Noncommutative numerical motives and the Tannakian formalism

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Motives and Noncommutative motives

• Motives (pure): smooth projective algebraic varieties X
cohomology theories H_{dR}, H_{Betti}, H_{etale}, $\dots$
universal cohomology theory: motives $\Rightarrow$ realizations

• NC Motives (pure): smooth propert dg-categories $\mathscr{A}$
homological invariants: K-theory, Hochschild and cyclic cohomology
universal homological invariant: NC motives

dg-categories

$\mathscr{A}$ category whose morphism sets $\mathscr{A}(x, y)$ are complexes of k-modules ($k =$ base ring or field) with composition satisfying Leibniz rule

$$d(f \circ g) = df \circ g + (-1)^{\deg(f)} f \circ dg$$

$\texttt{dgcat} =$ category of (small) dg-categories with dg-functors (preserving dg-structure)

From varieties to dg-categories

$$X \Rightarrow \mathscr{D}^{dg}_{perf}(X)$$

dg-category of perfect complexes

H^0 gives derived category $\mathscr{D}_{perf}(X)$ of perfect complexes of $\mathscr{O}_X$-modules

saturated dg-categories (Kontsevich)

- smooth dgcat: perfect as a bimodule over itself
- proper dgcat: if the complexes $\mathscr{A}(x, y)$ are perfect
- saturated = smooth + perfect

smooth projective variety $X \Rightarrow$ smooth proper dgcat $\mathscr{D}^{dg}_{perf}(X)$
(but also smooth proper dgcat not from smooth proj varieties)

derived Morita equivalences

• $\mathscr{A}^{op}$ same objects and morphisms $\mathscr{A}^{op}(x,y) = \mathscr{A}(y,x)$; right dg $\mathscr{A}$-module: dg-functor $\mathscr{A}^{op} \to \mathscr{C}_{dg}(k)$ (dg-cat of complexes of k-modules); $\mathscr{C}(\mathscr{A})$ cat of $\mathscr{A}$-modules; $\mathscr{D}(\mathscr{A})$ (derived cat of $\mathscr{A}$) localization of $\mathscr{C}(\mathscr{A})$ w/ resp to quasi-isom

• functor $F : \mathscr{A} \to \mathscr{B}$ is derived Morita equivalence iff induced functor $\mathscr{D}(\mathscr{B}) \to \mathscr{D}(\mathscr{A})$ (restriction of scalars) is an equivalence of triangulated categories

• cohomological invariants (K-theory, Hochschild and cyclic cohomologies) are derived Morita invariant: send derived Morita equivalences to isomorphisms symmetric monoidal category $\mathtt{Hmo}$

• homotopy category: dg-categories up to derived Morita equivalences

• $\otimes$ extends from k-algebras to dg-categories

• can be derived with respect to derived Morita equivalences (gives symmetric monoidal structure on $\mathtt{Hmo}$)

• saturated dg-categories = dualizable objects in $\mathtt{Hmo}$ (Cisinski–Tabuada)

• Euler characteristic of dualizable object: $\chi(\mathscr{A}) = HH(\mathscr{A})$ Hochschild homology complex (Cisinski–Tabuada)

Further refinement: $\mathtt{Hmo}_0$

• all cohomological invariants listed are “additive invariants":

$$E : \mathtt{dgcat} \to A, \quad E(\mathscr{A}) \oplus E(\mathscr{B}) = E(|M|)$$

where A additive category and $|M|$ dg-category $Obj(|M|) = Obj(\mathscr{A}) \cup Obj(\mathscr{B})$ morphisms $\mathscr{A}(x, y)$, $\mathscr{B}(x, y)$, $X(x, y)$ with X a $\mathscr{A}$–$\mathscr{B}$ bimodule

• $\mathtt{Hmo}_0$: objects dg-categories, morphisms $K_0\mathrm{rep}(\mathscr{A}, \mathscr{B})$ with $\mathrm{rep}(\mathscr{A}, \mathscr{B}) \subset \mathscr{D}(\mathscr{A}^{op} \otimes^{\mathbb{L}} \mathscr{B})$ full triang subcat of $\mathscr{A}$–$\mathscr{B}$ bimodules X with $X(a, -) \in \mathscr{D}_{perf}(\mathscr{B})$; composition = (derived) tensor product of bimodules

