1)^n \, a_i \cdot b_j \cdot \dim HH_n(\mathcal{A}, X_i \otimes_{\mathcal{B}} Y_j) \in F\,,$$

where $HH_n(\mathcal{A}, X_i \otimes_{\mathcal{B}} Y_j)$ denotes the $n^{\mathrm{th}}$-Hochschild homology group of $\mathcal{A}$ with coefficients in the $\mathcal{A}$-$\mathcal{A}$-bimodule $X_i \otimes_{\mathcal{B}} Y_j$. This procedure gives rise to a well-defined bilinear pairing

$$\langle - \cdot - \rangle : \mathsf{Hom}_{\mathrm{NChow}(k)_F}((\mathcal{A}, e), (\mathcal{B}, e')) \otimes_F \mathsf{Hom}_{\mathrm{NChow}(k)_F}((\mathcal{B}, e'), (\mathcal{A}, e)) \longrightarrow F\,.$$

In contrast with $\chi(-,-)$, this bilinear pairing is symmetric. A correspondence $\underline{X}$ is *numerically equivalent to zero* if for every correspondence $\underline{Y}$ the intersection number $\langle \underline{X} \cdot \underline{Y} \rangle$ is zero. As proved in [14, Thm. 1.5], the correspondences which are numerically equivalent to zero form a $\otimes$-ideal $\mathcal{N}$ of the category $\mathrm{NChow}(k)_F$. The *category of noncommutative numerical motives* $\mathrm{NNum}(k)_F$ is then defined as the pseudo-abelian envelope of the quotient category $\mathrm{NChow}(k)_F/\mathcal{N}$.

## 6. Proof of Theorem 1.1

The proof will consist on showing that the $\otimes$-ideals $\mathcal{K}\mathrm{er}(\chi)$ and $\mathcal{N}$, described respectively in §4 and §5, are exactly the same. As explained in the proof of Theorem 4.3 it is equivalent to work with smooth and proper dg categories or with smooth and proper dg algebras. In what follows we will use the latter approach.

Let $A$ be a dg algebra and $M$ a right dg $A$-module. We will denote by $D(M)$ its *dual*, *i.e.* the left dg $A$-module $\mathcal{C}_{\mathsf{dg}}(A)(M, A)$. This procedure is (contravariantly) functorial in $M$, and thus gives rise to a triangulated functor $\mathcal{D}(A) \to \mathcal{D}(A^{\mathsf{op}})^{\mathsf{op}}$ which restricts to an equivalence $\mathcal{D}_c(A) \stackrel{\sim}{\to} \mathcal{D}_c(A^{\mathsf{op}})^{\mathsf{op}}$. Since the Grothendieck group of a triangulated category is canonically isomorphic to the one of the opposite category, we obtain then an induced isomorphism $K_0(A)_F \stackrel{\sim}{\to} K_0(A^{\mathsf{op}})_F$.

**Proposition 6.1.** *Let $A$ and $B$ be two smooth and proper dg algebras and $X, Y \in \mathcal{D}_c(A^{\mathsf{op}} \otimes B)$. Then, $\chi(X,Y) \in F$ agrees with the categorical trace of the correspondence $[Y \otimes_B D(X)] \in \mathsf{End}_{\mathrm{NChow}(k)_F}((A, \mathrm{id}_A))$.*

*Proof.* The $A$-$B$-bimodules $X$ and $Y$ give rise, respectively, to correspondences $[X] : (A, \mathrm{id}_A) \to (B, \mathrm{id}_B)$ and $[Y] : (A, \mathrm{id}_A) \to (B, \mathrm{id}_B)$ in $\mathrm{NChow}(k)_F$. On the other hand, the $B$-$A$-bimodule $D(X) := \widehat{(A^{\mathsf{op}} \otimes B)}_{\mathsf{pe}}(X, A^{\mathsf{op}} \otimes B) \in \mathcal{D}_c(B^{\mathsf{op}} \otimes A)$ (see Notation 2.1) gives rise to a correspondence $[D(X)] : (B, \mathrm{id}_B) \to (A, \mathrm{id}_A)$. We can then consider the following composition

$$[Y \otimes_B D(X)] : (A, \mathrm{id}_A) \xrightarrow{[Y]} (B, \mathrm{id}_B) \xrightarrow{[D(X)]} (A, \mathrm{id}_A)\,. \tag{6.2}$$