• $\mathscr{U}_A : \mathtt{dgcat} \to \mathtt{Hmo}_0$, id on objects, sends dg-functor to class in Grothendieck group of associated bimodule

• all additive invariants factor through $\mathtt{Hmo}_0$

noncommutative Chow motives (Kontsevich) $\mathrm{NChow}_F(k)$

• $\mathrm{Hmo}_{0;F}$ = the F-linearization of additive category Hmo_0

• $\mathrm{Hmo}_{0;F}^{\natural}$ = idempotent completion of $\mathrm{Hmo}_{0;F}$

• $\mathrm{NChow}_F(k)$ = idempotent complete full subcategory gen by saturated dg-categories

$\mathrm{NChow}_F(k)$:

- Objects: $(\mathscr{A}, e)$ smooth proper dg-categories (and idempotents)
- Morphisms $K_0(\mathscr{A}^{op} \otimes_k^{\mathbb{L}} \mathscr{B})_F$ (correspondences)
- Composition: induced by derived tensor product of bimodules

relation to commutative Chow motives (Tabuada):

$$\mathrm{Chow}_{\mathbb{Q}}(k)/_{-\otimes\mathbb{Q}(1)} \hookrightarrow \mathrm{NChow}_{\mathbb{Q}}(k)$$

commutative motives embed as noncommutative motives after moding out by the Tate motives

orbit category $\mathrm{Chow}_{\mathbb{Q}}(k)/_{-\otimes\mathbb{Q}(1)}$

$(\mathscr{C}, \otimes, \mathbf{1})$ additive, $F-$ *linear*, rigid symmetric monoidal;
$\mathscr{O} \in \mathrm{Obj}(\mathscr{C})$ $\otimes$-invertible object:
orbit category $\mathscr{C}/_{-\otimes\mathscr{O}}$ same objects and morphisms

$$\mathrm{Hom}_{\mathscr{C}/_{-\otimes\mathscr{O}}}(X, Y) = \oplus_{j\in\mathbb{Z}}\mathrm{Hom}_{\mathscr{C}}(X, Y \otimes \mathscr{O}^{\otimes j})$$

Numerical noncommutative motives

M.M., G.Tabuada, *Noncommutative motives, numerical equivalence, and semi-simplicity*, arXiv:1105.2950

$(\mathscr{A}, e)$ and $(\mathscr{B}, e')$ objects in $\mathrm{NChow}_F(k)$ and correspondences

$$\underline{X} = e \circ [\sum_i a_i X_i] \circ e', \qquad \underline{Y} = e' \circ [\sum_j b_j Y_j] \circ e$$

X_i and Y_j bimodules

$\Rightarrow$ intersection number:

$$\langle \underline{X}, \underline{Y} \rangle = \sum_{ij} [HH(\mathscr{A}; X_i \otimes^{\mathbb{L}}_{\mathscr{B}} Y_j)] \in K_0(k)_F$$

with $[HH(\mathscr{A}; X_i \otimes^{\mathbb{L}}_{\mathscr{B}} Y_j)]$ class in $K_0(k)_F$ of Hochschild homology complex of $\mathscr{A}$ with coefficients in the $\mathscr{A}$–$\mathscr{A}$ bimodule $X_i \otimes^{\mathbb{L}}_{\mathscr{B}} Y_j$

numerically trivial: $\underline{X}$ if $\langle \underline{X}, \underline{Y} \rangle = 0$ for all $\underline{Y}$

- $\otimes$-ideal $\mathscr{N}$ in the category $\mathrm{NChow}_F(k)$
- $\mathscr{N}$ largest $\otimes$-ideal strictly contained in $\mathrm{NChow}_F(k)$

numerical motives: $\mathrm{NNum}_F(k)$

$$\mathrm{NNum}_F(k) = \mathrm{NChow}_F(k)/\mathscr{N}$$

abelian semisimple (M.M., G.Tabuada, arXiv:1105.2950)

- $\mathrm{NNum}_F(k)$ is abelian semisimple

analog of Jannsen's result for commutative numerical pure motives

What about Tannakian structures and motivic Galois groups?

For commutative motives this involves standard conjectures (C = Künneth and D = homological and numerical equivalence)

Questions:

- is $\mathrm{NNum}_F(k)$ (neutral) super-Tannakian?
- is there a good analog of the standard conjecture C (Künneth)?
- does this make the category Tannakian?
- is there a good analog of standard conjecture D (numerical = homological)?
- does this neutralize the Tannakian category?
- relation between motivic Galois groups for commutative and noncommutative motives?

Tannakian categories $(\mathscr{C}, \otimes, \mathbf{1})$

F-linear, abelian, rigid symmetric monoidal with $\mathrm{End}(\mathbf{1}) = F$

• Tannakian: $\exists$ K-valued *fiber functor*, K field ext of F: exact faithful $\otimes$-functor $\omega : \mathscr{C} \to \mathrm{Vect}(K)$; neutral if $K = F$

$\omega \Rightarrow$ equivalence $\mathscr{C} \simeq \mathrm{Rep}_F(\mathrm{Gal}(\mathscr{C}))$ affine group scheme (Galois group) $\quad \mathrm{Gal}(\mathscr{C}) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$

• intrinsic characterization (Deligne): F char zero, $\mathscr{C}$ Tannakian iff $\mathrm{Tr}(id_X)$ non-negative integer for each object X

super-Tannakian categories $(\mathscr{C}, \otimes, \mathbf{1})$

F-linear, abelian, rigid symmetric monoidal with $\mathrm{End}(\mathbf{1}) = F$

$s\mathrm{Vect}(K)$ super-vector spaces $\mathbb{Z}/2\mathbb{Z}$-graded

- super-Tannakian: $\exists$ K-valued *super fiber functor*, K field ext of F: exact faithful $\otimes$-functor $\omega : \mathscr{C} \to s\mathrm{Vect}(K)$; neutral if $K = F$

$\omega \Rightarrow$ equivalence $\mathscr{C} \simeq \mathrm{Rep}_F(s\mathrm{Gal}(\mathscr{C}), \epsilon)$ super-reps of affine super-group-scheme (super-Galois group)
$s\mathrm{Gal}(\mathscr{C}) = \underline{\mathrm{Aut}}^{\otimes}(\omega)$ $\quad \epsilon =$ parity automorphism

- intrinsic characterization (Deligne) F char zero, $\mathscr{C}$ super-Tannakian iff Shur finite (if F alg closed then neutral super-Tannakian iff Schur finite)

- Schur finite: symm grp S_n, idempotent $c_\lambda \in \mathbb{Q}[S_n]$ for partition λ of n (irreps of S_n), Schur functors $S_\lambda : \mathscr{C} \to \mathscr{C}$, $\quad S_\lambda(X) = c_\lambda(X^{\otimes n})$
$\mathscr{C} =$ Schur finite iff all objects X annihilated by some Schur functor $S_\lambda(X) = 0$

Main results

M.M., G.Tabuada, *Noncommutative numerical motives, Tannakian structures, and motivic Galois groups*, arXiv:1110.2438

assume either: (i) $K_0(k) = \mathbb{Z}$, F is k-algebra; (ii) k and F both field extensions of a field K

- **Thm 1**: $\mathrm{NNum}_F(k)$ is super-Tannakian; if F alg closed also neutral
- **Thm 2**: standard conjecture $C_{NC}(\mathscr{A})$: the Künneth projectors

$$\pi^{\pm}_{\mathscr{A}} : \overline{HP}_*(\mathscr{A}) \twoheadrightarrow \overline{HP}^{\pm}_*(\mathscr{A}) \hookrightarrow \overline{HP}_*(\mathscr{A})$$

are algebraic: $\pi^{\pm}_{\mathscr{A}} = \overline{HP}_*(\underline{\pi}^{\pm}_{\mathscr{A}})$ with $\underline{\pi}^{\pm}_{\mathscr{A}}$ correspondences. If k field ext of F char 0, sign conjecture implies