Recall from [17] that the $\otimes$-unit of $\mathrm{NChow}(k)_F$ is the noncommutative motive $(k, \mathrm{id}_k)$, where $k$ is the ground field considered as a dg algebra concentrated in degree zero. Recall also that the dual of $(A, \mathrm{id}_A)$ is $(A^{\mathsf{op}}, \mathrm{id}_{A^{\mathsf{op}}})$ and that the evaluation map $(A, \mathrm{id}_A) \otimes (A^{\mathsf{op}}, \mathrm{id}_{A^{\mathsf{op}}}) \stackrel{\mathrm{ev}}{\to} (k, \mathrm{id}_k)$ is given by the class in $K_0(A^{\mathsf{op}} \otimes A)_F$ of $A$ considered as a $A$-$A$-bimodule. Hence, the categorical trace of the correspondence (6.2) is the following composition

$$(k, \mathrm{id}_k) \xrightarrow{[Y \otimes_B D(X)]} (A^{\mathsf{op}}, \mathrm{id}_{A^{\mathsf{op}}}) \otimes (A, \mathrm{id}_A) \simeq (A, \mathrm{id}_A) \otimes (A^{\mathsf{op}}, \mathrm{id}_{A^{\mathsf{op}}}) \xrightarrow{[A]} (k, \mathrm{id}_k)\,.$$

Since the composition operation in $\mathrm{NChow}(k)_F$ is given by the tensor product of bimodules, the above composition corresponds to the class in $K_0(k)_F \simeq F$ of the complex of $k$-vector spaces

$$(Y \otimes_B D(X)) \otimes_{A^{\mathsf{op}} \otimes A} A^{\mathsf{op}}\,. \tag{6.3}$$

Thanks to the natural isomorphisms

$$(Y \otimes_B D(X)) \otimes_{A^{\mathsf{op}} \otimes A} A^{\mathsf{op}} \simeq Y \otimes_{A^{\mathsf{op}} \otimes B} D(X) \simeq \widehat{(A^{\mathsf{op}} \otimes B)}_{\mathsf{pe}}(X, Y)$$

we conclude that (6.3) is naturally isomorphic to $\widehat{(A^{\mathsf{op}} \otimes B)}_{\mathsf{pe}}(X, Y)$. As a consequence they have the same Euler characteristic

$$\sum_i (-1)^i \dim \mathrm{H}^i((Y \otimes_B D(X)) \otimes_{A^{\mathsf{op}} \otimes A} A^{\mathsf{op}}) = \sum_i (-1)^i \dim \mathrm{H}^i(\widehat{(A^{\mathsf{op}} \otimes B)}_{\mathsf{pe}}(X, Y))\,.$$

The natural isomorphisms of $k$-vector spaces (2.2) (applied to $\mathcal{A} = A^{\mathsf{op}} \otimes B$, $M = X$ and $N = Y$) allow us then to conclude that the right hand-side of the above equality agrees with $\chi(X, Y) \in \mathbb{Z}$. On the other hand, the left hand-side is simply the class of the complex (6.3) in the Grothendieck group $K_0(k) = \mathbb{Z}$. As a consequence, this equality holds also on the $F$-linearized Grothendieck group $K_0(k)_F \simeq F$ and so the proof is finished. □

Now, let $(A, e)$ and $(B, e')$ be two noncommutative Chow motives (with $A$ and $B$ dg algebras). As explained above, the duality functor induces an isomorphism $K_0(A^{\mathsf{op}} \otimes B)_F \simeq K_0(B^{\mathsf{op}} \otimes A)_F$ on the $F$-linearized Grothendieck groups. Via the description (4.10) of the Hom-sets of $\mathrm{NChow}(k)_F$, we obtain then an induced duality isomorphism

$$D(-) : \mathsf{Hom}_{\mathrm{NChow}(k)_F}((A, e), (B, e')) \xrightarrow{\sim} \mathsf{Hom}_{\mathrm{NChow}(k)_F}((B, e'), (A, e))\,. \tag{6.4}$$