$$C^+(Z) \Rightarrow C_{NC}(\mathscr{D}^{dg}_{perf}(Z))$$

i.e. on commutative motives more likely to hold than sign conjecture

- **Thm 3**: k and F char 0, one extension of other: if C_{NC} holds then change of symmetry isomorphism in tensor structure gives category $\mathrm{NNum}^{\dagger}_F(k)$ Tannakian
- **Thm 4**: standard conjecture $D_{NC}(\mathscr{A})$:

$$K_0(\mathscr{A})_F/\sim_{hom} = K_0(\mathscr{A})_F/\sim_{num}$$

homological defined by periodic cyclic homology: kernel of

$$K_0(\mathscr{A})_F = \mathrm{Hom}_{\mathrm{NChow}_F(k)}(k,\mathscr{A}) \xrightarrow{\overline{HP}_*} \mathrm{Hom}_{sVect(K)}(\overline{HP}_*(k), \overline{HP}_*(\mathscr{A}))$$

when k field ext of F char 0: $D(Z) \Rightarrow D_{NC}(\mathscr{D}^{dg}_{perf}(Z))$
i.e. for commutative motives more likely to hold than D conjecture

• **Thm 5**: F ext of k char 0: if C_{NC} and D_{NC} hold then $\mathrm{NNum}^{\dagger}_{F}(k)$ is a neutral Tannakian category with periodic cyclic homology as fiber functor

• **Thm 6**: k char 0: if C, D and C_{NC}, D_{NC} hold then

$$s\mathrm{Gal}(\mathrm{NNum}_k(k) \twoheadrightarrow \mathrm{Ker}(t : s\mathrm{Gal}(\mathrm{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$$

$$\mathrm{Gal}(\mathrm{NNum}^{\dagger}_k(k) \twoheadrightarrow \mathrm{Ker}(t : \mathrm{Gal}(\mathrm{Num}^{\dagger}_k(k)) \twoheadrightarrow \mathbb{G}_m)$$

where t induced by inclusion of Tate motives in the category of (commutative) numerical motives

(using periodic cyclic homology and de Rham cohomology)

Thm 1: Schur finiteness $\overline{HH} : \mathrm{NChow}_F(k) \to \mathscr{D}_c(F)$
F-linear symmetric monoidal functor (Hochschild homology)

$$(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HH}))^\natural \to \mathscr{D}_c(F)$$

faithful F-linear symmetric monoidal

$\mathscr{D}_c(\mathscr{A})$ = full triang subcat of compact objects in $\mathscr{D}(\mathscr{A}) \Rightarrow \mathscr{D}_c(F)$ identified with fin-dim $\mathbb{Z}$-graded F-vector spaces: Shur finite

general fact: $L : \mathscr{C}_1 \to \mathscr{C}_2$ F-linear symmetric monoidal functor: $X \in \mathscr{C}_1$ Schur finite $\Rightarrow L(X) \in \mathscr{C}_2$ Schur finite; L faithful then also converse: $L(X) \in \mathscr{C}_2$ Schur finite $\Rightarrow X \in \mathscr{C}_1$ Schur finite

conclusion: $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HH}))^\natural$ is Schur finite

also $\mathrm{Ker}(\overline{HH}) \subset \mathscr{N}$ with F-linear symmetric monoidal functor
$(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HH}))^\natural \to (\mathrm{NChow}_F(k)/\mathscr{N})^\natural = \mathrm{NNum}_F(k)$

$\Rightarrow \mathrm{NNum}_F(k)$ Schur finite $\Rightarrow$ super-Tannakian

Thm 2: periodic cyclic homology

mixed complex (M, b, B) with $b^2 = B^2 = Bb + bB = 0$, $\deg(b) = 1 = -\deg(B)$: *periodized*

$$\cdots \prod_{n\ even} M_n \stackrel{b+B}{\to} \prod_{n\ odd} M_n \stackrel{b+B}{\to} \prod_{n\ even} M_n \cdots$$

periodic cyclic homology (the derived cat of $\mathbb{Z}/2\mathbb{Z}$-graded complexes

$$HP : \mathtt{dgcat} \to \mathscr{D}_{\mathbb{Z}/2\mathbb{Z}}(k)$$

induces F-linear symmetric monoidal functor

$$\overline{HP}_* : \mathrm{NChow}_F(k) \to s\mathrm{Vect}(F)$$

or to $s\mathrm{Vect}(k)$ if k field ext of F

Note the issue here:

• mixed complex functor symmetric monoidal but 2-periodization not (infinite product don't commute with $\otimes$)

• *lax symmetric monoidal* with $\mathscr{D}_{\mathbb{Z}/2\mathbb{Z}}(k) \simeq S\mathrm{Vect}(k)$ (not fin dim)

• $HP : \mathtt{dgcat} \to S\mathrm{Vect}(k)$ *additive invariant*: through $\mathtt{Hmo}_0(k)$

• $\mathrm{NChow}_F(k) = (\mathtt{Hmo}_0(k)^{sp})^{\natural}_F$ (sp = gen by smooth proper dgcats)

• periodic cyclic hom *finite dimensional* for smooth proper dgcats + a result of Emmanouil

$\Rightarrow$ lax symmetric monoidal $\overline{HP}_* : \mathtt{Hmo}_0(k)^{sp} \to s\mathrm{Vect}(k)$ is symmetric monoidal

standard conjecture C_{NC} (Künneth type)

- $C_{NC}(\mathscr{A})$: Künneth projections

$$\pi^{\pm}_{\mathscr{A}} : \overline{HP}_*(\mathscr{A}) \twoheadrightarrow \overline{HP}^{\pm}_*(\mathscr{A}) \hookrightarrow \overline{HP}_*(\mathscr{A})$$

are algebraic: $\pi^{\pm}_{\mathscr{A}} = \overline{HP}_*(\underline{\pi}^{\pm}_{\mathscr{A}})$ image of correspondences

- then from Keller have

$\overline{HP}_*(\mathscr{D}^{dg}_{perf}(Z)) = HP_*(\mathscr{D}^{dg}_{perf}(Z)) = HP_*(Z) = \oplus_{n\, even} H^n_{dR}(Z)$

- hence $C^+(Z) \Rightarrow C_{NC}(\mathscr{D}^{dg}_{perf}(Z))$ with $\underline{\pi}^{\pm}_{\mathscr{D}^{dg}_{perf}(Z)}$ image of $\underline{\pi}^{\pm}_Z$ under

$\mathrm{Chow}(k) \to \mathrm{Chow}(k)/_{-\otimes\mathbb{Q}(1)} \hookrightarrow \mathrm{NChow}(k)$

classical: (using deRham as Weil cohomology) $C(Z)$ for Z correspondence, the Künneth projections $\pi^n_Z : H^*_{dR}(Z) \twoheadrightarrow H^n_{dR}(Z)$ are algebraic, $\pi^n_Z = H^*_{dR}(\underline{\pi}^n_Z)$, with $\underline{\pi}^n_Z$ correspondences

sign conjecture: $C^+(Z)$: Künneth projectors $\pi^+_Z = \sum_{n=0}^{\dim Z} \pi^{2n}_Z$ are algebraic, $\pi^+_Z = H^*_{dR}(\underline{\pi}^+_Z)$ (hence π^-_Z also)

Thm 3: Tannakian category first steps

• have F-linear symmetric monoidal and also full and essentially surjective functor: $\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*) \to \mathrm{NChow}_F(k)/\mathscr{N}$

• assuming $C_{NC}(\mathscr{A})$: have $\underline{\pi}^{\pm}_{(\mathscr{A},e)} = e \circ \underline{\pi}^{\pm}_{\mathscr{A}} \circ e$; if $\underline{X}$ trivial in $\mathrm{NChow}_F(k)/\mathscr{N}$ intersection numbers $\langle \underline{X}^n, \underline{\pi}^{\pm}_{(\mathscr{A},e)} \rangle$ vanishes ($\mathscr{N}$ is $\otimes$-ideal)

• intersection number is categorical trace of $\underline{X}^n \circ \underline{\pi}^{\pm}_{(\mathscr{A},e)}$ (M.M., G.Tabuada, 1105.2950)

$$\Rightarrow \mathrm{Tr}(\overline{HP}_*(\underline{X}^n \circ \underline{\pi}^{\pm}_{(\mathscr{A},e)}) = \mathrm{Tr}(\overline{HP}^{\pm}_*(\underline{X})^n) = 0$$

trace all n-compositions vanish $\Rightarrow$ nilpotent $\overline{HP}^{\pm}_*(\underline{X})$