**Proposition 6.5.** *The following square*

$$\begin{array}{ccc} \mathsf{Hom}_{\mathrm{NChow}(k)_F}((A,e),(B,e')) \otimes_F \mathsf{Hom}_{\mathrm{NChow}(k)_F}((A,e),(B,e')) & \xrightarrow{\chi(-,-)} & F \\ {\scriptstyle (6.4)\otimes \mathrm{id}} \downarrow \simeq & & \| \\ \mathsf{Hom}_{\mathrm{NChow}(k)_F}((B,e'),(A,e)) \otimes_F \mathsf{Hom}_{\mathrm{NChow}(k)_F}((A,e),(B,e')) & \xrightarrow[\langle - \cdot - \rangle]{} & F \end{array}$$

*is commutative.*

*Proof.* Since the $F$-linearized Grothendieck group $K_0(A^{\mathsf{op}} \otimes B)_F$ is generated by the elements of shape $[X]$, with $X \in \mathcal{D}_c(A^{\mathsf{op}} \otimes B)$, and $\chi(-,-)$ and $\langle - \cdot - \rangle$ are bilinear, it suffices to show the commutativity of the above square with respect to the correspondences $\underline{X} = (e \circ [X] \circ e')$ and $\underline{Y} = (e \circ [Y] \circ e')$. By Proposition 6.1, $\chi(\underline{X}, \underline{Y}) = \chi(X, Y) \in F$ agrees with the categorical trace in $\mathrm{NChow}(k)_F$ of the correspondence $[Y \otimes_B D(X)] \in \mathsf{End}_{\mathrm{NChow}(k)_F}((A, \mathrm{id}_A))$.

On the other hand, since the bilinear pairing $\langle - \cdot - \rangle$ is symmetric, we have the following equality $\langle D(\underline{X}) \cdot \underline{Y} \rangle = \langle \underline{Y} \cdot D(\underline{X}) \rangle$. By [14, Corollary 4.4], we then conclude that the intersection number $\langle \underline{Y} \cdot D(\underline{X}) \rangle$ agrees also with the categorical trace of the correspondence $[Y \otimes_B D(X)]$. The proof is then achieved. □

We now have all the ingredients needed to prove Theorem 1.1. We will show that a correspondence $\underline{X} \in \mathsf{Hom}_{\mathrm{NChow}(k)_F}((A,e),(B,e'))$ belongs to $\mathrm{Ker}(\chi)$ if and only if it is numerically equivalent to zero. Assume first that $\underline{X} \in \mathrm{Ker}_R(\chi) = \mathrm{Ker}(\chi)$. Then, by Proposition 6.5, the intersection number $\langle D(\underline{Y}) \cdot \underline{X} \rangle$ is trivial for every correspondence $\underline{Y} \in \mathsf{Hom}_{\mathrm{NChow}(k)_F}((A, \mathrm{id}_A), (B, \mathrm{id}_B))$. The symmetry of the bilinear pairing $\langle - \cdot - \rangle$, combined with isomorphism (6.4), allow us then to conclude that $\underline{X}$ is numerically equivalent to zero.

Now, assume that $\underline{X}$ is numerically equivalent to zero. Once again the symmetry of the bilinear pairing $\langle - \cdot - \rangle$, combined with isomorphism (6.4), implies that $\chi(\underline{Y}, \underline{X}) = 0$ for every correspondence $\underline{Y} \in \mathsf{Hom}_{\mathrm{NChow}(k)_F}((A, \mathrm{id}_A), (B, \mathrm{id}_B))$. As a consequence, $\underline{X} \in \mathrm{Ker}_R(\chi) = \mathrm{Ker}(\chi)$. The above arguments hold for all noncommutative Chow motives and correspondences. Therefore, the $\otimes$-ideals $\mathcal{K}\mathrm{er}(\chi)$ and $\mathcal{N}$, described respectively in §4 and §5, are exactly the same and so the proof of Theorem 1.1 is finished.

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