• conclude: nilpotent ideal as kernel of

$$\mathrm{End}_{\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*)}(\mathscr{A}, e) \twoheadrightarrow \mathrm{End}_{\mathrm{NChow}_F(k)/\mathscr{N}}(\mathscr{A}, e)$$

• then functor $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^\natural \to \mathrm{NNum}_F(k)$ full conservative essentially surjective: (quotient by $\mathscr{N}$ full and ess surj; idempotents can be lifted along surj F-linear homom with nilpotent

Tannakian category: modification of tensor structure

• $H : \mathscr{C} \to s\mathrm{Vect}(K)$ symmetric monoidal F-linear (K ext of F) faithful, Künneth projectors $\pi_N^{\pm} = H(\underline{\pi}_N^{\pm})$ for $\underline{\pi}_N^{\pm} \in \mathrm{End}_{\mathscr{C}}(N)$ for all $N \in \mathscr{C}$ then modify symmetry isomorphism

$$c^{\dagger}_{N_1,N_2} = c_{N_1,N_2} \circ (e_{N_1} \otimes e_{N_2}) \quad \text{with } e_N = 2\underline{\pi}_N^{+} - id_N$$

• get F-linear symmetric monoidal functor $\mathscr{C}^{\dagger} \stackrel{H}{\to} s\mathrm{Vect}(K) \to \mathrm{Vect}(K)$

• if $P : \mathscr{C} \to \mathscr{D}$, F-linear symmetric monoidal (essentially) surjective, then $P : \mathscr{C}^{\dagger} \to \mathscr{D}^{\dagger}$ (use image of e_N to modify $\mathscr{D}$ compatibly)

• apply to functors $\overline{HP}_* : (\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^{\natural} \to s\mathrm{Vect}(K)$ and $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^{\natural} \to \mathrm{NNum}_F(k)$

$\Rightarrow$ obtain $\mathrm{NNum}^{\dagger}_F(k)$ satisfying Deligne's intrinsic characterization for Tannakian: with $\tilde{N}$ lift to $(\mathrm{NChow}_F(k)/\mathrm{Ker}(\overline{HP}_*))^{\natural,\dagger}$ have

$$\mathrm{rk}(N) = \mathrm{rk}(\overline{HP}_*(\tilde{N})) = \dim(\overline{HP}^{+}_*(\tilde{N})) + \dim(\overline{HP}^{-}_*(\tilde{N})) \geq 0$$

Thm 4: Noncommutative homological motives

$$\overline{HP}_* : \mathrm{NChow}_F(k) \to s\mathrm{Vect}(K)$$

$$K_0(\mathscr{A})_F = \mathrm{Hom}_{\mathrm{NChow}_F(k)}(k, \mathscr{A}) \stackrel{\overline{HP}_*}{\to} \mathrm{Hom}_{s\mathrm{Vect}(K)}(\overline{HP}_*(k), \overline{HP}_*(\mathscr{A}))$$

kernel gives homological equivalence $K_0(\mathscr{A})_F$ mod $\sim_{hom}$

- $D_{NC}(\mathscr{A})$ standard conjecture:

$$K_0(\mathscr{A})_F / \sim_{hom} = K_0(\mathscr{A})_F / \sim_{num}$$

- on $\mathrm{Chow}_F(k)/_{-\otimes \mathbb{Q}(1)}$ induces homological equivalence with sH_{dR} (de Rham even/odd) $\Rightarrow \mathscr{Z}^*_{hom}(Z)_F \twoheadrightarrow K_0(\mathscr{D}^{dg}_{perf}(Z))_F / \sim_{hom}$
- classical cycles $\mathscr{Z}^*_{hom}(Z)_F \simeq \mathscr{Z}^*_{num}(Z)_F$; for numerical $\mathscr{Z}^*_{num}(Z)_F \stackrel{\sim}{\to} K_0(\mathscr{D}^{dg}_{perf}(Z))_F / \sim_{num}$; then get

$$D(Z) \Rightarrow D_{NC}(\mathscr{D}^{dg}_{perf}(Z))$$

Thm 5: assume C_{NC} and D_{NC} then

$$\overline{HP}_* : \mathrm{NNum}^\dagger_F(k) \to \mathrm{Vect}(F)$$

exact faithful $\otimes$-functor: fiber functor $\Rightarrow$ *neutral* Tannakian category $\mathrm{NNum}^\dagger_F(k)$

Thm 6: Motivic Galois groups

- Galois group of neutral Tannakian category $\mathrm{Gal}(\mathrm{NNum}^\dagger_F(k))$ want to compare with commutative case $\mathrm{Gal}(\mathrm{Num}^\dagger_F(k))$
- super-Galois group of super-Tannakian category $s\mathrm{Gal}(\mathrm{NNum}_F(k))$ compare with commutative motives case $s\mathrm{Gal}(\mathrm{Num}_F(k))$
- related question: what are truly noncommutative motives?

Tate triples (Deligne–Milne)

• For $A = \mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$ and $B = \mathbb{G}_m$ or μ_2, Tannakian cat $\mathscr{C}$ with A-grading: A-grading on objects with $(X \otimes Y)^a = \oplus_{a=b+c} X^b \otimes Y^c$; homom $w : B \to \underline{\mathrm{Aut}}^{\otimes}(id_{\mathscr{C}})$ (weight); central hom $B \to \underline{\mathrm{Aut}}^{\otimes}(\omega)$

• Tate triple $(\mathscr{C}, w, T)$: $\mathbb{Z}$-graded Tannakian $\mathscr{C}$ with weight w, invertible object T (Tate object) weight -2

• Tate triple $\Rightarrow$ central homom $w : \mathbb{G}_m \to \mathrm{Gal}(\mathscr{C})$ and homom $t : \mathrm{Gal}(\mathscr{C}) \to \mathbb{G}_m$ with $t \circ w = -2$.

• $H = Ker(t : \mathrm{Gal}(\mathscr{C}) \to \mathbb{G}_m)$ defines Tannakian category $\simeq \mathrm{Rep}(H)$. It is the “quotient Tannakian category" (Milne) of inclusion of subcategory gen by Tate object into $\mathscr{C}$

Galois group and orbit category

• $\mathscr{T} = (\mathscr{C}, w, T)$ Tate triple, $\mathscr{S} \subset \mathscr{C}$ gen by T, pseudo-ab envelope $(\mathscr{C}/_{-\otimes T})^\natural$ of orbit cat $\mathscr{C}/_{-\otimes T}$ is neutral Tannakian with

$$\mathrm{Gal}((\mathscr{C}/_{-\otimes T})^\natural) \simeq \mathrm{Ker}(t : \mathrm{Gal}(\mathscr{C}) \twoheadrightarrow \mathbb{G}_m)$$

• Quotient Tannakian categories with resp to a fiber functor (Milne): $\omega_0 : \mathscr{S} \to \mathrm{Vect}(F)$ then $\mathscr{C}/\omega_0$ pseudo-ab envelope of $\mathscr{C}'$ with same objects as $\mathscr{C}$ and morphisms $\mathrm{Hom}_{\mathscr{C}'}(X, Y) = \omega_0(\underline{Hom}_{\mathscr{C}}(X, Y)^H)$ with X^H largest subobject where H acts trivially

• fiber functor $\omega_0 : X \mapsto \mathrm{colim}_n \mathrm{Hom}_{\mathscr{C}}(\oplus_{r=-n}^n \mathbf{1}(r), X) \in \mathrm{Vect}(F)$
$\Rightarrow$ get $\mathscr{C}' = \mathscr{C}/_{-\otimes T}$

super-Tannakian case: super Tate triples

• Need a super-Tannakian version of Tate triples

• super Tate triple: $\mathscr{S}\mathscr{T} = (\mathscr{C}, \omega, \underline{\pi}^{\pm}, \mathscr{T}^{\dagger})$ with $\mathscr{C}$ = neutral super-Tannakian; $\omega : \mathscr{C} \to s\mathrm{Vect}(F)$ super-fiber functor; idempotent endos: $\omega(\underline{\pi}_X^{\pm}) = \pi_X^{\pm}$ Künneth proj.; neutral Tate triple $\mathscr{T}^{\dagger} = (\mathscr{C}^{\dagger}, w, T)$ with $\mathscr{C}^{\dagger}$ modified symmetry constraint from $\mathscr{C}$ using $\underline{\pi}^{\pm}$

• assuming C and D: a super Tate triple for (comm) num motives

$$(\mathrm{Num}_k(k), \overline{sH^*_{dR}}, \underline{\pi}_X^{\pm}, (\mathrm{Num}_k^{\dagger}(k), w, \mathbb{Q}(1)))$$

super-Tannakian case: orbit category

• $\mathscr{S}\mathscr{T} = (\mathscr{C}, \omega, \underline{\pi}^{\pm}, \mathscr{T}^{\dagger})$ super Tate triple; $\mathscr{S} \subset \mathscr{C}$ full neutral super-Tannakian subcat gen by T

• Assume: $\underline{\pi}^{-}_{T}(T) = 0$; for $K = \mathrm{Ker}(t : \mathrm{Gal}(\mathscr{C}^{\dagger}) \to \mathbb{G}_m)$ of Tate triple $\mathscr{T}^{\dagger}$, if $\epsilon : \mu_2 \to H$ induced $\mathbb{Z}/2\mathbb{Z}$ grading from $t \circ w = -2$; then (H, ϵ) super-affine group scheme is Ker of $s\mathrm{Gal}(\mathscr{C}) \to s\mathrm{Gal}(\mathscr{S})$ and $\mathrm{Rep}_F(H, \epsilon) = \mathrm{Rep}^{\dagger}_F(H)$.

• Conclusion: pseudoabelian envelope of $\mathscr{C}/_{-\otimes T}$ is neutral super-Tannakian and seq of exact $\otimes$-functors $\mathscr{S} \subset \mathscr{C} \to (\mathscr{C}/_{-\otimes T})^{\natural}$ gives

$$s\mathrm{Gal}((\mathscr{C}/_{-\otimes T})^{\natural}) \xrightarrow{\sim} \mathrm{Ker}(t : s\mathrm{Gal}(\mathscr{C}) \to \mathbb{G}_m)$$

• have also $(\mathscr{C}^{\dagger}/_{-\otimes T})^{\natural} \simeq (\mathscr{C}/_{-\otimes T})^{\natural,\dagger} \simeq \mathrm{Rep}^{\dagger}_F(H, \epsilon) \simeq \mathrm{Rep}_F(H)$

Then for Galois groups:

• then surjective $\mathrm{Gal}(\mathrm{NNum}_k^{\dagger}(k)) \twoheadrightarrow \mathrm{Gal}((\mathrm{Num}_k^{\dagger}(k)/_{-\otimes\mathbb{Q}(1)})^{\natural})$ from embedding of subcategory and $\mathrm{Gal}((\mathrm{Num}_k^{\dagger}(k)/_{-\otimes\mathbb{Q}(1)})^{\natural}) = \mathrm{Ker}(t : \mathrm{Num}_k^{\dagger}(k) \to \mathbb{G}_m)$

• for super-Tannakian: surjective (from subcategory) $s\mathrm{Gal}(\mathrm{NNum}_k(k)) \twoheadrightarrow s\mathrm{Gal}((\mathrm{Num}_k(k)/_{-\otimes\mathbb{Q}(1)})^{\natural})$ and $s\mathrm{Gal}((\mathrm{Num}_k(k)/_{-\otimes\mathbb{Q}(1)})^{\natural}) \simeq \mathrm{Ker}(t : s\mathrm{Gal}(\mathrm{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$

• What is kernel? Ker = "truly noncommutative motives"

$$\mathrm{Gal}(\mathrm{NNum}_k^{\dagger}(k)) \twoheadrightarrow \mathrm{Ker}(t : \mathrm{Num}_k^{\dagger}(k) \to \mathbb{G}_m)$$

$$s\mathrm{Gal}(\mathrm{NNum}_k(k)) \twoheadrightarrow \mathrm{Ker}(t : s\mathrm{Gal}(\mathrm{Num}_k(k)) \twoheadrightarrow \mathbb{G}_m)$$

what do they look line? examples (nc tori, ...)? general properties?